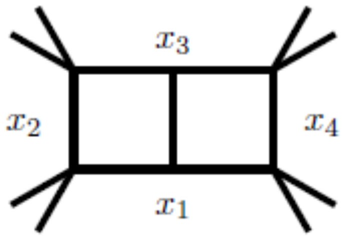

LOOPS AND SPLINES

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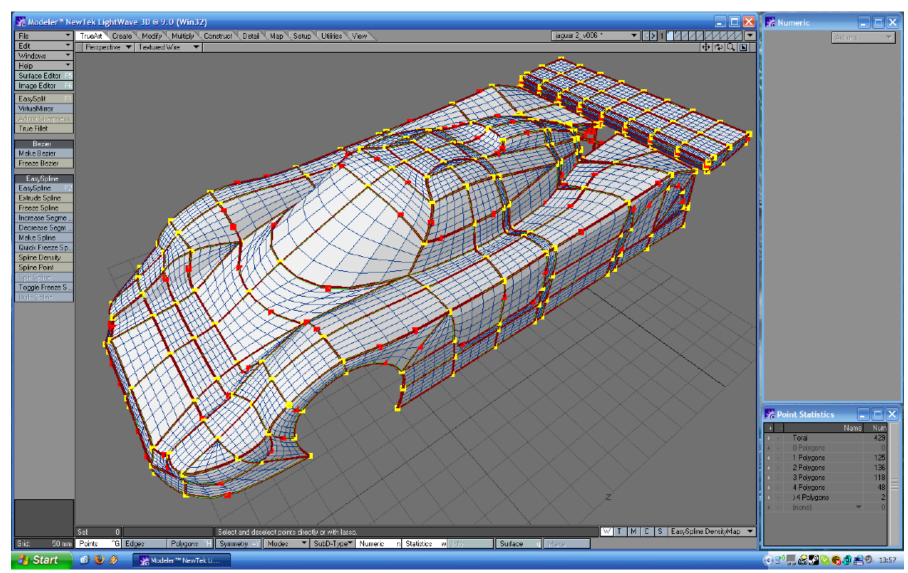
Brown University

Non-Perturbative QCD
June 2013



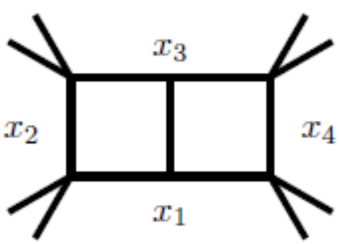
Conformal Integrals

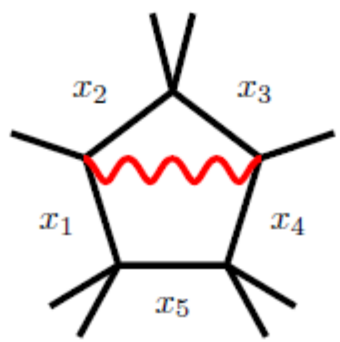
- Relevant for CFT computations, Witten diagrams, and dual conformal invariant loop integrals.

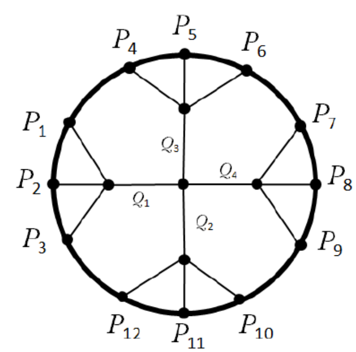
- Definition: integrals which lead to expressions depending only on cross-ratios.
  - E.g. at four points

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- Examples:

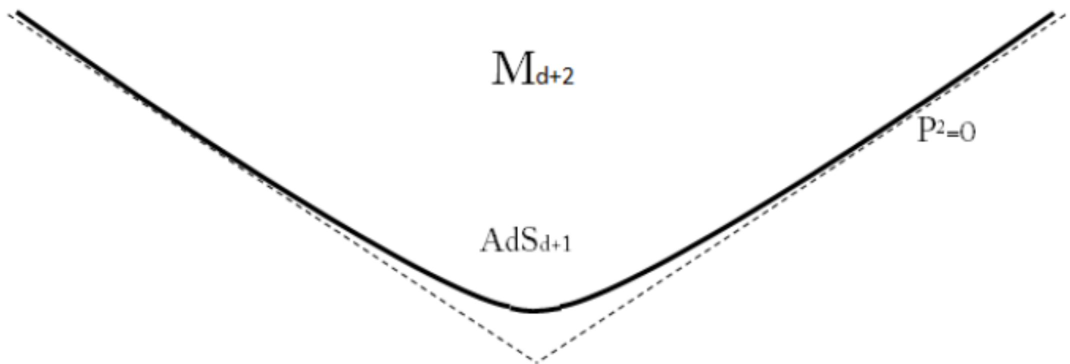






Embedding Space

Idea: AdS_{d+1} and its boundary as hypersurfaces in $d+2$ Minkowski space. Conformal group $SO(d+1,1)$ is simply Lorentz group.



