Topology and θ dependence in nonabelian gauge theories: recent results from the lattice

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Outline

The θ term

$$\mathcal{L}^{\theta}_{QCD} = \mathcal{L}_{QCD} + \theta q(x); \qquad q(x) = \frac{g^2}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{a\,\mu\nu}F^{a\,\rho\sigma}$$

Main properties of the θ term:

- θ is a dimensionless RG invariant parameter in $[0, 2\pi)$
- the θ term is a four-divergence: no effect on the classical equations of motion, purely quantistic and nonperturbative effects
- on smooth configurations $Q = \int q(x)\mathrm{d}^4x \in \mathbb{Z}$
- behaviour under $U(1)_A$: if $\psi_j \to e^{i\alpha\gamma_5}\psi_j$ and $\bar{\psi}_j \to \bar{\psi}_j e^{i\alpha\gamma_5}$ then $\theta \to \theta - 2\alpha N_f$ and $m_j \to m_j e^{2i\alpha}$
- the θ term becomes imaginary in the Euclidean formulation
- it breaks explicitly P and CP simmetry
- experimentally θ is compatible with zero ($|\theta| \lesssim 10^{-9}$ from neutron electric dipole moment). Strong CP problem.

Why studying θ dependence

θ-dependence = dependence on θ of the vacuum energy $E(\theta)$ (at $T = 0$) or of the free energy $F(\theta, T)$

Why studying θ dependence if $\theta \approx 0$ experimentally?

theory: to better understand some nonperturbative features of Yang-Mills theory and QCD. To investigate the range of validity of different approximation schemes.

SM phenomenology: to get informations on some hadronic properties e.g. $m^2_{\eta'} = \frac{2N_f}{f^2_\pi}\chi_{top}^{N_f=0}$ Witten 1979, Veneziano 1979.

BSM phenomenology: the introduction of a new pseudoscalar particle (axion) was proposed to solve the strong CP problem; such a particle is also a natural DM candidate. The QCD θ dependence can be used to obtain (under specific cosmological assumptions) lower bounds for the axion coupling $1/f_a$.

The general form of $F(\theta, T)$

$$F(\theta, T) = -\frac{1}{V_4} \log \int [\mathscr{D}A][\mathscr{D}\bar{\psi}][\mathscr{D}\psi] \exp\left(-\int_0^{1/T} \mathrm{d}t \int \mathrm{d}^3x \, \mathcal{L}_\theta^E \right)$$

Assuming analiticity at $\theta = 0$ the free energy density can be written as:

$$F(\theta, T) - F(0, T) = \frac{1}{2}\chi(T)\theta^2 \left[1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \cdots \right] ,$$

and it is easy to see that

$$\chi = \frac{1}{V_4}\langle Q^2 \rangle_0 \qquad b_2 = -\frac{\langle Q^4 \rangle_0 - 3\langle Q^2 \rangle_0^2}{12\langle Q^2 \rangle_0}$$

$$b_4 = \frac{\langle Q^6 \rangle_0 - 15\langle Q^2 \rangle_0 \langle Q^4 \rangle_0 + 30\langle Q^2 \rangle_0^3}{360\langle Q^2 \rangle_0}$$

and so on, where $\langle \quad \rangle_0$ denotes the average at $\theta = 0$.

Large N_c and χPT

Large N_c

The scaling variable to keep fixed is $\bar{\theta} \equiv \theta/N_c$ and one gets (Witten 1980)

$$\chi = \bar{\chi} + \cdots$$
$$b_{2n} = \bar{b}_{2n}/N_c^{2n} + \cdots$$

Chiral perturbation theory

LO at $T = 0$ (Di Vecchia, Veneziano 1980)

$$E_0(\theta) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} \sin^2 \frac{\theta}{2}}$$

NLO (Grilli di Cortona, Hardy, Pardo Vega, Villadoro 1511.02867)

$$z \equiv m_u/m_d = 0.48(3) \qquad \chi^{1/4} = 75.5(5)\,\mathrm{MeV} \qquad b_2 = -0.029(2)$$
$$z = 1 \qquad \chi^{1/4} = 77.8(4)\,\mathrm{MeV} \qquad b_2 = -0.022(1)$$

Dilute Instanton Gas Approximation (1)

Hypothesis: the dynamic of the system is dominated by weakly interacting objects of topological charge ± 1.

