Holographic relations at finite radius

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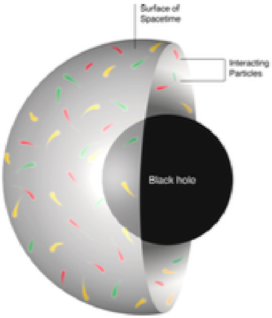
June 11, 2018

- The original example of holography in string theory is the famous AdS/CFT conjecture of Maldacena:

  - String theory on a background with $(d+1)$*-dimensional Anti-de Sitter asymptotics is dual to a* d*-dimensional conformal field theory.*

- Many examples of gauge/gravity dualities involving various spacetime asymptotics.

Introduction



- Original argument for holography: maximum entropy associated with a given spacetime volume scales as the surface area in Planck units.
- Follows from black holes being the most entropic objects for a given mass.
- No dependence on asymptotics!

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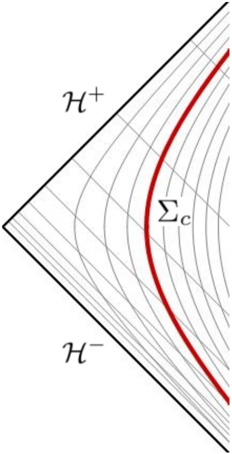
Introduction

- Consider a timelike **hypersurface** Σ_c, in a spacetime with generic asymptotics.
- Can we define a QFT on Σ_c, holographically dual to the interior of the spacetime?



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References

1. M.T. “TT deformations in general dimensions",
1805.10287.
2. Old work: Compère, McFadden, Skenderis and M.T.,
2011-2012.

Holographic reconstruction

- **Top-down** models postulate a complete relationship between string theory in a given background and a specific QFT e.g. $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM.
- In **bottom-up** models, we instead engineer the gravity theory to capture defining features of the QFT.

Holographic reconstruction

- Consider an **RG flow to a UV fixed point**, driven by a single operator $\mathcal{O}$.
- The minimal ingredients required to describe this holographically are:

$$S = \int d^{d+1}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 + V(\phi)\right)$$

where ϕ is the bulk scalar dual to $\mathcal{O}$ and the potential is such that the action admits AdS_{d+1} extrema.

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Holographic dictionary

- More precisely, one can extract from asymptotic expansions near the conformal boundary $\rho = 0$:

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}\left(g_{(0)ij} + \rho^2 g_{(2)ij} \cdots + \rho^d g_{(d)ij} \cdots\right) dx^i dx^j$$

and

$$\phi = \rho^{d-\Delta}(\phi_{(d-\Delta)} + \cdots) + \rho^{\Delta}(\phi_{(\Delta)} + \cdots)$$

the **dilatation Ward identity** for $\langle T_{ij} \rangle \sim g_{(d)ij}$ and $\langle \mathcal{O} \rangle \sim \phi_{(\Delta)}$

$$\langle T^i_i \rangle + \phi_{(d-\Delta)} \langle \mathcal{O} \rangle \sim 0$$

Holographic renormalization

- Use radial foliation near the conformal boundary

$$ds^2 = dr^2 + \gamma_{ij}(r,x)dx^i dx^j$$

where for AAdS $\gamma_{ij}(r,x) \sim e^{2r} g_{(0)ij} + \cdots$ as $r \to \infty$.

- The conjugate momentum to γ is the **Brown-York quasi-local stress tensor**

$$\mathcal{T}_{ij} = \left(K_{ij} - K\gamma_{ij} \right)$$

where the extrinsic curvature $K_{ij} = \frac{1}{2}\partial_r \gamma_{ij}$.