$$-X^+X^- + X^\mu X_\mu = -1 \qquad (R_{\text{AdS}} = 1) \qquad\qquad P^2 = 0, \qquad P \simeq \lambda P, \qquad \lambda > 0$$

$$X^A(x^a) = \frac{1}{x_0}(1, x_0^2 + x^2, x^\mu), \qquad P^M(x^\mu) = (1, y^2, y^\mu)$$

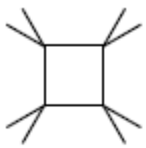
$$P_{ij} \equiv -2P_i \cdot P_j = (y_i - y_j)^2 \longrightarrow \text{Massless}$$

$$-2P \cdot X = \frac{1}{x_0}(x_0^2 + (x-y)^2). \longrightarrow \text{Massive}$$

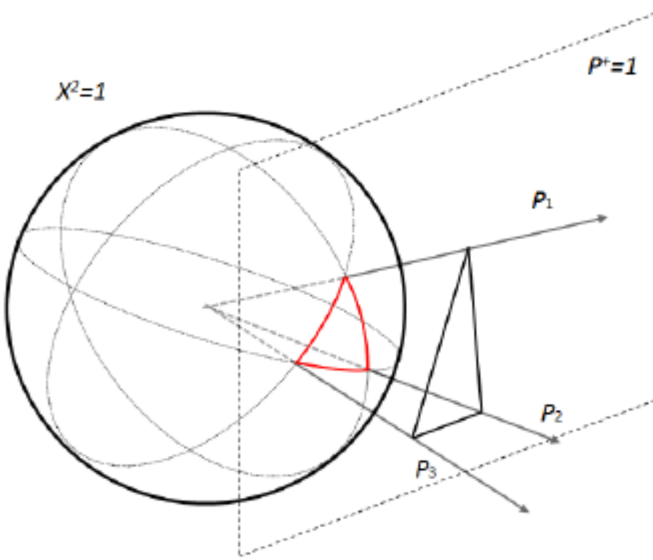
Hyperbolic Simplices

- The 4d box integral is given by the volume of a hyperbolic tetrahedron in AdS_5

Mason, Skinner;
Davydychev, Delbourgo;
Schnetz



$$= \int \frac{d^4Q}{2\pi^2} \frac{1}{(P_1 \cdot Q)(P_2 \cdot Q)(P_3 \cdot Q)(P_4 \cdot Q)}$$



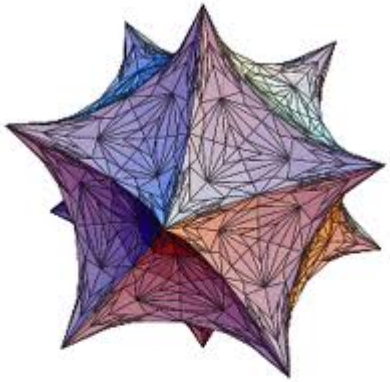
$$\simeq \frac{\mathrm{Vol}_H(S_3)}{\mathrm{Vol}_E(S_3)}$$

S for Simplex

Hyperbolic Simplices

- The 4d box integral is given by the volume of a hyperbolic tetrahedron in AdS_5
- One-loop MHV amplitudes: hyperbolic tetrahedra glue up to form a closed polytope.

Something like this...



Hyperbolic Simplices

- The 4d box integral is given by the volume of a hyperbolic tetrahedron in AdS_5
- One-loop MHV amplitudes: hyperbolic tetrahedra glue up to form a closed polytope.

Something like this...

Coincidence?!

Hyperbolic Simplices

- How much of this story generalizes? (other dimensions, propagator weights, masses, AdS/CFT integrals...)