This is surely true in the weak coupling approximation ($T \gg \Lambda_{QCD}$).

In the DIGA approximation we thus have (Gross, Pisarski, Yaffe 1981)

$$Z_\theta = \mathrm{Tr} e^{-H_\theta/T} \approx \sum \frac{1}{n_+! n_-!} (V_4 D)^{n_+ + n_-} e^{-S_0(n_+ + n_-) + i\theta(n_+ - n_-)}$$
$$= \exp\left[2 V_4 D e^{-S_0} \cos\theta\right]$$

where $1/D$ is a typical 4−volume, that in perturbation theory is related to the functional determinants of the fields in the instanton background. Using DIGA without perturbation theory we thus have:

$$F(\theta, T) - F(0, T) \approx \chi(T)(1 - \cos\theta)$$

Dilute Instanton Gas Approximation (2)

Using only the DIGA hypothesis we have informations on the explicit values of $b_{2n}(T)$ but not on $\chi(T)$:

$$b_2 = -\frac{1}{12} \quad b_4 = \frac{1}{360} \quad b_{2n} = (-1)^n \frac{2}{(2n+2)!}$$

Using also perturbation theory $S_0 = \frac{8\pi^2}{g^2(T)} \simeq (\frac{11}{3}N - \frac{2}{3}N_f)\log(T/\Lambda)$ and close to the chiral limit $D \propto T^4(m/T)^{N_f}$, so that

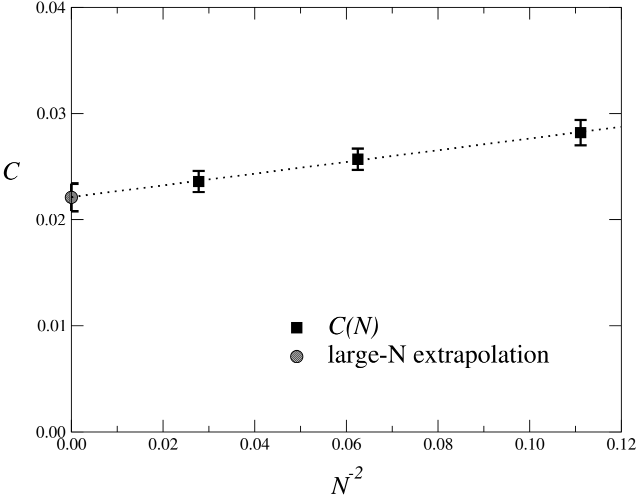
$$\chi(T) \propto m^{N_f} T^{4-\frac{11}{3}N-\frac{1}{3}N_f}$$

(Gross, Pisarski, Yaffe 1981)

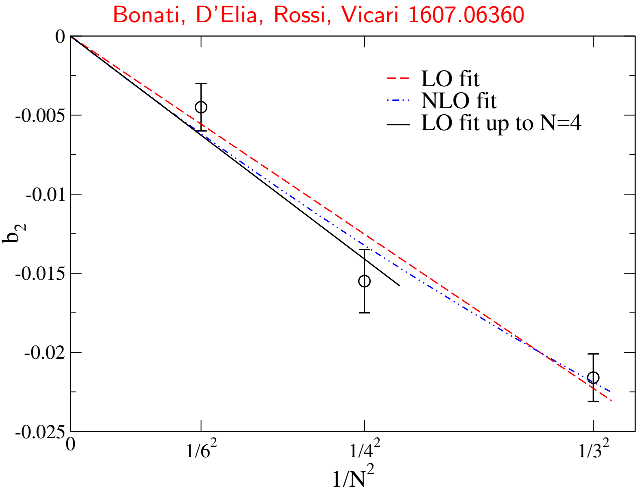
$SU(N)$ theories at $T = 0$ (1)

