Discontinuities of Feynman Integrals

Ruth Britto

CEA Saclay

Brown University, Twelfth Workshop on Non-Perturbative Quantum Chromodynamics
June 10, 2013

Outline

- Landau and Cutkosky
- Classic unitarity cuts
  - Dispersion relations
  - Modern unitarity method, with master integrals
  - Dimensional regularization and masses [recent work with Mirabella, Ochirov]
- Generalized or iterated cuts
  - Double dispersion relations
  - Cut integrals and discontinuities [work in progress with Abreu, Duhr, Gardi]

Singularities of Feynman integrals: Landau conditions

Denominators: $A_i \equiv M_i^2 - q_i^2$

Feynman parameters α_i.

1st Landau condition:

$$\alpha_i A_i = 0 \qquad \forall i,$$

2nd Landau condition:

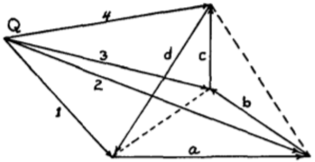
$$\sum \alpha_i q_i = 0, \qquad \text{for each closed loop.}$$

Cutkosky cuts

Discontinuities = Landau singularities = replace propagators by delta functions in integral

Any number of delta functions!

At one loop: geometric interpretation of 2nd Landau condition.



Polytope volume $\to 0$. Point Q falls into hyperplane of external momenta.

Unitarity Cuts

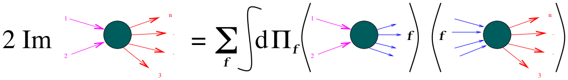
Scattering and interaction matrices:

$$S = 1 + iT$$

The unitarity condition: $S^\dagger S = 1$.

$$-i(T - T^\dagger) = T^\dagger T$$

$$2\mathrm{Im}\, T = T^\dagger T$$



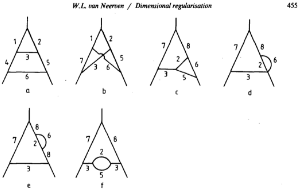
Cut across one channel, with any number of loops.

Dispersion relations

From the imaginary part, reconstruct the integral:

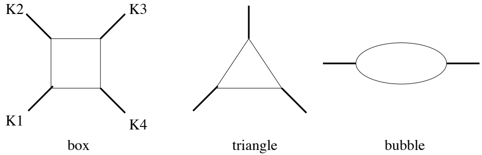
$$A(K^2) = \frac{1}{\pi}\int_0^\infty ds\, \frac{\mathrm{Im}\; A(s)}{s - K^2}$$

Classic example: On-shell vertex function, 2 loops. [Van Neerven, 1986]



Integration is still hard work. At least at one loop, one can do much better.

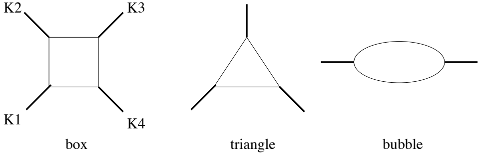
Master integrals



$A^{1-\mathrm{loop}} = \sum_i c_i I_i + r,$ $\quad c_i, r$ are rational functions.

Analytically known at 1-loop, some special cases beyond.

Master integrals

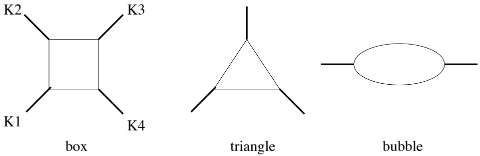


$A^{1-\text{loop}} = \sum_i c_i I_i + r,$ c_i, r are rational functions.

Analytically known at 1-loop, some special cases beyond.

- e.g. box $\int d^{4-2\epsilon}k \, \frac{1}{(\ell^2)(\ell-K_1)^2(\ell-K_1-K_2)^2(\ell-K_1-K_2-K_3)^2}$
- scalar numerators
- max. 4 propagators in 4d
- can include masses

Master integrals



$A^{1-\mathrm{loop}} = \sum_i c_i I_i + r,$ c_i, r are rational functions.