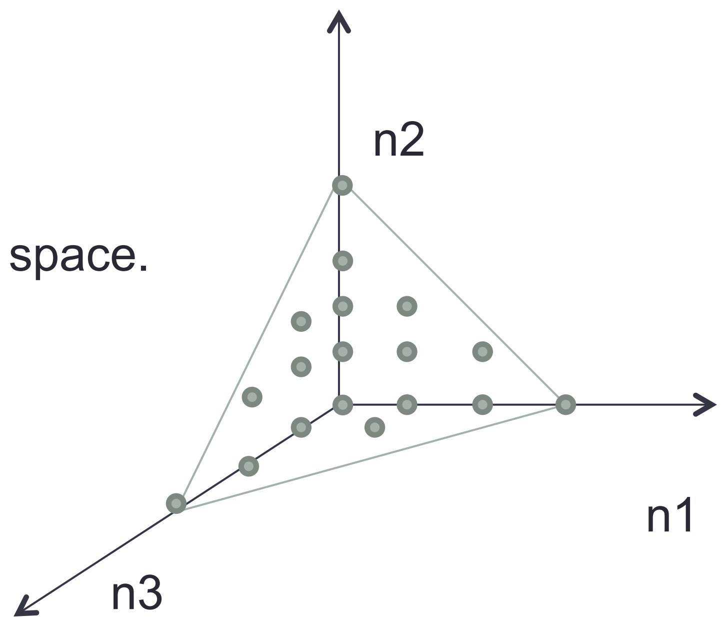
- Is there a geometric interpretation of higher loop integrals?

- ... Ultimately, can this lead to a better understanding of the geometry of *integrated* N=4 amplitudes?

Detour: counting points in polytopes

$$d_{\{q_i\}}(Q) = \#\left\{ \{n_i\} | \sum_i n_i q_i = Q \right\}$$

- Set of linear equations define surface in `n' space.



$$\left(q_1 \middle| \cdots \middle| q_m \right) \cdot \begin{pmatrix} n_1 \\ \vdots \\ n_m \end{pmatrix} = \begin{pmatrix} Q \end{pmatrix}$$

- In the continuum limit, get volume of polytope in 't' space.

$$d_{\{q_i\}}(Q) \to \mathcal{T}(Q, \{q_i\}) = \int_0^{+\infty} \prod_{i=1}^{n} \mathrm{d}t_i \, \delta(Q - \sum t_i q_i)$$

SPLINE!

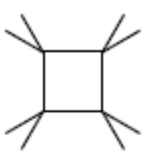
Generating function

- To compute degeneracy, introduce generating function

$$\sum_{Q} d_{\{q_i\}}(Q) e^{-Q\cdot\mu} = \prod_{i=1}^{n} \frac{1}{1-e^{-q_i\cdot\mu}}$$

- Continuum limit:

$$\int dQ\, \mathcal{T}_{\{q_i\}}(Q)\, e^{-Q\cdot\mu} \to \prod_{i=1}^{n} \frac{1}{q_i\cdot\mu}$$

( $= \int \frac{d^4Q}{2\pi^2} \frac{1}{(P_1\cdot Q)(P_2\cdot Q)(P_3\cdot Q)(P_4\cdot Q)}$)

Spline as Laplace transform

$$\prod_{i=1}^{n} \frac{\Gamma(\Delta_i)}{(P_i \cdot Q)^{\Delta_i}} = \int_0^{+\infty} D_\Delta^n t \, e^{Q\cdot(\sum_i t_i P_i)} = \int_{\mathbb{M}^D} \mathrm{d}X \, e^{Q\cdot X} \, \mathcal{T}_{\{\Delta_i\}}(X; \{P_i\})$$

$$\mathcal{T}_{\{\Delta_i\}}(X; \{P_i\}) = \int_0^{+\infty} D_\Delta^n t \; \delta(X - \sum_{i=1}^{n} t_i P_i)$$

$$D_\Delta^n t \;\equiv\; \prod_{i=1}^{n} \mathrm{d}t_i \, t_i^{\Delta_i - 1}$$

- The spline captures the *geometry* associated to the integrand.
- Rational function identities map onto geometrical identities

