LECTURE 2: VECTORS AND DOT PRODUCTS

Welcome to the magical world of vectors, which are useful companions in our multivariable adventure. This topic calls for an obligatory Skyrim joke: “Today I learned about vectors, but then I took an *arrow* to the knee”

1. Definition

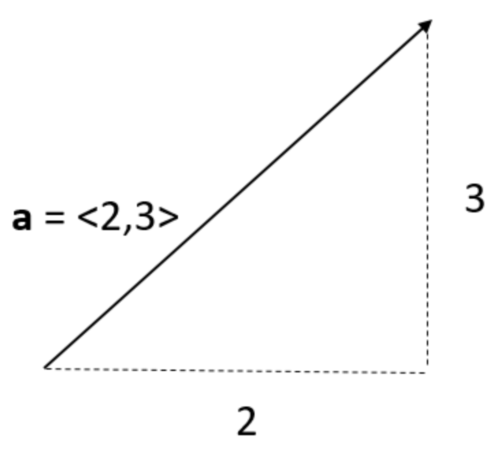
Definition:

A **vector** is an arrow with 2 (or 3) components

Example 1:

Draw $\mathbf{a} = \langle 2, 3 \rangle$

All you need to do is draw an arrow that goes 2 units to the right and 3 units up:



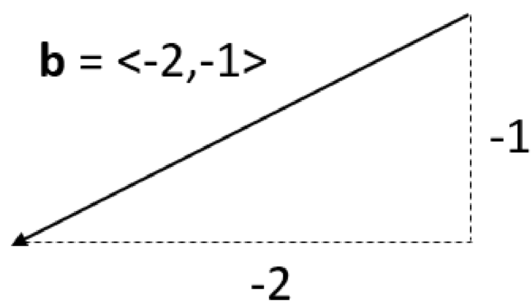
Date: Wednesday, September 1, 2021.

Example 2:

Draw $\mathbf{b} = \langle -2, -1 \rangle$

This time you go 2 units to the left and 1 unit down

Warning: Do not confuse the <u>vector</u> $\langle -2, -1 \rangle$ with the <u>point</u> $(-2, -1)$. Unlike points, vectors have a sense of direction (here right/left and up/down).

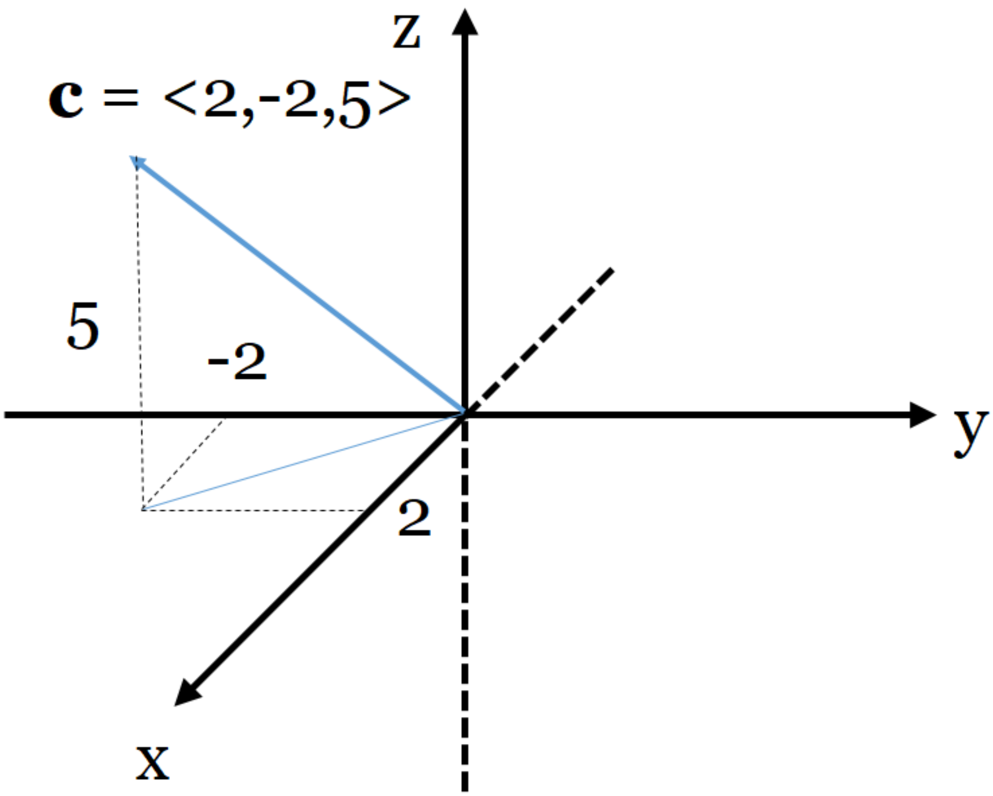


Of course, you can do the same thing in 3 dimensions

Example 3:

Draw $\mathbf{c} = \langle 2, -2, 5 \rangle$

Here you go 2 units to the front, 2 units to the left, and then 5 units up.

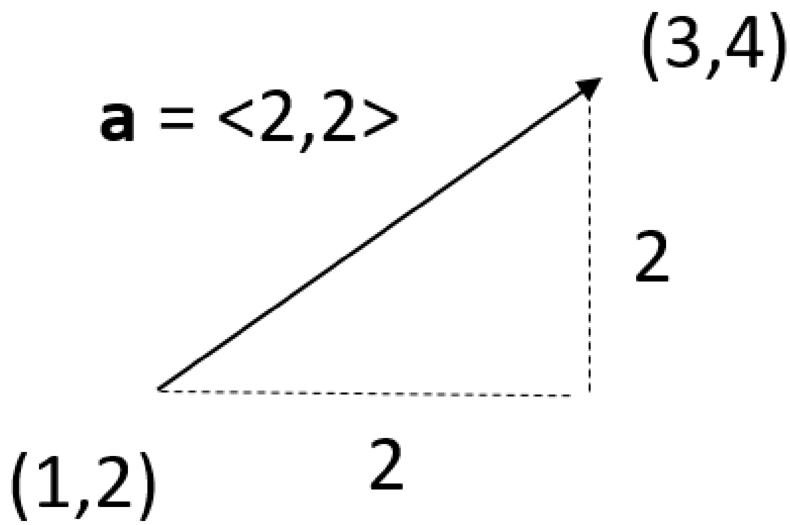


You can even find vectors connecting two points

Example 4:

Draw the vector **a** from $(1, 2)$ to $(3, 4)$

$$\mathbf{a} = \langle 3 - 1, 4 - 2 \rangle = \langle 2, 2 \rangle$$

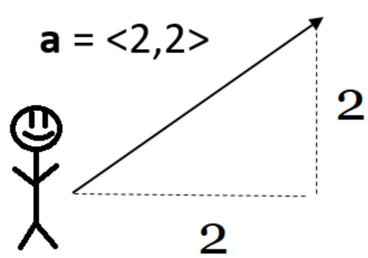


Note: The order matters here; do not confuse this with $\langle 1-3, 2-4 \rangle = \langle -2, -2 \rangle$, which goes the other way around

2. Applications

The world of vectors is filled with applications:

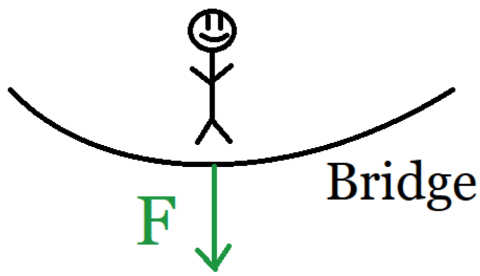
(1) A **velocity vector** represents the direction and magnitude in which a person or an object is moving



Here the person is walking northeast with a speed of 2 mph both to the right and up.