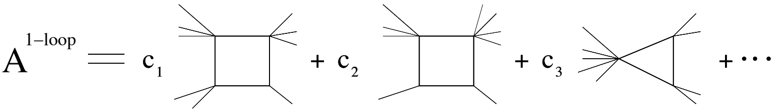
Analytically known at 1-loop, some special cases beyond.

e.g.: If $K_3^2 = K_4^2 = 0$,

$$\begin{aligned} I_4^{2m\ h} \quad = \quad & \frac{2r_\Gamma}{st}\frac{1}{\epsilon^2}\left[\frac{1}{2}(-s)^{-\epsilon} + (-t)^{-\epsilon} - \frac{1}{2}(-K_1^2)^{-\epsilon} - \frac{1}{2}(-K_2^2)^{-\epsilon}\right] \\ & -\frac{2r_\Gamma}{st}\left[-\frac{1}{2}\ln\left(\frac{s}{K_1^2}\right)\ln\left(\frac{s}{K_2^2}\right) + \frac{1}{2}\ln^2\left(\frac{s}{t}\right)\right. \\ & \left. + \mathrm{Li}_2\left(1-\frac{K_1^2}{t}\right) + \mathrm{Li}_2\left(1-\frac{K_2^2}{t}\right)\right] + \mathcal{O}(\epsilon). \end{aligned}$$

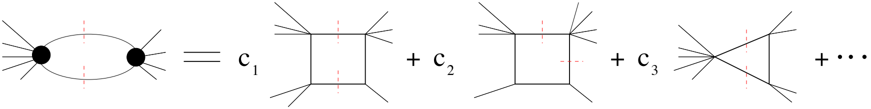
Amplitudes from unitarity cuts

$$A^{1-\text{loop}} = \sum c_i I_i$$



Amplitudes from unitarity cuts

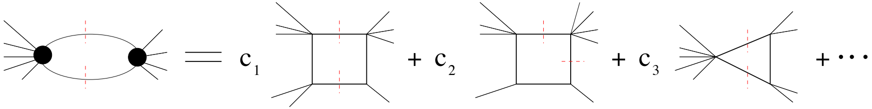
$$\Delta A^{1-\mathrm{loop}} = \sum c_i \Delta I_i$$



LHS: work at the level of tree amplitudes. RHS: known masters.

Amplitudes from unitarity cuts

$$\Delta A^{1-\text{loop}} = \sum c_i \Delta I_i$$



Matching 4-dimensional cuts can suffice to determine reduction coefficients! Logarithms with unique arguments.
"cut-constructibility"

[Bern, Dixon, Dunbar, Kosower]

But: we still get several coefficients together in the same equation.

How do we evaluate a unitarity cut?

Cut integrals

$$\Delta A^{1-\mathrm{loop}} = \int d\mu \; A^{\mathrm{tree}}(-\ell, i, \ldots, j, \ell - K) \; A^{\mathrm{tree}}(K - \ell, j+1, \ldots, i-1, \ell)$$

$$d\mu = d^4\ell \;\; \delta(\ell^2) \; \delta((\ell - K)^2)$$

Change to homogeneous (CP^1) spinor variables with

$$\ell_{a\dot{a}} = t \; \lambda_a \tilde{\lambda}_{\dot{a}}.$$

Integration measure:

$$\int d^4\ell \; \delta(\ell^2) \; (\bullet) = \int_0^\infty dt \;\, t \int_{\tilde{\lambda}=\bar{\lambda}} \langle \lambda \; d\lambda \rangle \; [\tilde{\lambda} \; d\tilde{\lambda}] \; (\bullet)$$

[Cachazo, Svrček, Witten]

Systematic procedure: spinor integration

[Anastasiou, RB, Buchbinder, Cachazo, Feng, Kunszt, Mastrolia]

- Change variables, $\ell = t\lambda\tilde{\lambda}$, and use the spinor measure,

$$\int d^4\ell \; \delta(\ell^2)\delta((\ell - K)^2) = \int dt \; t \int \langle\lambda \; d\lambda\rangle[\tilde{\lambda} \; d\tilde{\lambda}]\delta((t\lambda\tilde{\lambda} - K)^2)$$

- Use 2nd delta function to perform t-integral.
- $\lambda, \tilde{\lambda} \to z, \bar{z}$ familiar complex variables.
- Evaluate with residue theorem.
- Identify cuts of basis integrals and read off coefficients.
  D-dimensional cuts also treated, for complete amplitudes.
- We have given formulas for the resulting coefficients.