Computing the Spline

- The computation depends crucially on the number of nodes vs dimension

$$\mathcal{T}_{\{\Delta_i\}}(X;\{P_i\}) = \int_0^{+\infty} D^n_\Delta t\ \delta(X - \sum_{i=1}^{n} t_i P_i)$$

$$\underbrace{\left(P_1 \middle| \ldots \middle| P_n\right)}_{M} \cdot \underbrace{\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}}_{T} = \begin{pmatrix} X \end{pmatrix} \longrightarrow$$

$$T = M^{-1} X$$

$$t_i = W_i \cdot X$$

$$W_i \cdot P_j = \delta_{ij}$$

$$\mathcal{T}_{\{\Delta_i\}}(X;\{X_i\}) = \frac{\prod_{i=1}^{D} (W_i \cdot X)^{\Delta_i - 1}\, \Theta(W_i \cdot X)}{\sqrt{\det P_i \cdot P_j}}$$

(n=D=d+2)

Geometric picture

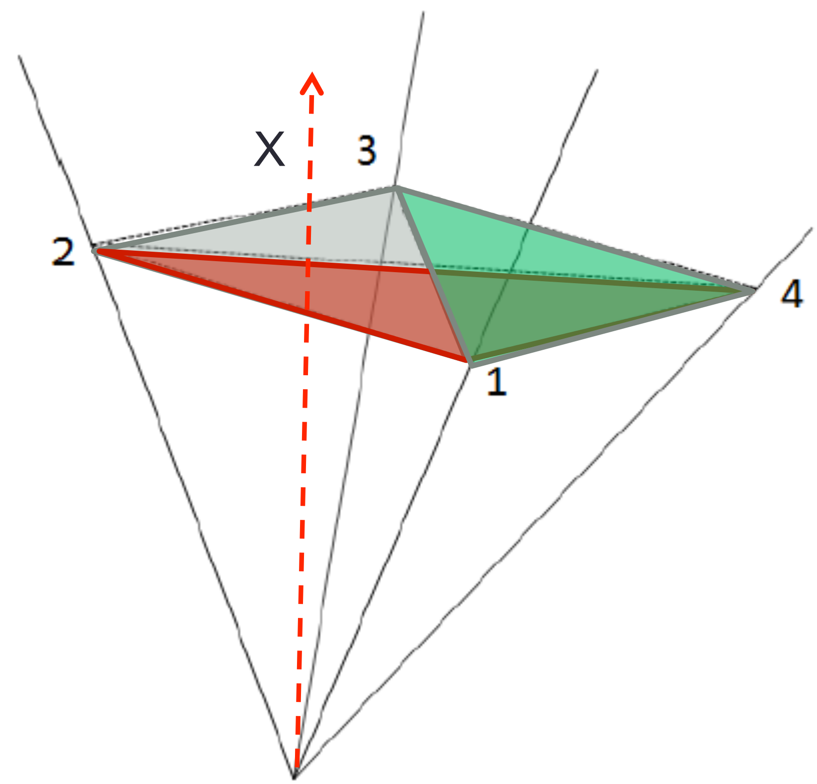
$$\mathcal{T}(X;\{P_i\}) = \int_0^{+\infty} \prod_{i=1}^{n} \mathrm{d}t_i \, \delta^{(D)}(X - \sum_{i=1}^{n} t_i P_i)$$

- **Worldsheet**: the spline computes the volume of a polytope in *t* space.
- **Target space**: The spline is a distribution in X, with support on the *polyhedral cone* spanned by the vectors P_i.
- *n=D* – worldsheet polytope is a point – "volume" is a constant. Target space gives the characteristic function of the cone.
- Characteristic function: it is the intersection of several halfspaces defined by hyperplanes – the W_i vectors.

$$\mathcal{T}(X;\{P_i\}) = \frac{\prod_{i=1}^{D} \Theta(W_i \cdot X)}{\sqrt{\det P_i \cdot P_j}}$$