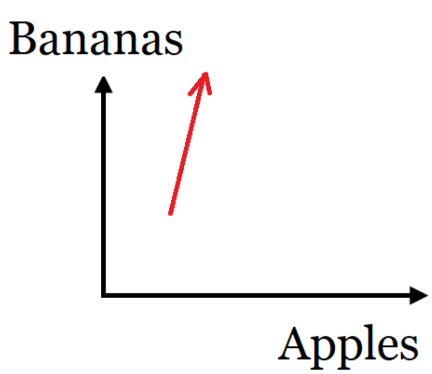
(2) The **force** that an object exerts on another can be represented by a **force** vector; think gravity for example

For example, in engineering, if the force acting on a bridge is too big, it might collapse!



(3) Also appears in electricity and magnetism

(4) Even appears in economics, describes the "trend" of a certain company. For instance, if a company sells Apples and Bananas, the graph below shows that the current trend is for the company to produce more Bananas than Apples



3. Basic Operations

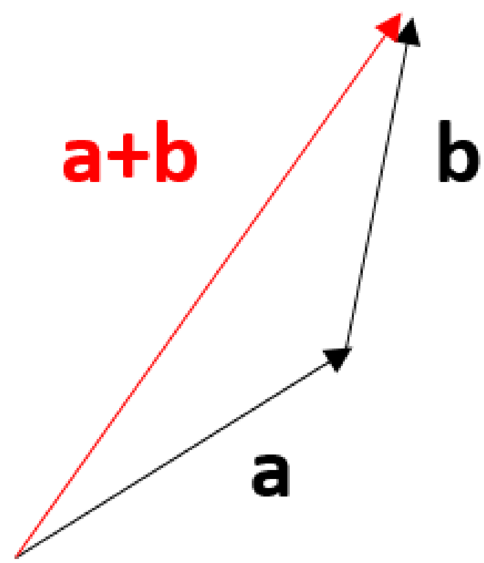
Given two vectors, what can we do to them? Just like for points, we can add them:

Example 5: (Addition)

If $\mathbf{a} = \langle 1, 2 \rangle$ and $\mathbf{b} = \langle 3, 4 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle 1+3, 2+4 \rangle = \langle 4, 6 \rangle$$

You can represent this as gluing the two vectors together (not drawn to scale)

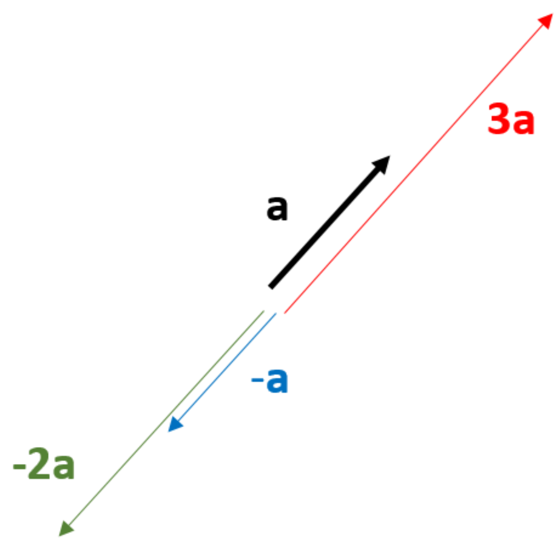


You can also multiply a vector by a number

Example 6: (Scalar Multiplication)

If $\mathbf{a} = \langle 1, 2 \rangle$, then:

$$\begin{aligned} 3\mathbf{a} =& 3\langle 1, 2 \rangle = \langle 3, 6 \rangle \\ -\mathbf{a} =& -\langle 1, 2 \rangle = \langle -1, -2 \rangle \\ -2\mathbf{a} =& -2\langle 1, 2 \rangle = \langle -2, -4 \rangle \end{aligned}$$



Notice all those vectors lie on the same line, but $-\mathbf{a}$ and $-2\mathbf{a}$ go in the opposite direction.

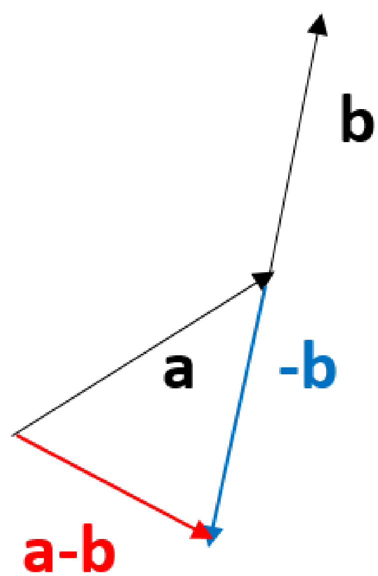
Note: Facts like $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ are still true for vectors.

You can also subtract two vectors, which has a nice geometric interpretation

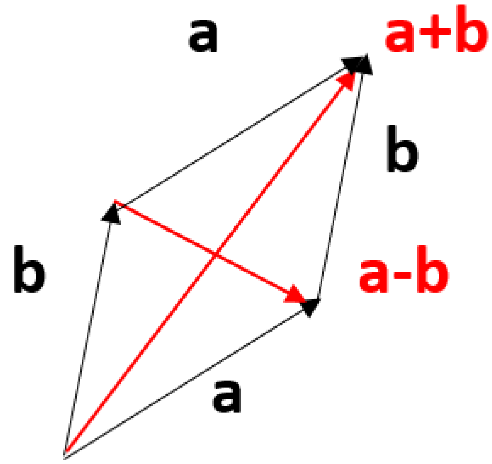
Example 7: (Subtraction)

If $\mathbf{a} = \langle 1, 2 \rangle$ and $\mathbf{b} = \langle 3, 4 \rangle$, then

$$\mathbf{a} - \mathbf{b} = \langle 1 - 3, 2 - 4 \rangle = \langle -2, -2 \rangle$$



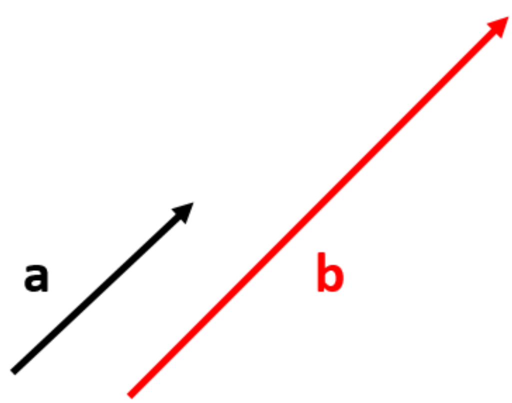
Interpretation: If you compare the picture with the one with $\mathbf{a} + \mathbf{b}$, notice that $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are the diagonals of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$.