Dimensional regularization at one loop

In $D = 4 - 2\epsilon$ dimensions, the result of reduction is

$$A = \sum_i e_i \,(\text{pentagon}) + \sum_i d_i \,(\text{box}) + \sum_i c_i \,(\text{triangle}) + \sum_i b_i \,(\text{bubble})$$

No extra rational term.

Unitarity in $D = 4 - 2\epsilon$ dimensions

Orthogonal decomposition, keeping external momenta in 4 dimensions. [Bern, Chalmers, Mahlon, Morgan]

$$\int d^{4-2\epsilon}\ell_{4-2\epsilon} \quad = \quad \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int_0^1 du\ u^{-1-\epsilon} \int d^4\ell_4.$$

where $\ell^2_{-2\epsilon} = \frac{K^2}{4} u$.

The integral over u will remain. The u-dependence is controlled:

$$\Delta A \quad = \quad \int_0^1 du\ u^{-1-\epsilon} \int d^4\ell \ \ \delta(\ell^2)\ \delta(\sqrt{1-u}\ K^2 - 2K\cdot\ell)$$

Recognize and perform the 4-d integral as before.

(Cf. methods by Ossola, Papadopoulos, Pittau; Forde; Ellis, Giele, Kunszt; Kilgore; Giele, Kunszt, Melnikov; Badger)

Massive particles

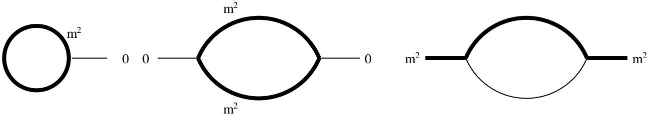
Cut amplitude:

$$\int \langle \lambda \; d\lambda \rangle [\tilde{\lambda} \; d\tilde{\lambda}] \left(\frac{\sqrt{\Delta[K^2, M_1^2, M_2^2]}}{K^2} \right) \frac{(K^2)^{n+1}}{\langle \lambda | K | \tilde{\lambda}]^{n+2}} \frac{\prod_{j=1}^{n+k} \langle \lambda | R_j | \tilde{\lambda}]}{\prod_{i=1}^{k} \langle \lambda | Q_i | \tilde{\lambda}]}$$

- The integral coefficients have the same form.
  [RB, Feng, Mastrolia, Yang]

- New master integrals.

The special "massive" master integrals



These integrals do not have kinematic cuts.

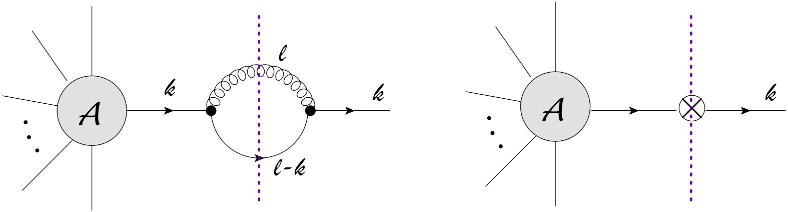
$$I_1 = m^{2-2\epsilon}\frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)}$$

$$I_2(0;m^2,m^2) = m^{-2\epsilon}\frac{\Gamma(1+\epsilon)}{\epsilon}$$

$$I_2(m^2;0,m^2) = m^{-2\epsilon}\frac{\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}$$

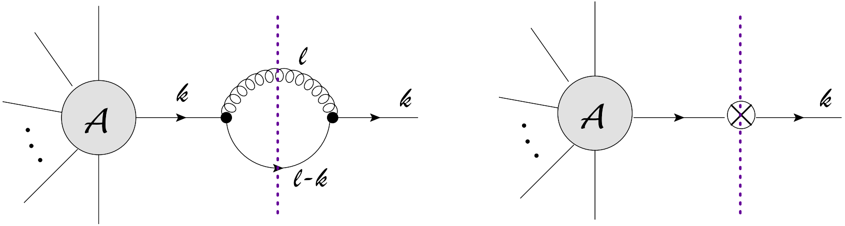
Divergent cuts for on-shell bubbles

- Try to apply unitary cuts to the special massive master integrals
- Cut of massless on-shell bubble diverges, due to internal on-shell propagator
- Must include the counterterms.