General Geometric picture

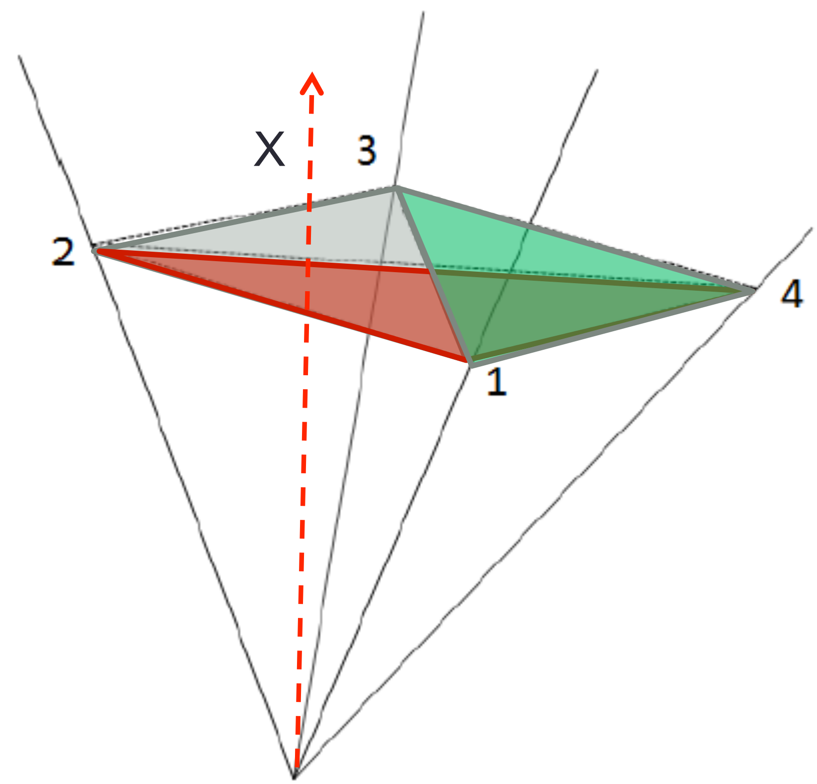
- n<D: spline is a distribution in X.
- n>D: worldsheet polytope is non trivial, with a volume that scales homogeneously with X.
- Spline is a sum of local homogeneous forms in X of degree n-D. One form per simplicial cone decomposing the full polyhedral cone.
- Spline continuous with discontinuous (n-D) derivative across simplicial cells. At the cell walls, worldsheet volume vanishes.



Target space picture

General Geometric picture

- More generally, the P_i vectors fit into some (D x n) matrix M
- The ``shape" of the spline (cell structure) only depends on the class of M in the matroid stratification of the Grassmannian Gr(D,n).
- The spline itself depends on M as an element of
- GL(n,D)/SO(d+1,1)



Target space picture

Computing the spline

- How to determine the spline?
- Use Laplace transform+partial fractions:
- Ex: n=4 in D=3

$$P_4 = (W^1_{23}.P_4)P_1 + (W^2_{31}.P_4)P_2 + (W^3_{12}.P_4)P_3$$

$$\prod_{i=1}^{4} \frac{1}{P_i \cdot Q} = \frac{W^1_{23} \cdot P_4}{(P_2 \cdot Q)(P_3 \cdot Q)(P_4 \cdot Q)^2} + \frac{W^2_{31} \cdot P_4}{(P_1 \cdot Q)(P_3 \cdot Q)(P_4 \cdot Q)^2} + \frac{W^3_{12} \cdot P_4}{(P_1 \cdot Q)(P_2 \cdot Q)(P_4 \cdot Q)^2}$$

$$\frac{W^1_{23} \cdot P_4}{(P_2 \cdot Q)(P_3 \cdot Q)(P_4 \cdot Q)^2} = W^1_{23} \cdot \partial_Q \left[\frac{1}{(P_2 \cdot Q)(P_3 \cdot Q)(P_4 \cdot Q)} \right]$$

$$= \int \mathrm{d}X \, e^{-Q \cdot X} \underbrace{(W^1_{23} \cdot X) \, \mathcal{T}_{1,1,1}(X, \{P_2, P_3, P_4\})}$$

n=D we already know this case!

A piece of the new spline, which is now linear in X

Computing integrals

$$\prod_{i=1}^{n} \frac{\Gamma(\Delta_i)}{(P_i \cdot Q)^{\Delta_i}} = \int_0^{+\infty} D^n_\Delta t \, e^{Q\cdot(\sum_i t_i P_i)} = \int_{\mathbb{M}^D} \mathrm{d}X \, e^{Q\cdot X} \, \mathcal{T}_{\{\Delta_i\}}(X;\{P_i\})$$

$$\int \frac{\mathrm{d}^d Q}{2\pi^{d/2}} \prod_{i=1}^{n} \frac{\Gamma(\Delta_i)}{(P_i \cdot Q)^{\Delta_i}} \longrightarrow \int_{\mathbb{M}^D} \mathrm{d}X \, e^{X^2} \, \mathcal{T}_{\{\Delta_i\}}(X;\{P_i\})$$

- One-loop integrals are Gaussian integrals over cones.
- But! Spline is homogeneous in |X|; integral can be performed.