Example 8: (Parallel Vectors)

Consider $\mathbf{a} = \langle 1, 2 \rangle$ and $\mathbf{b} = \langle 3, 6 \rangle$

Notice $\mathbf{b} = 3\mathbf{a}$. Also, geometrically, $\mathbf{a}$ and $\mathbf{b}$ are parallel, as in the following figure



Definition: (Parallel Vectors)

$\mathbf{a}$ and $\mathbf{b}$ are **parallel** if $\mathbf{b} = c\,\mathbf{a}$ for some real number c (or $\mathbf{a} = c\,\mathbf{b}$ for some c)

In other words, one vector is a multiple of the other one.

Example 9:

Are the following vectors parallel?

(a) $\langle 2, 4\rangle$ and $\langle -4, -8\rangle$

(b) $\langle 3, 5\rangle$ and $\langle 2, 9\rangle$

(c) $\langle 1, 5, 2\rangle$ and $\langle 3, 15, 6\rangle$

Answers:

(a) Yes, $\langle -4, -8\rangle = (-2)\,\langle 2, 4\rangle$ (negative numbers are ok)

(b) No, $\langle 2, 9\rangle$ is not a multiple of $\langle 3, 5\rangle$

(c) Yes, $\langle 3, 15, 6\rangle = 3\,\langle 1, 5, 2\rangle$

4. Lengths

Another special operation you can do to a vector is to take its length

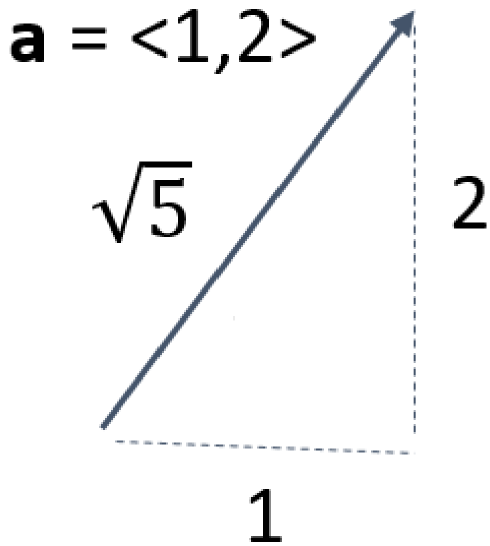
Example 10:

Find the length $\|\mathbf{a}\|$ of $\mathbf{a} = \langle 1, 2\rangle$

Definition:

$$\|\mathbf{a}\| = \|\langle 1, 2\rangle\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Think Pythagorean theorem, it's the length of the hypotenuse of a triangle with sides 1 and 2



Note: The book uses $|\mathbf{a}|$ instead of $\|\mathbf{a}\|$, but this can be easily confused with $|x|$ (absolute value).

Example 11:

Find $\|\mathbf{b}\|$, where $\mathbf{b} = \langle -3, 4\rangle$

$$\|\mathbf{b}\| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

This also works in higher dimensions

Example 12:

Find $\|\mathbf{a}\|$, where $\mathbf{a} = \langle 1, 2, 4 \rangle$

$$\|\mathbf{a}\| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{1 + 4 + 16} = \sqrt{21}$$

It is sometimes useful to produce vectors of length 1 (called unit vectors). Luckily, this is easy to do:

Fact:

$\frac{\mathbf{a}}{\|\mathbf{a}\|}$ always has length 1

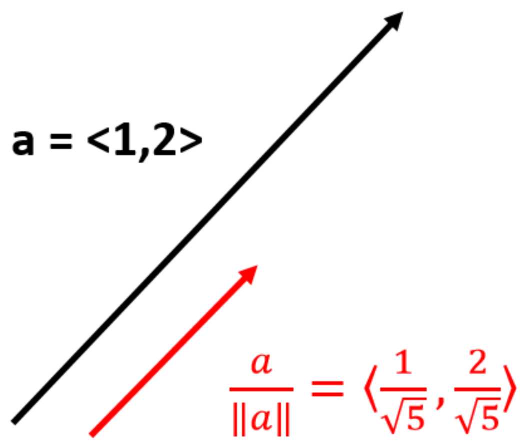
Example 13:

If $\mathbf{a} = \langle 1, 2 \rangle$, then

$$\|\mathbf{a}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

What makes this special is that the new "normalized" vector has the **same** direction as the original one, but now it has length 1 (think same direction, but smaller magnitude)



Example 14:

Normalize $\mathbf{b} = \langle 2, 0, 3 \rangle$

$$\|\mathbf{b}\| = \sqrt{2^2 + 0 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

$$\frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{1}{\sqrt{13}} \langle 2, 0, 3 \rangle = \left\langle \frac{2}{\sqrt{13}}, 0, \frac{3}{\sqrt{13}} \right\rangle$$

Proof of Fact: It's a one-liner! Just calculate the length of:

$$\left\| \frac{\mathbf{a}}{\|\mathbf{a}\|} \right\| = \left\| \frac{1}{\|\mathbf{a}\|} \mathbf{a} \right\| = \frac{1}{\|\mathbf{a}\|} \|\mathbf{a}\| = 1$$

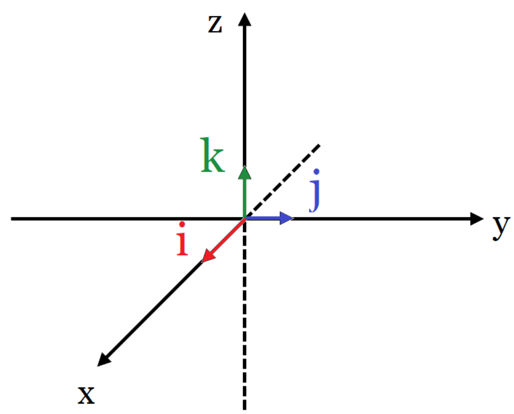
Definition:

The **standard** unit vectors (in 2 dimensions) are

$$\mathbf{i} = \langle 1, 0 \rangle \text{ and } \mathbf{j} = \langle 0, 1 \rangle$$

And in 3 dimensions they are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle, \mathbf{k} = \langle 0, 0, 1 \rangle$$



Example 15:

$$5\mathbf{i} + 6\mathbf{j} - 7\mathbf{k} = \langle 5, 6, -7 \rangle$$

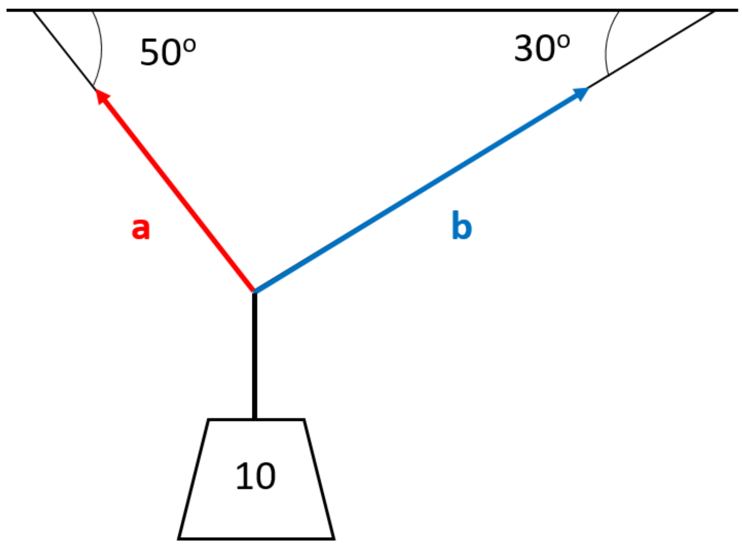
5. Crazy Physics Problem

Finally, since vectors arise in physics and engineering, let's solve an application problem with them.