Holographic renormalization

- $\mathcal{T}_{ij}$ is not finite as $r \to \infty$.
- **Boundary counterterms** added to the Einstein-Hilbert action

$$S_{\rm ct} = -\int d^d x \sqrt{-h}((d-1) + \cdots)$$

render the onshell action finite and give additional contributions to the quasi-local stress tensor:

$$T_{ij} = \left(K_{ij} - K\gamma_{ij} + (d-1)\gamma_{ij} + \cdots\right)$$

(Balasubramanian and Kraus; de Haro, Skenderis and Solodukhin)

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Holographic renormalization

- T_{ij} does have a finite limit as $r \to \infty$:

$$\mathcal{L}_{r\to\infty}\left(T_{ij}\right) = \langle T_{ij}\rangle \sim g_{(d)ij}.$$

- The renormalized stress tensor satisfies the expected CFT identities e.g. for $d = 2$

$$\langle T^i_i\rangle = \frac{c}{6}\mathcal{R}(g_{(0)})$$

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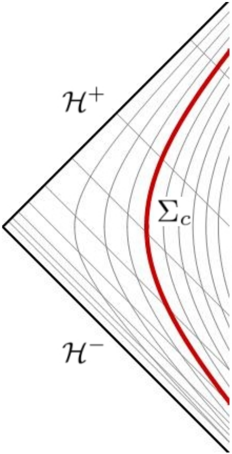
Finite radius hypersurface

- Natural to ask about duality for finite radius hypersurface.
- From QFT perspective: radial evolution is RG flow.
- In presence of horizons, one obtains a fluid/gravity relation.



(Minwalla et al; Polchinski et al; Strominger et al; Compère, McFadden, Skenderis and Taylor; ....)

Gauss-Codazzi relations

- In the radial Hamiltonian decomposition, one can write the Einstein equations in Gauss-Codazzi form.
- In particular, for AdS gravity

$$K^2 - K^{ij}K_{ij} = \mathcal{R}(\gamma) + d(d-1)$$

which implies that, for flat hypersurfaces at finite radius,

$$T^i_i = -4\pi G \left(T_{ij} T^{ij} - \frac{1}{(d-1)} (T^i_i)^2 \right)$$

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QFT interpretation

- We view this relation as a dilatation Ward identity:

$$T_i^i = -\lambda \mathcal{T}$$

where

$$\mathcal{T} = \left(T_{ij} T^{ij} - \frac{1}{(d-1)} (T_i^i)^2 \right)$$

- In $d = 2$, $\mathcal{T}$ is the $T\bar{T}$ operator explored by Zamoldchikov.
- Holographic relation in $d = 2$ proposed by (McGough et al).

$T\bar{T}$ operator in 2d

- Zamoldchikov showed that this operator has a remarkable **OPE structure** as $x \to y$:

$$T\bar{T}(x, y) = \mathcal{T}(y) + \sum_{\alpha} A_{\alpha}(x - y)\nabla_y \mathcal{O}_{\alpha}(x)$$

  i.e. we can identify the operator as local, modulo derivatives of other local operators.

- Smirnov and Zamoldchikov also explored the behaviour of a CFT under deformations by $\mathcal{T}$ i.e.

$$S_{\rm CFT} \to S_{\rm CFT} + \lambda \int d^2x \, \mathcal{T}.$$

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Energy spectrum

- Consider the (Euclidean) theory on a cylinder of radius R.
- In a **stationary** state such that

$$\langle T_{\tau\tau} \rangle = -\frac{E}{R}$$

the defining relation for the family of QFTs implies that

$$\frac{\partial E}{\partial \lambda} + 2E\frac{\partial E}{\partial R} = 0$$

Energy spectrum

- This can be re-expressed in terms of dimensionless quantities (ϵ, α) using

$$\alpha = \frac{\lambda}{R^2} \qquad E = \frac{1}{R}\epsilon$$

with

$$\partial_\alpha \epsilon = 2\epsilon\left(\epsilon + 2\alpha\partial_\alpha \epsilon\right)$$

- This is the defining ODE for the **energy spectrum** $\epsilon(\alpha)$.

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Generalization to $d > 2$

In general dimensions:

$$\mathcal{T} = \left(T_{ij} T^{ij} - \frac{1}{(d-1)} (T_i^i)^2 \right)$$

- Definite of composite operator more subtle; **renormalization** required as operators approach each other.
- Details of operator definition not required for energy spectrum, but would be needed for **correlation functions, entanglement entropy** etc.