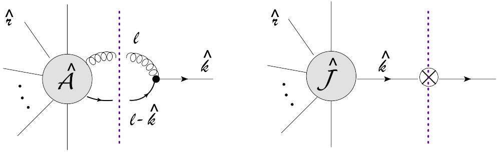
Masses, fermions and unitarity [EGKM]

[Ellis, Giele, Kunszt, Melnikov]



- Isolate and remove the divergent diagrams
- Implicit use of counterterm
- Feynman-diagram decomposition is gauge dependent
- Embedded in a numerical algorithm

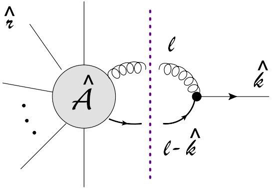
Our method [RB, Mirabella]



Use an off-shell continuation of the fermion mass. The cut is finite until we take the on-shell limit.

- Power series expansion in the off-shell parameter
- In the on-shell limit, divergences are guaranteed to cancel: keep only finite terms
- Explicit use of counterterms. Gauge dependence enters only in tree level currents.
- Clean analytic results

Off-shell continuation



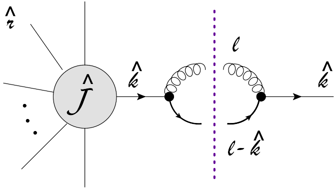
Momentum-conserving shift: $k \to \hat{k} = k + \xi r, \qquad r \to \hat{r} = r - \xi r.$

Can choose $\bar{k} = r$ for some null external momentum r, so it stays on shell: $\hat{r}^2 = r^2 = 0$.

Propagator of interest: $k^2 - m^2 \to \xi(2k \cdot r)$

Cut diverges as $1/\xi$.

Reduction of the shifted divergent diagram



$$\mathcal{A}_L = \frac{1}{\hat{k}^2 - m^2} \left(\bar{u}_{\hat{k}-\ell} \, \not{\epsilon}^*_\ell \, (m + \hat{\not{k}}) \, \hat{\mathcal{J}} \right),$$
$$\mathcal{A}_3 = \bar{u}_k \, \not{\epsilon}_\ell \, u_{\hat{k}-\ell}$$

For internal helicity sum, use Feynman gauge as in EGKM:

$$\sum_{\lambda=\pm} \varepsilon^\lambda_\mu \left(\varepsilon^\lambda_\nu \right)^\star = -g_{\mu\nu}$$

Reduction of the shifted divergent diagram

Simple reduction of linear bubble gives

$$\int \mathcal{A}_3 \mathcal{A}_L \to \frac{1}{\xi(2k \cdot \bar{k})} (4m^2 \, \bar{u}_k \hat{\mathcal{J}} + 2\xi m \, \bar{u}_k \bar{\not{k}} \hat{\mathcal{J}}) \; B_0(\hat{k}^2).$$

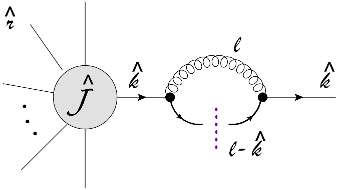
Also expand off-shell current and scalar bubble to 1st order:

$$\begin{aligned} \hat{\mathcal{J}} &= \mathcal{J} + \xi \mathcal{J}' \\ B_0(\hat{k}^2) &= B_0(m^2) + \xi(2k \cdot \bar{k}) B_0'(m^2) \end{aligned}$$