Del Debbio, Panagopoulos, Vicari 0204125

$C = \chi/\sigma^2,\ T = 0$



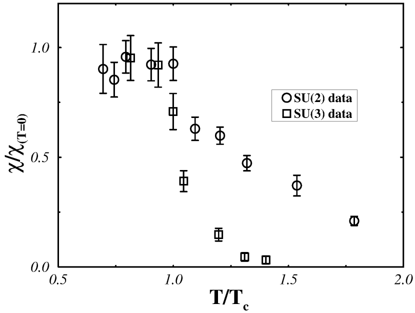
$SU(N)$ theories at $T = 0$ (2)

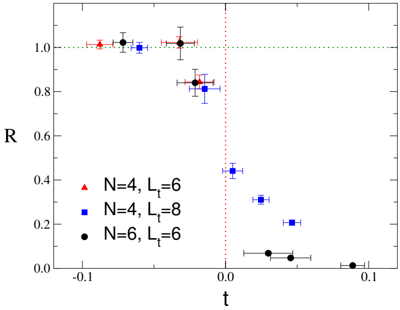


$SU(N)$ theories across T_c (1)

Alles, D'Elia, Di Giacomo 9706016

Del Debbio, Panagopoulos, Vicari 0407068





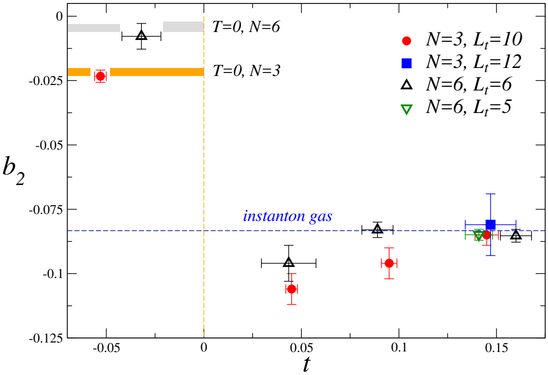
The topological susceptibility is constant for $T \lesssim T_c$ and then abruptly decreases ($t = (T - T_c)/T_c$).

$SU(N)$ theories across T_c (2)

Bonati, D'Elia, Panagopoulos, Vicari 1301.7640

(Bonati, D'Elia, Scapellato 1512.01544

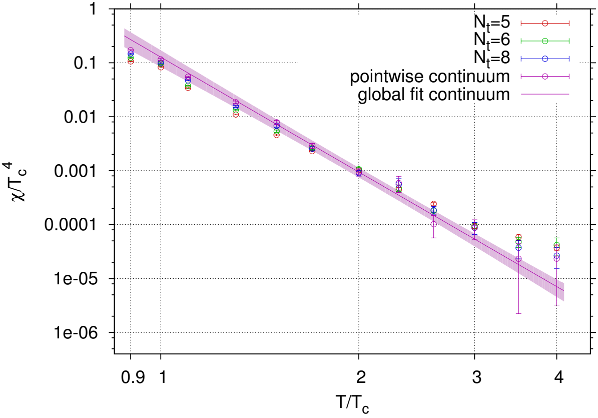
Bonati, D'Elia, Rossi, Vicari 1607.06360)



- large-N_c scaling for $T < T_c$, b_2 independent of N_c for $T > T_c$
- DIGA values ($b_2 = -1/12$, $b_4 = 1/360$) reached for $T \gtrsim 1.1 T_c$

$SU(3)$ theory for $T > T_c$

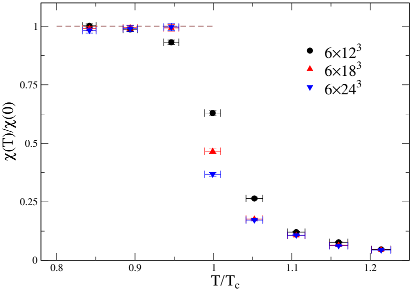
Borsanyi et al. 1508.06917

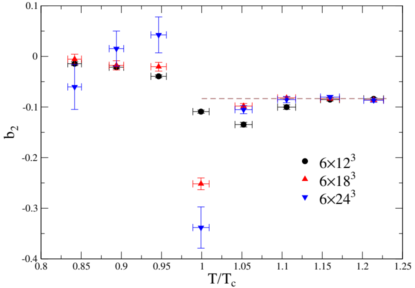


$\chi(T) \propto 1/T^b$, where $b = 7.1(4)(2)$ (DIGA prediction $b = 7$).