Warning: This problem is a little bit involved and likely optional for the quizzes and exams (unless mentioned otherwise)

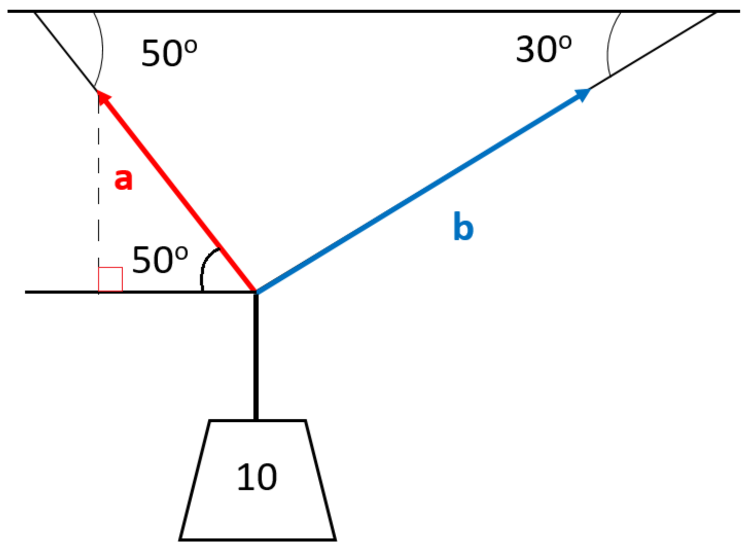
Example 16:

A 10-lb weight hangs from two wires as shown in the picture below. Find the forces $\mathbf{a}$ and $\mathbf{b}$, as well as their lengths $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$.



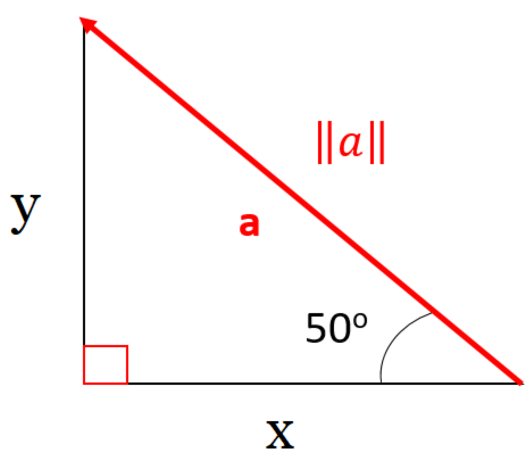
STEP 1: Find $\mathbf{a}$

Let's focus on the left-hand-side of the picture:



(Because of alternating angles, the angle on the bottom left of the picture is also 50°).

Now focus on the right triangle formed by **a** in the picture below. Let's call the sides x and y, and the hypotenuse is by definition $\|\mathbf{a}\|$ (the length of the vector **a**)



By SOHCAHTOA, we have:

$$\cos(50) = \frac{x}{\|\mathbf{a}\|} \Rightarrow x = \|\mathbf{a}\| \cos(50)$$
$$\sin(50) = \frac{y}{\|\mathbf{a}\|} \Rightarrow y = \|\mathbf{a}\| \sin(50)$$

And therefore, we get

$$\mathbf{a} = \langle -x, y \rangle = \langle - \|\mathbf{a}\| \cos(50), \|\mathbf{a}\| \sin(50) \rangle$$

(We put a minus sign because **a** goes to the left, not to the right)

STEP 2: Find **b**

Similarly, we get

$$\mathbf{b} = \langle \|\mathbf{b}\| \cos(30), \|\mathbf{b}\| \sin(30) \rangle$$

(Here we have a plus sign because **b** goes to the right)

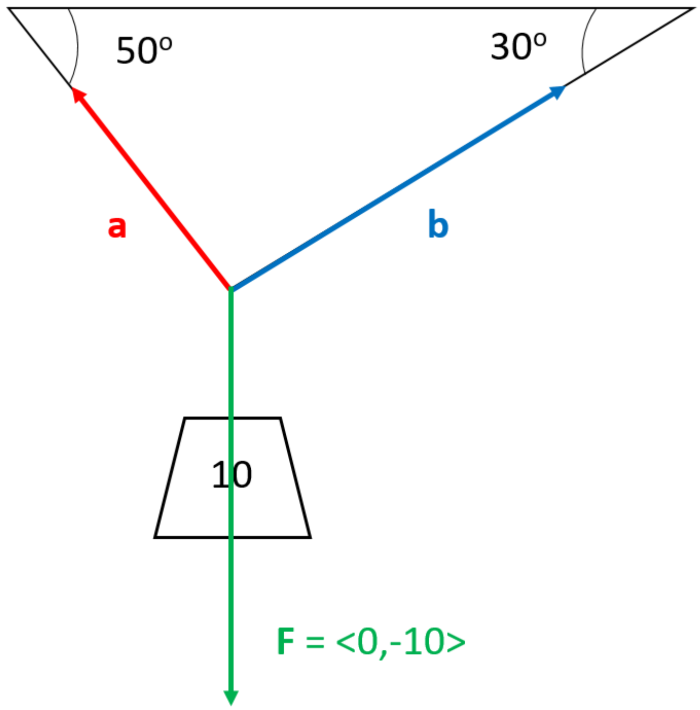
So all that is left to find is $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$. Once we found those, then we're done since the above equations give us **a** and **b**.

STEP 3: Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$.