Result:

$$\frac{1}{\xi} \frac{4m^2 \bar{u}_k \mathcal{J} \, B_0}{(2k \cdot \bar{k})} + \frac{4m^2 (2k \cdot \bar{k}) \, \bar{u}_k \mathcal{J} \, B_0' + 4m^2 \, \bar{u}_k \mathcal{J}' \, B_0 + 2m \, \bar{u}_k \bar{\not{k}} \mathcal{J} \, B_0}{(2k \cdot \bar{k})}.$$

Reduction: tadpole part

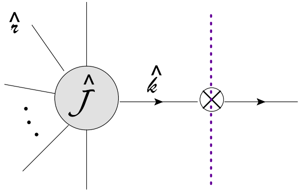


Similar, except gluon polarization sum $\to$ propagator, $-ig_{\mu\nu}/\ell^2$.

Result:

$$\left[\left(\frac{2}{\xi(2k \cdot \bar{k})} - \frac{1}{m^2} \right) \bar{u}_k \, \mathcal{J} + \frac{1}{(2k \cdot \bar{k})m} \bar{u}_k \, \bar{\not{k}} \, \mathcal{J} + \frac{2}{(2k \cdot \bar{k})} \bar{u}_k \, \mathcal{J}' \right] A_0 .$$

Counterterm



$$-\frac{1}{\xi(2k\cdot\bar{k})}\bar{u}_k\left(\not{k}\delta Z_\psi+\xi\not{\bar{k}}\delta Z_\psi-m\delta Z_\psi-m\delta Z_m\right)\left(\not{k}+\xi\not{\bar{k}}+m\right)\hat{\mathcal{J}}$$

Renormalization constants in on-shell scheme:

$$\begin{aligned}\delta Z_m &= \frac{A_0}{m^2}+2B_0,\\ \delta Z_\psi &= \frac{A_0}{m^2}-4m^2B_0'.\end{aligned}$$

Verify total cancellation of divergent diagram.

Small examples with Feynman Diagrams

- $H \to b\bar{b}$
  3 loop diagrams + 2 counterterm diagrams
- $q\bar{q} \to t\bar{t}$
  12 loop diagrams + 2 counterterm diagrams

1. Implemented momentum shift

2. Computed bubble and tadpole coefficients from unitarity cut

3. Checked cancellation of divergences against counterterm and agreement of finite result with Passarino-Veltman reduction.

The fermion-channel cut in the spinor-helicity formalism

The spinor-helicity convention for the polarization vectors requires axial gauge:

$$\varepsilon^-(p) = -\sqrt{2}\,\frac{|p\rangle\,[q| + |q]\,\langle p|}{[qp]}, \qquad \varepsilon^+(p) = -\sqrt{2}\,\frac{|p]\,\langle q| + |q\rangle\,[p|}{\langle qp\rangle}.$$

The completeness relation is

$$\sum_{\lambda=\pm} \varepsilon^\lambda_\mu(p)\left(\varepsilon^\lambda_\nu(p)\right)^\star = -g_{\mu\nu} + \frac{p_\mu q_\nu + q_\mu p_\nu}{p\cdot q}.$$

Specific gauge choice = choice of q for each p.

Additional counterterm in axial gauge, for spinors

The double cut gets an extra $\mathcal{O}(\xi^0)$ contribution:

$$\frac{1}{\xi(2k\cdot\bar{k})}\int d\mu_{2,k}\left[\frac{\left(\bar{u}_k\,\not{\ell}\,u_{\hat{k}-\ell}\right)\left(\bar{u}_{\hat{k}-\ell}\,\not{q}\,(m+\hat{\not{k}})\,\hat{\mathcal{J}}\right)}{q\cdot\ell}\right.$$
$$\left.+\frac{\left(\bar{u}_k\,\not{q}\,u_{\hat{k}-\ell}\right)\left(\bar{u}_{\hat{k}-\ell}\,\not{\ell}\,(m+\hat{\not{k}})\,\hat{\mathcal{J}}\right)}{q\cdot\ell}.\right]$$