Intermezzo: G_2 theory across T_c

Bonati 1501.01172





Everything looks the same as in $SU(N)$ theories, but in G_2 no large-N_c limit exists! Alternative explanation? Relation to confinement?

The QCD case

In the last couple of years several lattice studies investigated θ dependence in QCD with physical or almost physical quark masses:

Trunin et al. 1510.02265 Bonati et al. 1512.06746
Petreczky et al. 1606.03145 Borsanyi et al. 1606.07494
Burger et al. 1805.06001

At $T = 0$ χPT provides reliable results and lattice studies give results compatible with it.

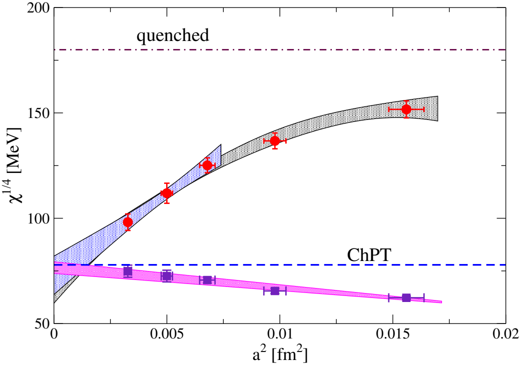
Most of the effort was devoted to the high temperature phase: the value of $\chi(T)$ for $200\mathrm{MeV} \lesssim T \lesssim 2\mathrm{GeV}$ is relevant for axion phenomenology and lattice QCD appears to be the only first principle methods to reliably investigate this range of temperatures (Berkowitz et al. 1505.07455).

General consensus: $\chi(T)$ is basically constant up to $T \simeq T_c \simeq 155\,\mathrm{MeV}$ then suddenly decreases.
Still no general consensus: details of the behaviour of $\chi(T)$ for $T > T_c$ (in particular: when DIGA sets in?)

QCD at $T = 0$

Most lattice discretizations of the fermion action introduce an explicit breaking of chiral symmetry, that is recovered only in the continuum limit. On the other hand topology is extremely sensitive to chiral symmetry.

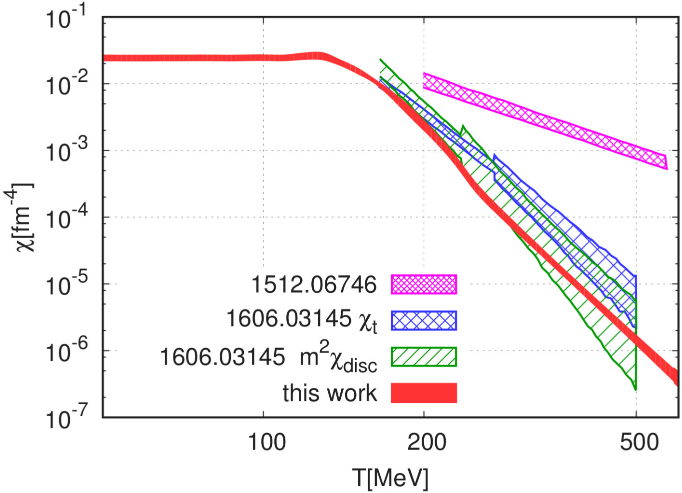
Consequence: large lattice artefacts



from Bonati et al. 1512.06746, see also Borsanyi et al. 1606.07494

Purple points have been corrected by using the mass of the non-Goldstone pions on the lattice to rescale the results for χ using ChPT.