(Notice so far we haven't used the weight at all)

Let **F** be the force that the weight exerts on the wire. Since the weight is 10lbs and is pulling down the wire, we have

$$\mathbf{F} = \langle 0, -10 \rangle$$



Important Observation:

Since the weight counterbalances forces **a** and **b** the two wires, we must have:

$$\mathbf{a} + \mathbf{b} = -\mathbf{F} = -\langle 0, -10\rangle = \langle 0, 10\rangle$$

And using our equations for **a** and **b** from **STEPS 1 and 2**, this gives:

$$\underbrace{\langle -\|\mathbf{a}\|\cos(50), \|\mathbf{a}\|\sin(50)\rangle}_{\mathbf{a}} + \underbrace{\langle \|\mathbf{b}\|\cos(30), \|\mathbf{b}\|\sin(30)\rangle}_{\mathbf{b}} = \langle 0, 10\rangle$$

Comparing components, this tells us we need to solve the system:

$$\begin{cases} -\|\mathbf{a}\|\cos(50) + \|\mathbf{b}\|\cos(30) = 0 \\ \|\mathbf{a}\|\sin(50) + \|\mathbf{b}\|\sin(30) = 10 \end{cases}$$

Using this and some algebra (which I'll skip here), we can solve for $\|\mathbf{b}\|$ in terms of $\|\mathbf{a}\|$ and ultimately find:

$$\|\mathbf{a}\| = \frac{10}{\sin(50) + \tan(30)\cos(50)} \approx 8.79 \text{ lbs}$$

$$\|\mathbf{b}\| = \frac{\|\mathbf{a}\|\cos(50)}{\cos(30)} \approx 6.53 \text{ lbs}$$

STEP 4: Using the equations in **STEPS 1 and 2**, we ultimately get

$$\mathbf{a} = \langle -\|\mathbf{a}\|\cos(50), \|\mathbf{a}\|\sin(50)\rangle \approx \langle -5.65, 6.73\rangle$$

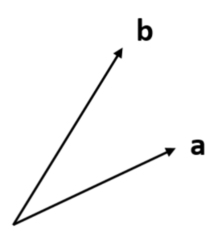
$$\mathbf{b} = \langle \|\mathbf{b}\|\cos(30), \|\mathbf{b}\|\sin(30)\rangle \approx \langle 5.65, 3.27\rangle$$

6. The Dot Product

Let's move on to a useful way of multiplying vectors, called the dot product

Example 17:

Let $\mathbf{a} = \langle 1, 2\rangle$ and $\mathbf{b} = \langle 3, 4\rangle$



Definition:

$$\mathbf{a} \cdot \mathbf{b} = (1)(3) + (2)(4) = 3 + 8 = 11$$

Note: $\mathbf{a} \cdot \mathbf{b}$ is a **number**, not a vector. It intuitively measures how "close" $\mathbf{a}$ and $\mathbf{b}$ are (see below)

Example 18:

$$\langle 2, -3 \rangle \cdot \langle 4, 8 \rangle = (2)(4) + (-3)(8) = 8 - 24 = -16$$

(The dot product can be negative)

Example 19:

$$\langle 1, 2, 3 \rangle \cdot \langle 4, 5, 6 \rangle = (1)(4) + (2)(5) + (3)(6) = 4 + 10 + 18 = 32$$

Example 20:

Calculate $\mathbf{a} \cdot \mathbf{a}$ and $\|\mathbf{a}\|$, where $\mathbf{a} = \langle 1, 2, 3 \rangle$

$$\mathbf{a} \cdot \mathbf{a} = \langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle = 1^2 + 2^2 + 3^2 = 14$$

$$\|\mathbf{a}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

Notice those two are related! In fact:

Fact:

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$$

(We'll use this fact at the end)

7. Applications

There are many ways in which the dot product is useful:

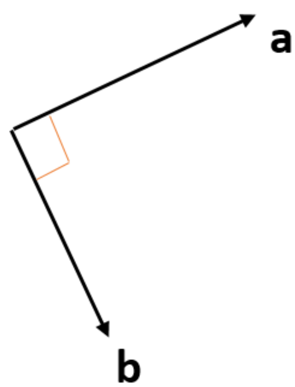
(1) Perpendicular: First of all, it gives us a 1 second way of checking if two vectors are perpendicular.

Example 21:

Calculate $\mathbf{a} \cdot \mathbf{b}$, where $\mathbf{a} = \langle 1, 1 \rangle$ and $\mathbf{b} = \langle 1, -1 \rangle$

$$\langle 1, 1 \rangle \cdot \langle 1, -1 \rangle = (1)(1) + (1)(-1) = 0$$

But also notice that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular!



This is always true:

Fact:

$\mathbf{a}$ and $\mathbf{b}$ are perpendicular $\Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$

Example 22:

Are the following two vectors perpendicular?

$$\mathbf{a} = \langle 1, 2, 7 \rangle, \mathbf{b} = \langle 3, 2, -1 \rangle$$

$$\langle 1, 2, 7 \rangle \cdot \langle 3, 2, -1 \rangle = (1)(3) + (2)(2) + (7)(-1) = 3 + 4 - 7 = 0 \checkmark$$

Hence they are perpendicular.

Example 23:

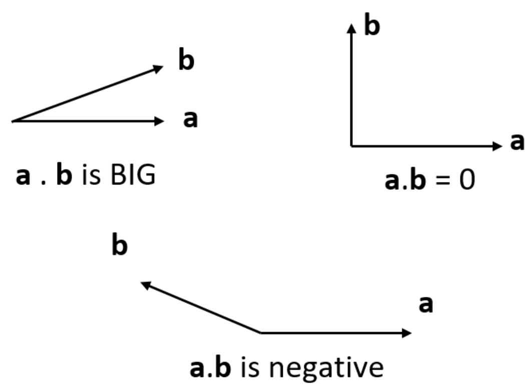
For which t are the following vectors perpendicular

$$\mathbf{a} = \langle t, 5, -1 \rangle, \mathbf{b} = \langle t, t, -6 \rangle$$

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= \langle t, 5, -1 \rangle \cdot \langle t, t, -6 \rangle \\
&= (t)(t) + (5)(t) + (-1)(-6) \\
&= t^2 + 5t + 6 \\
&= (t + 2)(t + 3) \\
&= 0
\end{aligned}$$

Which is true if and only if $t = -2$ or $t = -3$.

(2) Geometric Interpretation: As mentioned above, the dot product measures how "close" two vectors are in terms of directions. In fact, consider the following 3 scenarios:



In the first scenario, **a** and **b** are close to each other, so their dot product is big

In the second scenario, **a** and **b** are perpendicular, so their dot product is 0

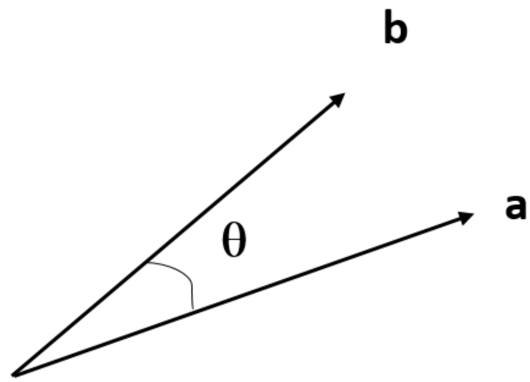
Finally, in the last scenario, **a** and **b** point away from each other so their dot product is large and negative (think -10000)

(the length of **a** and **b** also play a role, as in the formula below)

(3) Angles: In fact, one can even use the dot product to find the **angle** between two vectors.