Second term vanishes by Ward identity with cut gluon.
First term is cancelled by a new (non-divergent) counterterm:

$$\mathcal{M}^k = -\frac{1}{2k\cdot\bar{k}}\bar{u}_k\left[(\bar{\not{k}}-m)\,\hat{\not{k}}\not{q}\,\delta Z_k'\right](\hat{\not{k}}+m)\hat{\mathcal{J}},$$
$$\delta Z_k' = \frac{B_0}{q\cdot k}$$

Example from $t\bar{t} \to gg$ amplitude

Full analytic result computed previously by other methods. [Körner, Merabashvili; Badger, Sattler, Yundin]

- Color decomposition
- 3-point tree $\times$ 5-point tree for m^2 on-shell bubbles
- Checked cancellation of divergence and evaluated on-shell bubble coefficient, for equal-helicity gluons

On-shell recursion for off-shell currents [RB, Ochirov]

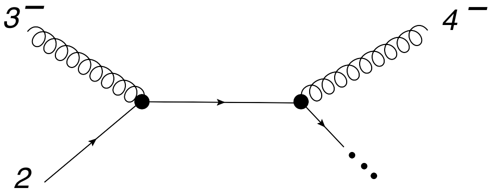
Finite parts of the counterterm need off-shell $\widehat{\mathcal{J}}$ explicitly, not the gauge-invariant $\mathcal{A}_{\mathcal{L}}\mathcal{A}_{\mathcal{R}}$.

The current $\widehat{\mathcal{J}}$ depends on gauge choices of external gluons—these cancel among counterterms.

Generate $\widehat{\mathcal{J}}$ by BCFW-type relations, starting purely from 3-point vertices with the polarizations

$$\varepsilon^-(p) = -\sqrt{2}\,\frac{|p\rangle\,[q| + |q]\,\langle p|}{[qp]}, \qquad \varepsilon^+(p) = -\sqrt{2}\,\frac{|p]\,\langle q| + |q\rangle\,[p|}{\langle qp\rangle}.$$

BCFW shifts available for any pair of massless quarks/gluons.



Example: Take $q_3 = q_4 = q$.

$$iJ = -i\frac{|q]\langle 3| + |3\rangle [q|}{[q\hat{3}]} \frac{p_2 - \widehat{p_3} + m}{(p_2 - p_3)^2 - m^2} \frac{|q]\langle \hat{4}| + |\hat{4}\rangle [q|}{[q4]} |2)$$

$$= \frac{i}{[q3][q4]}\Bigg\{ \frac{1}{\langle 3|2|3]}\Big(|4\rangle[q|2|3\rangle[q| - |q]\langle 4|1|q]\langle 3| + m|q]\langle 43\rangle[q| \Big) - \frac{1}{[34]}\Big([q3]\big(|q]\langle 3| + |3\rangle[q|\big) + [q4]\big(|q]\langle 4| + |4\rangle[q|\big)\Big)\Bigg\}|2)$$

On-shell recursion for off-shell currents [RB, Ochirov]

For a nice recursion, we need

- Residue at infinity $= 0$
- Poles from propagators only

Residue at infinity can be made zero, no worse than on-shell case. Argument from groups of Feynman diagrams.

Reference spinors generically introduce "unphysical poles" which can be avoided for some gauge choices.

$$\varepsilon^{-}(\hat{p}) = -\sqrt{2}\,\frac{|p\rangle\,[q| + |q]\,\langle p|}{[qp] - z[qp']}$$

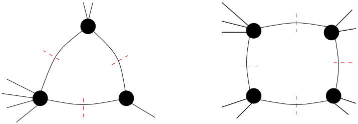
Result: recursion established for certain helicities, with preferred gauge choice.

More loops?

Conceptual challenges at two loops and beyond:

- Master integrals more numerous, not canonical, and not all known analytically
- Nonplanar topologies
- Need D-dimensional ingredients for cuts

Generalized cuts



- One-loop box coefficients "quadruple cuts"

  [RB, Cachazo, Feng]

- Typically require complex momenta.