QCD at $T \gtrsim T_c$

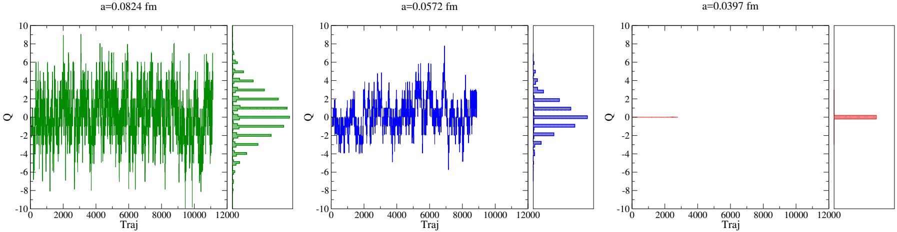


from Borsanyi et al. 1606.07494

Numerical problems: “freezing” of the topological charge

As the continuum limit is approached it gets increasingly difficult to correctly sample the various topological sectors.

exponential critical slowing-down

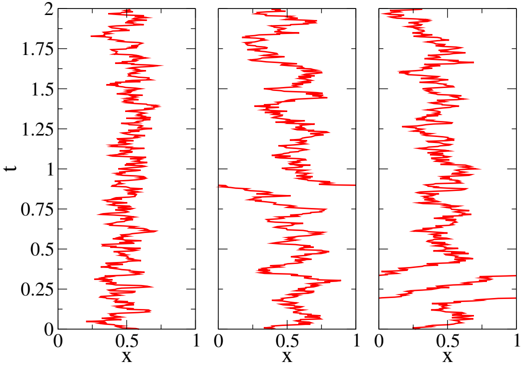


from Bonati, D’Elia, Mariti, Martinelli, Mesiti, Negro, Sanfilippo, Villadoro 1512.06746

Origin of the problem

Basically all the update schemes used in lattice simulations changes the configuration in a way that becomes almost continuous when the lattice spacing gets small.

To change the topological sector we need “large” updates, that are very difficult to achieve.



Examples of configurations from a QM toy model:

$$L[x] = \frac{1}{2}\dot{x}^2 - \theta\dot{x}$$

$x \in [0, 1)$ with periodic b.c.
(left) $Q = 0$
(center) $Q = -1$
(right) $Q = 2$
Bonati, D'Elia 1709.10034

Numerical problems: "small box" effect

As T gets large we have $\chi(T) \to 0$ and the typical amount of topological fluctuation in a system of volume V_4, $\langle Q^2 \rangle = V_4 \chi(T)$, goes to zero.

The probability $P(Q)$ of observing a configuration with charge Q gets strongly peaked at $Q = 0$ and the sampling becomes very difficult.

This is *not* an algorithmic problem

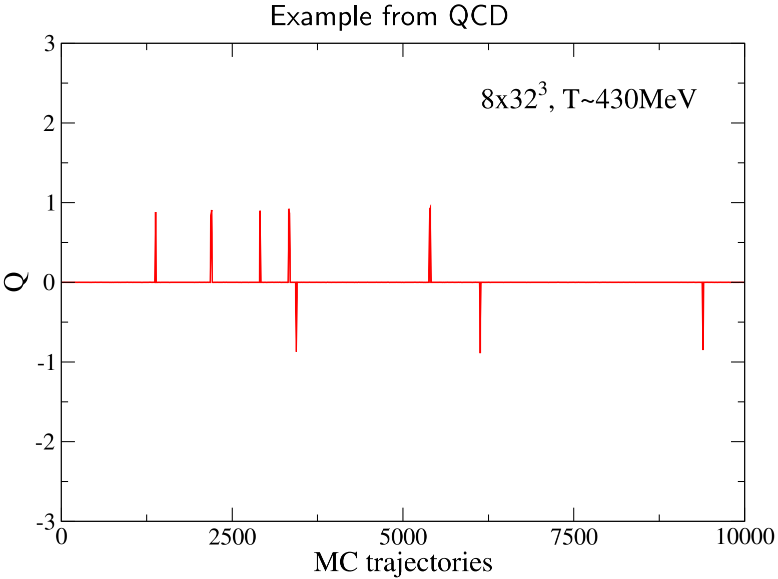
although from the practical point of view it looks like freezing.

For the QM toy model it can be seen that

$$P(Q) = \frac{\exp(-TQ^2/2)}{\sum_{\mathcal{Q}\in\mathbb{Z}} \exp(-T\mathcal{Q}^2/2)}$$

and $P(1)/P(0) = \exp(-T/2)$: exponentially large (in T) statistics are needed to estimate $\chi(T)$.