Example 24:

Find the angle between $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle 0, 1, -2 \rangle$



Angle Formula:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\theta)$$

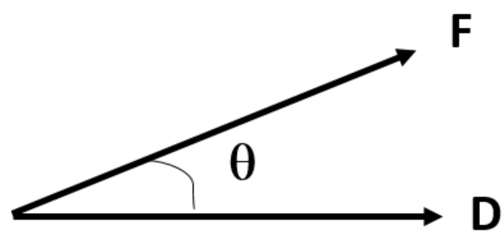
Here:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (1)(0) + (2)(1) + (3)(-2) = 2 - 6 = -4 \\ \|\mathbf{a}\| &= \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \\ \|\mathbf{b}\| &= \sqrt{0^2 + 1^2 + (-2)^2} = \sqrt{5} \end{aligned}$$

Therefore the angle formula gives:

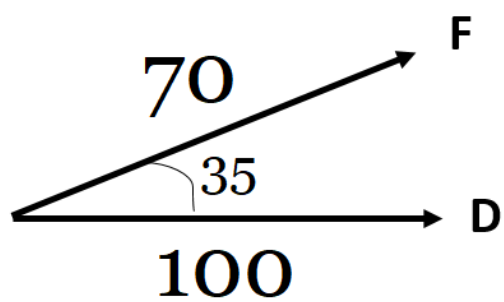
$$\begin{aligned} -4 &= \sqrt{14}\sqrt{5}\cos(\theta) \\ -4 &= \sqrt{70}\cos(\theta) \\ \cos(\theta) &= \frac{-4}{\sqrt{70}} \\ \theta &= \cos^{-1}\left(\frac{-4}{\sqrt{70}}\right) \\ \theta &\approx 119^\circ \end{aligned}$$

(4) Physical Interpretation: If $\mathbf{F}$ is a Force and $\mathbf{D}$ is a displacement vector, then $\mathbf{F} \cdot \mathbf{D}$ is the **work** done of $\mathbf{F}$ on $\mathbf{D}$.



Example 25:

Find the work done of a force $\mathbf{F}$ of 70 N on a displacement $\mathbf{D}$ of 100 m, given that the angle is 35°



By the definition of work and the angle formula, we have

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{D} \\ &= \|\mathbf{F}\| \, \|\mathbf{D}\| \cos(\theta) \\ &= 70 \times 100 \times \cos(35^\circ) \\ &\approx 5734 \text{ N } \cdot \text{ m} \end{aligned}$$

(And N · m is sometimes called Joules, J)

Example 26:

Find the work done when a force $\mathbf{F} = \langle 2, 0, 3 \rangle$ moves an object from $A = (1, 2, 4)$ to $B = (3, 5, 0)$

The displacement is $\mathbf{D} = \overrightarrow{AB} = \langle 3 - 1, 5 - 2, 0 - 4 \rangle = \langle 2, 3, -4 \rangle$.

Therefore, by the definition of work, we have

$$W = \mathbf{F} \cdot \mathbf{D} = \langle 2, 0, 3 \rangle \cdot \langle 2, 3, -4 \rangle = 4 + 0 - 12 = -8$$

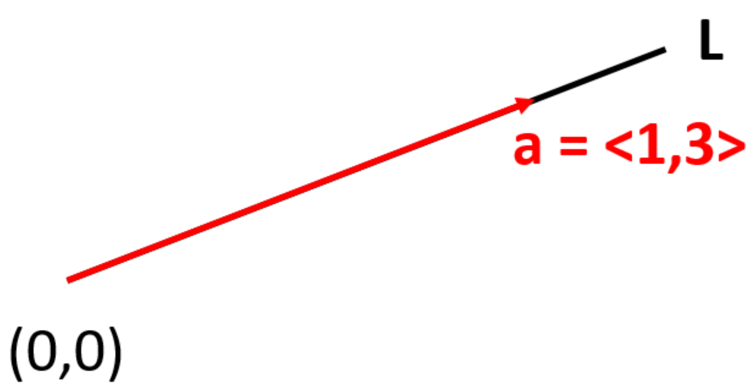
8. Vector Projection

Here is the most important application of dot products: It allows us to project (or squish) a vector on another one

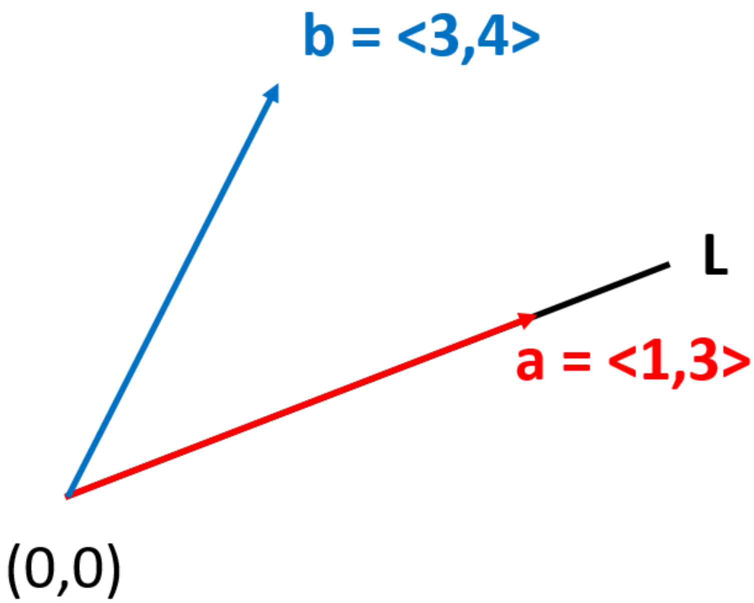
Motivation:

Let $\mathbf{b} = \langle 3, 4 \rangle$ and $\mathbf{a} = \langle 1, 3 \rangle$ be two vectors

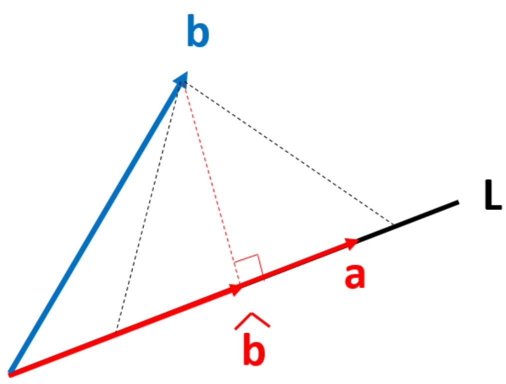
Consider the line L that goes through $(0, 0)$ and with slope $\mathbf{a} = \langle 1, 3 \rangle$:



Now look at $\mathbf{b}$ (which is not L).



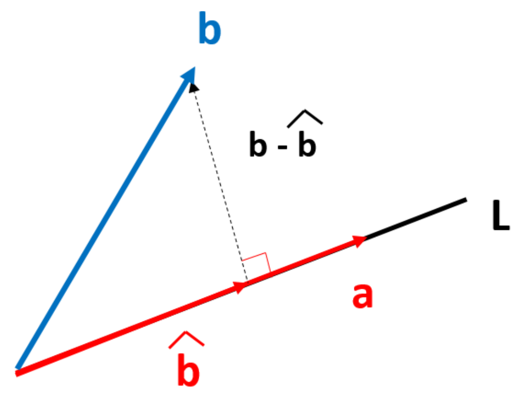
Notice: There are many ways of projecting (squishing) **b** on the line L, but only one that seems optimal, which is called $\widehat{\mathbf{b}}$



Definition:

$\widehat{\mathbf{b}}$ (or $\text{proj}_{\mathbf{a}}\,\mathbf{b}$) is the **vector projection** of **b** on **a**

Note: This is sometimes called the *orthogonal* projection. Why orthogonal? Because it's precisely the vector on L such that $\mathbf{b} - \widehat{\mathbf{b}}$ and $\mathbf{a}$ are *orthogonal*.