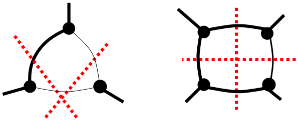
- One-loop: sequence of quadruple, triple, double, single cuts. "OPP method" underlies all state-of-the-art numerical codes. Samples complex momenta. [Ossola, Papadopoulos, Pittau; Mastrolia; Forde; Kilgore; Ellis, Giele, Kunszt; Giele, Kunszt, Melnikov; RB, Mirabella]

Generalized cuts beyond one loop

- Extension of OPP method at 2 loops and beyond. Algebraic-geometric analysis of integrands and their relations. [Mastrolia, Mirabella, Ossola, Peraro; Badger, Frellesvig, Zhang]
- "Maximal cuts." Multi-dimensional complex residues = leading singularities. [Buchbinder, Cachazo; Bern, Carrasco, Johansson, Kosower; Larsen, Kosower; Caron-Huot, Larsen; Johansson, Kosower, Larsen]
- Can we make use of non-maximal cuts? Work without master integrals?

Double dispersion relations

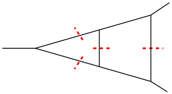
Previously computed at one loop, with strictly real momenta.
[Mandelstam; Ball, Braun, Dosch]. From iterated cuts.



Spectral function: 3-cut of box with real momenta = 4-cut with complex momenta = volume of tetrahedron

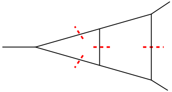
Check with more dimensions or more loops.

Two or more loops [with Abreu, Duhr, Gardi]



- Use symbol of multiple polylogarithms. [Goncharov, Spradlin, Vergu, Volovich] Encodes discontinuities in various channels. [Gaiotto, Maldacena, Sever, Viera]
- Deeper entries in the symbol appear in terms of more natural variables
- Can match discontinuities to iterated cuts

Two or more loops [with Abreu, Duhr, Gardi]

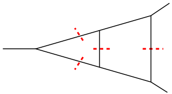


- Use symbol of multiple polylogarithms. [Goncharov, Spradlin, Vergu, Volovich] Encodes discontinuities in various channels. [Gaiotto, Maldacena, Sever, Viera]
- Deeper entries in the symbol appear in terms of more natural variables
- Can match discontinuities to iterated cuts

$$
\begin{aligned}
&y \otimes w \otimes (w \otimes \bar{w} + \bar{w} \otimes w - \bar{w} \otimes \bar{w}) + x \otimes (1-w) \otimes (w \otimes w - w \otimes \bar{w} - \bar{w} \otimes w) \\
&+y \otimes \bar{w} \otimes (w \otimes w - w \otimes \bar{w} - \bar{w} \otimes w) + x \otimes (1-\bar{w}) \otimes (w \otimes \bar{w} + \bar{w} \otimes w - \bar{w} \otimes \bar{w}) \\
&+x \otimes x \otimes ((1-\bar{w}) \otimes \bar{w} - (1-w) \otimes w))
\end{aligned}
$$

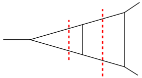
$$y = p_3^2/p_1^2, \qquad x = p_2^2/p_1^2, \qquad w = (p_1^2 + p_2^2 - p_3^2 + \sqrt{\lambda(p_1^2, p_2^2, p_3^2)})/p_1^2$$

Two or more loops [with Abreu, Duhr, Gardi]



- Use symbol of multiple polylogarithms. [Goncharov, Spradlin, Vergu, Volovich] Encodes discontinuities in various channels. [Gaiotto, Maldacena, Sever, Viera]
- Deeper entries in the symbol appear in terms of more natural variables
- Can match discontinuities to iterated cuts

The correspondence seems inexact: do cuts still have more information?



One of many remaining challenges: control of complexified momentum.

Summary and Outlook

- Discontinuities of Feynman integrals indicated by Landau and Cutkosky
- Not always easy to compute, but powerful:
  - Dispersion relations
  - Unitarity method with master integrals
  - Generalized cuts with master integrals
- Hard parts at one loop: rational parts, massive contributions. Under control, though some parts need improvement analytically. Clean method for divergent bubbles from off-shell momentum continuation.
- Beyond one loop: master integrals largely unknown. Seeking systematic approach via on-shell methods (generalized cuts). Using mathematics of multiple polylogarithms where applicable.