Numerical problems: "small box" effect



Conclusions

In $SU(N)$ Yang-Mills theory firm conclusions have been obtained in all the studied regimes

- at zero temperature χ and b_2 scale according to large-N_c
- at deconfinement there is a switch from the θ/N_c (large-N_c) behaviour to the θ (instanton) behaviour
- for T larger than T_c, $\chi(T)$ scales approximately as $\chi(T) \propto T^{-7}$, as in DIGA (although the prefactor is off by $\mathcal{O}(10)$)

In the "exotic example" of G_2 Yang-Mills, at a qualitative level, everything works as in $SU(N)$, but no large-N_c expansion.

In the case of QCD with light fermions

- the zero temperature case is well understood in χPT and lattice data reproduce the expected behaviour
- for $T \gtrsim 200\,\mathrm{MeV}$ formidable numerical problems are encountered and further study is needed (and on the way)

Thank you for your attention!

Backup with something more

Possible solutions of the strong CP problem

1. At least a massless quark ($m_u = 0$).
2. Assume a CP invariant lagrangian for the standard model and explain CP violation by CP SSB.
3. "Dynamical" θ angle.

Realization of mechanism 3: add to SM a pseudoscalar field a with coupling $\frac{a}{f_a} F\tilde{F}$ and only derivative interactions. Since the free energy has a minimum at $\theta = 0$, a will acquire a VEV such that $\theta + \frac{\langle a \rangle}{f_a} = 0$.

Goldstone bosons have only derivatives coupling, so the simplest possibility is to think of a as the GB of some $U(1)$ axial symmetry (Peccei-Quinn symmetry). The effective low-energy lagrangian is thus

$$\mathcal{L} = \mathcal{L}_{QCD} + \frac{1}{2}\partial_\mu a \partial^\mu a + \left(\theta + \frac{a(x)}{f_a}\right) q(x) + \frac{1}{f_a}\left(\begin{smallmatrix} model\ dependent \\ terms \end{smallmatrix}\right)$$

Axions as dark matter

Cosmological sources of axions: 1) thermal production 2) decay of topological objects 3) misalignment mechanism

Idea of the misalignment mechanism: the EoM of the axion is

$$\ddot{a}(t) + 3H(t)\dot{a}(t) + m_a^2(T)a(t) = 0$$

at $T \gg \Lambda_{QCD}$ the second term dominates and we have $a(t) \sim \mathrm{const}$ (assuming $\dot{a} \ll H$ initially); when $m_a \sim H$ the field start oscillating arount the minimum. When $m_a \gg H$ a WKB-like approx. can be used

$$a(t) \sim A(t) \cos \int^t m_a(\tilde{t})\mathrm{d}\tilde{t}; \qquad \frac{\mathrm{d}}{\mathrm{d}t}(m_a A^2) = -3H(t)(m_a A^2)$$

and thus the number of axions in the comoving frame $N_a = m_a A^2/R^3$ is conserved.

Overclosure bound: axion density $\leq$ dark matter density

Large-N_c argument

$$F^a_{\mu\nu}F^a_{\mu\nu} \text{ and } \epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \text{ scale as } N_c^2$$

To have a nontrivial θ dependence in the large-N_c limit we have to keep $\bar{\theta} \equiv \theta/N_c$ fixed, in such a way that θg^2 does not scale with N_c

The large-N_c scaling form of the free energy is thus (Witten 1980)

$$F(\theta, T) - F(0, T) = N_c^2 \bar{F}(\bar{\theta}, T)$$

where $\bar{F}$ is generically nontrivial for $N_c \to \infty$:

$$\bar{F}(\bar{\theta}, T) = \frac{1}{2}\bar{\chi}\bar{\theta}^2\left[1 + \bar{b}_2\bar{\theta}^2 + \bar{b}_4\bar{\theta}^4 + \cdots\right]$$

By matching the powers of θ we obtain

$$\chi = \bar{\chi} + \cdots$$
$$b_{2n} = \bar{b}_{2n}/N_c^{2n} + \cdots$$