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Energy spectrum

- Consider the (Euclidean) theory

$$S_{\rm CFT} \to S_{\rm CFT} + \lambda \int d^{D+1}x \, \mathcal{T}.$$

on a cylinder of spatial volume R^D. With

$$\alpha = \frac{\lambda}{R^d} \qquad E = \frac{1}{R}\epsilon$$

dimensionless energy $\epsilon(\alpha)$ satisfies

$$\partial_\alpha \epsilon = \left(1 + \frac{1}{D}\right)(\epsilon + 2\alpha\epsilon\partial_\alpha\epsilon)$$

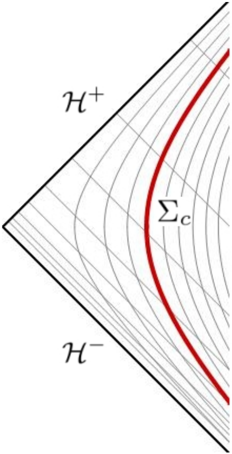
with $\epsilon(0)$ the CFT energy.

Back to gravity

- The conjectured holographic theory dual for finite radius is

$$S_{\rm CFT} \to S_{\rm CFT} + \lambda \int d^{D+1}x\, \mathcal{T}.$$

- Identifying the quasi-local stress tensor as the dual stress tensor, **Ward identity** matches by construction.
- Can we also reproduce **energy spectrum** in gravity?



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Black brane solutions

- Consider a static **black brane** in $(D+2)$ dimensions

$$ds^2 = (\rho^2 - \frac{\mu}{\rho^{D-1}})d\tau^2 + \frac{d\rho^2}{(\rho^2 - \frac{\mu}{\rho^{D-1}})} + \rho^2 dx^a dx_a$$

We can then read off from the quasi local stress tensor the dimensionless energy:

$$\epsilon = \frac{D\rho^d}{2\lambda}\left(1 - \left(1 - \frac{\lambda M}{\rho^d}\right)^{\frac{1}{2}}\right)$$

where $\mu = 4\pi GM$.

Black brane solutions

- In terms of dimensionless coupling $\alpha = \lambda / \rho^d$,

$$\epsilon = \frac{D}{2\alpha}\left(1 - (1 - \alpha M)^{\frac{1}{2}}\right)$$

- Note that the CFT energy is

$$\epsilon(0) = \frac{D}{4} M$$

and $\epsilon(\alpha)$ indeed satisfies:

$$\partial_\alpha \epsilon = \left(1 + \frac{1}{D}\right)(\epsilon + 2\alpha\epsilon\partial_\alpha \epsilon)$$

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Comments

1. Trivial to generalize to **boosted (spinning) branes.**
2. Addition of extra **bulk fields** (gauge fields, scalars etc) modifies CFT deformation e.g.

$$T^i_i = -\lambda \left(T^{ij} T_{ij} - \frac{1}{D} (T^i_i)^2 + 2 \mathcal{J}^i \mathcal{J}_i \right)$$

Also noticed in $d = 2$ by (Bzowski and Guica; Kraus et al).

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Conclusions and outlook

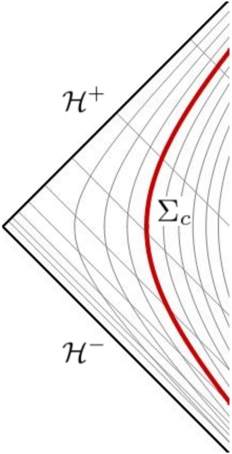
- The conjectured holographic theory dual for finite radius AdS is

$$S_{\rm CFT} \to S_{\rm CFT} + \lambda \int d^{D+1}x \, \mathcal{T}.$$

with

$$\mathcal{T} = \left(T^{ij} T_{ij} - \frac{1}{D} (T^i_i)^2 \right)$$

- Natural generalization of $d = 2$ proposal.



Conclusions and outlook

- Passes preliminary checks: **Ward identity, energy relations.**
- More detailed checks require **renormalized** definition of composite operator $\mathcal{T}$.
- Proposal can easily be extended beyond **AdS asymptotics** (but UV behaviour is required to fix integration constants).