How to calculate $\widehat{\mathbf{b}}$?

First of all, $\widehat{\mathbf{b}}$ is parallel to $\mathbf{a}$, and so

$$\widehat{\mathbf{b}} = \text{ (JUNK) } \mathbf{a}$$

This is important! $\widehat{\mathbf{b}}$ is a multiple of $\mathbf{a}$, **NOT** a multiple of $\mathbf{b}$.

Here is the formula for $\widehat{\mathbf{b}}$. We will derive it later.

Vector Projection Formula:

$$\text{proj}_{\mathbf{a}}\, \mathbf{b} = \widehat{\mathbf{b}} = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a}$$

How to remember this?

(1) $\widehat{\mathbf{b}}$ is a multiple of $\mathbf{a}$, so $\widehat{\mathbf{b}} = \text{ (JUNK) } \mathbf{a}$

(2) **Hugging Analogy:** $\mathbf{b}$ hugs $\mathbf{a}$ to get $\mathbf{b}\cdot\mathbf{a}$, and then $\mathbf{a}$ is so happy that it hugs itself to get $\mathbf{a}\cdot\mathbf{a}$

Example 27:

Calculate $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \widehat{\mathbf{b}}$, where $\mathbf{a} = \langle 1, 3\rangle$ and $\mathbf{b} = \langle 3, 4\rangle$

You're projecting/squishing $\mathbf{b}$ on $\mathbf{a}$ and so

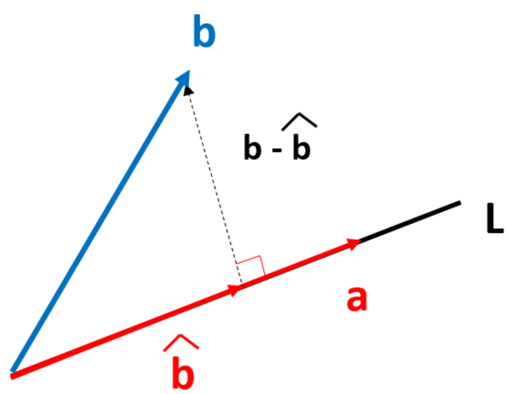
$$\begin{aligned}\widehat{\mathbf{b}} &= \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}\\ &= \left(\frac{\langle 3,4\rangle\cdot\langle 1,3\rangle}{\langle 1,3\rangle\cdot\langle 1,3\rangle}\right)\langle 1,3\rangle\\ &= \frac{15}{10}\langle 1,3\rangle\\ &= \frac{3}{2}\langle 1,3\rangle\\ &= \left\langle \frac{3}{2}, \frac{9}{2}\right\rangle\end{aligned}$$

Example 28:

Calculate $\text{proj}_{\mathbf{a}}\,\mathbf{b}$, where $\mathbf{a} = \langle 3, 6, -2\rangle$ and $\mathbf{b} = \langle 1, 2, 3\rangle$

$$\begin{aligned}\text{proj}_{\mathbf{a}}\,\mathbf{b} = \widehat{\mathbf{b}} &= \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}\\ &= \left(\frac{\langle 1,2,3\rangle\cdot\langle 3,6,-2\rangle}{\langle 3,6,-2\rangle\cdot\langle 3,6,-2\rangle}\right)\langle 3,6,-2\rangle\\ &= \frac{9}{49}\langle 3,6,-2\rangle\\ &= \left\langle \frac{27}{49}, \frac{54}{49}, \frac{-18}{49}\right\rangle\end{aligned}$$

Why this formula works: It's not too bad if you remember the following picture:



STEP 1: First of all, remember that $\widehat{\mathbf{b}}$ is a multiple of $\mathbf{a}$, so for some constant c, we have

$$\widehat{\mathbf{b}} = c\,\mathbf{a}$$

STEP 2: Now remember that $\mathbf{b} - \widehat{\mathbf{b}}$ and $\mathbf{a}$ are perpendicular, so

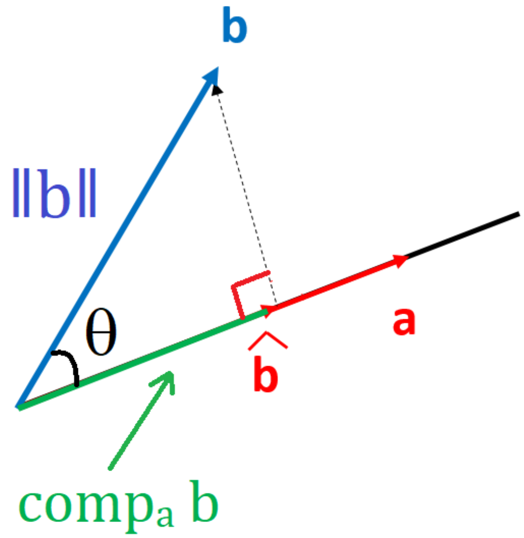
$$\begin{aligned}\left(\mathbf{b} - \widehat{\mathbf{b}}\right) \cdot \mathbf{a} =& 0\\ (\mathbf{b} - c\,\mathbf{a}) \cdot \mathbf{a} =& 0\\ \mathbf{b} \cdot \mathbf{a} - c(\mathbf{a} \cdot \mathbf{a}) =& 0\\ c(\mathbf{a} \cdot \mathbf{a}) =& \mathbf{b} \cdot \mathbf{a}\\ c =& \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\end{aligned}$$

Hence

$$\widehat{\mathbf{b}} = c\,\mathbf{a} = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a}$$

9. Scalar Projection

Related to vector projection, there is the concept of **scalar projection**, which we'll examine now. More precisely, let's look back at our picture with $\widehat{\mathbf{b}}$:



Intuitively: the scalar projection $\text{comp}_{\mathbf{a}}\,\mathbf{b}$ is defined to be the (green) leg of the triangle with hypotenuse $\|\mathbf{b}\|$ and angle θ above.

Derivation: By SOHCAHTOA, we have

$$\cos(\theta) = \frac{\text{comp}_{\mathbf{a}}\,\mathbf{b}}{\|\mathbf{b}\|} \Rightarrow \text{comp}_{\mathbf{a}}\,\mathbf{b} = \|\mathbf{b}\|\cos(\theta)$$

The only issue is that this depends on the unknown angle θ, but using the following trick we can get rid of it:

$$\begin{aligned}\text{comp}_{\mathbf{a}}\,\mathbf{b} &= \|\mathbf{b}\|\cos(\theta)\\ &= \|\mathbf{b}\|\cos(\theta)\frac{\|\mathbf{a}\|}{\|\mathbf{a}\|}\\ &= \frac{\|\mathbf{a}\|\,\|\mathbf{b}\|\cos(\theta)}{\|\mathbf{a}\|}\\ &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} \qquad \text{(Angle Formula)}\end{aligned}$$

Therefore, we obtain:

Scalar Projection Formula:

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$$

Example 29:

Calculate $\text{comp}_{\mathbf{a}}\,\mathbf{b}$, where $\mathbf{a} = \langle 1, 3\rangle$ and $\mathbf{b} = \langle 3, 4\rangle$

$$\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} = \frac{\langle 1,3\rangle\cdot\langle 3,4\rangle}{\|\langle 1,3\rangle\|} = \frac{(1)(3)+(3)(4)}{\sqrt{1^2+3^2}} = \frac{15}{\sqrt{10}} = \frac{15\sqrt{10}}{10} = \frac{3\sqrt{10}}{2}$$

Example 30:

Calculate $\text{comp}_{\mathbf{a}}\,\mathbf{b}$, where $\mathbf{a} = \langle 1, -4, 2\rangle$ and $\mathbf{b} = \langle 3, 8, 4\rangle$

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} = \frac{3-32+8}{\sqrt{1+16+4}} = \frac{-21}{\sqrt{21}} = -\sqrt{21}$$

10. Application of Projections

Why care about projections? Here are some nice applications

Example 31:

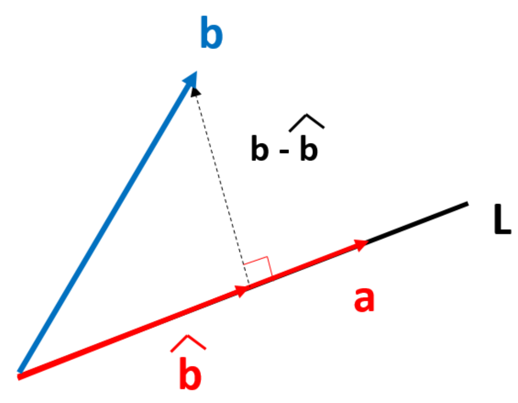
Let $\mathbf{b} = \langle 3, 4 \rangle$ and $\mathbf{a} = \langle 1, 3 \rangle$

Previously we found that:

$$\widehat{\mathbf{b}} = \text{proj}_{\mathbf{a}}\,\mathbf{b} = \left\langle \frac{3}{2}, \frac{9}{2} \right\rangle \text{ and } \text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{3\sqrt{10}}{2}$$

Application 1: Projections are useful to find a vector that's perpendicular to a given one.

(a) Find a vector that's perpendicular to $\mathbf{a} = \langle 1, 3 \rangle$



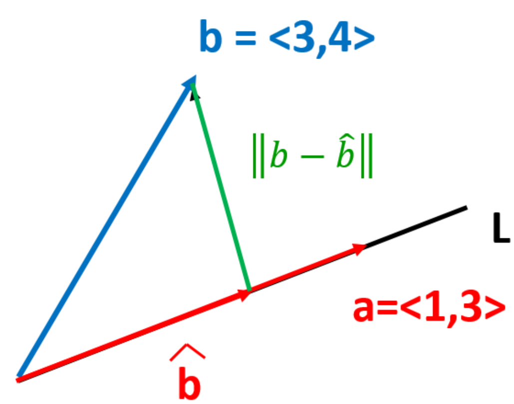
According to the picture above, the answer is *precisely* $\mathbf{b} - \widehat{\mathbf{b}}$

$$\text{Answer: } \mathbf{b} - \widehat{\mathbf{b}} = \langle 3, 4 \rangle - \left\langle \frac{3}{2}, \frac{9}{2} \right\rangle = \left\langle \frac{3}{2}, -\frac{1}{2} \right\rangle$$

Note: Of course here you could directly just guess $\langle -3, 1\rangle$, but the point is that this technique works in any dimensions.

Application 2: We can use projections to find the shortest distance from a point to a line.

(b) Find the (shortest) distance from the point $(3,4)$ to the line L containing $\mathbf{a} = \langle 1, 3\rangle$

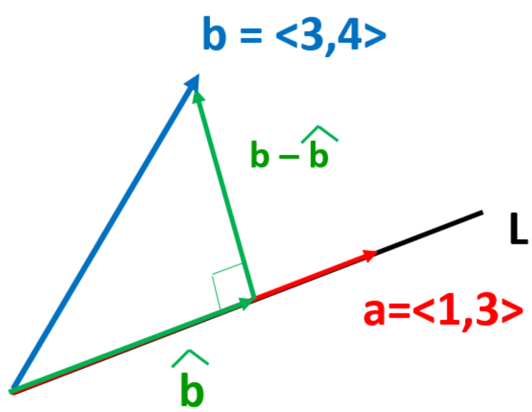


Again according to the picture, the answer is the length of the vector found in (a), that is $\left\|\mathbf{b} - \widehat{\mathbf{b}}\right\|$:

$$\text{Answer: } \left\|\mathbf{b} - \widehat{\mathbf{b}}\right\| = \left\|\left\langle \frac{3}{2}, -\frac{1}{2}\right\rangle\right\| = \frac{\sqrt{10}}{2}$$

Application 3: Finally, projections allow us to decompose vectors in a way that is especially useful in physics.

(c) Write $\mathbf{b} = \langle 3, 4 \rangle$ as the sum of two vectors, one parallel to $\mathbf{a} = \langle 1, 3 \rangle$ and one perpendicular to $\mathbf{a}$



$$\text{Trick: } \mathbf{b} = \widehat{\mathbf{b}} + (\mathbf{b} - \widehat{\mathbf{b}}) = \underbrace{\left\langle \frac{3}{2}, \frac{9}{2} \right\rangle}_{\text{parallel to } \mathbf{a}} + \underbrace{\left\langle \frac{3}{2}, -\frac{1}{2} \right\rangle}_{\text{perpendicular to } \mathbf{a}}$$

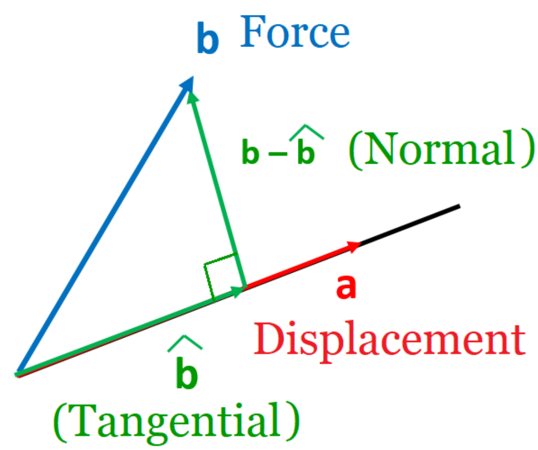
Physics Interpretation:

If $\mathbf{b}$ is force and $\mathbf{a}$ is displacement, then:

$\widehat{\mathbf{b}}$ is the **tangential component** of the force $\mathbf{b}$ (along $\mathbf{a}$)

$\mathbf{b} - \widehat{\mathbf{b}}$ is the **normal component** of the force (perpendicular to $\mathbf{a}$).

$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{3\sqrt{10}}{2}$ is the (signed) length of the tangential component of the force.



So in (c), we effectively rewrote the force as the sum of a tangential and normal components.