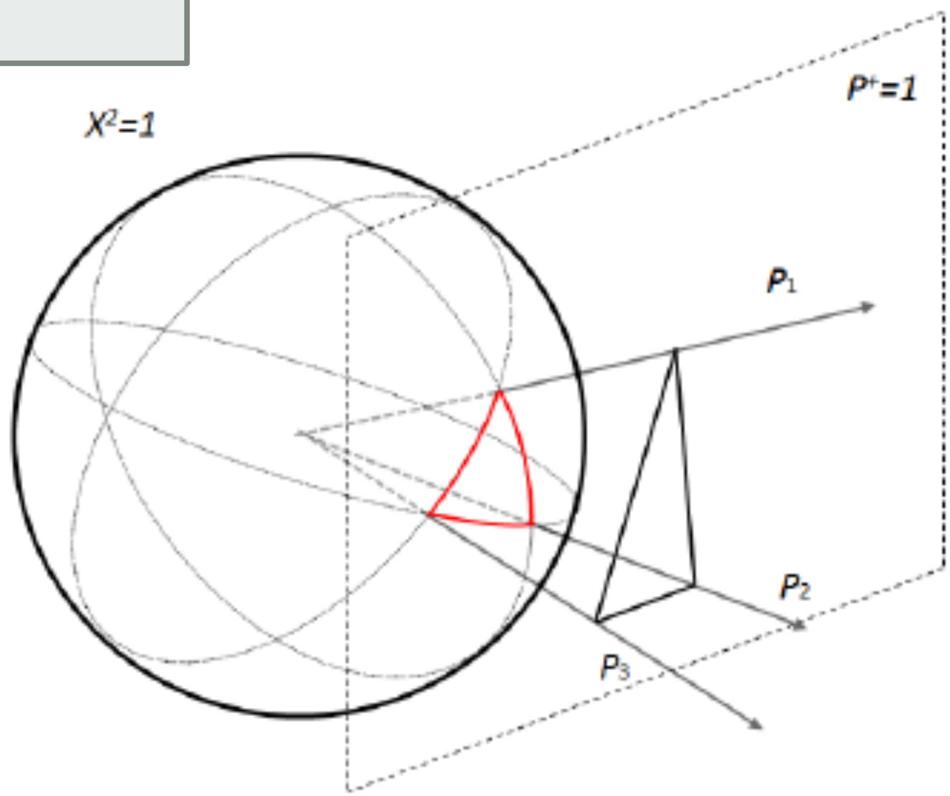
- **Result:**

$$\int_{\mathrm{AdS}_{d+1}} \mathrm{d}X \, \mathcal{T}_{\{\Delta\}}(X,\{P_i\})$$

$$\int_{\mathrm{AdS}_{d+1}} \mathrm{d}X \; \mathcal{T}_{\{\Delta\}}(X, \{P_i\})$$

Analog:

Intersecting a cone with a sphere, gives spherical angle



Hyperbolic simplices

- Applications:
  - The conformal ``star integrals'':

$$I^{(n)} = \int \frac{\mathrm{d}^d x}{i\pi^{d/2}} \prod_{i=1}^{n} \frac{1}{(x_i - x)^2} = \int \frac{\mathrm{d}^d Q}{i\pi^{d/2}} \prod_{i=1}^{n} \frac{1}{(-2P_i \cdot Q)}$$

$$\int_{\mathrm{AdS}_{d+1}} \mathrm{d}X \, \mathcal{T}_{\{\Delta\}}(X, \{P_i\}) = \int_{\mathrm{AdS}} dX \frac{\chi_C(X)}{\sqrt{\det P_i \cdot P_j}} = \frac{V_H(S_n)}{V_E(S_n)}$$

- Generalization of the star-triangle relation: stars get glued up into AdS hyperbolic simplices!

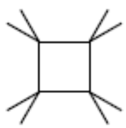
Hyperbolic simplices

$$V^{(n-1)} = \frac{\sqrt{|\det P_{ij}|}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} I^{(n)}$$

n=3 - triangle

$$I^{(3)} = \frac{\Gamma\left(\frac{1}{2}\right)^3}{\sqrt{P_{12}\, P_{13}\, P_{23}}} \quad \longrightarrow \quad V^{(2)} = \pi$$

n=4 - box



$$= \frac{\mathrm{Li}_2(x_+/x_-) - \mathrm{Li}_2\left(\frac{1-x_+}{1-x_-}\right) + \mathrm{Li}_2\left(\frac{1-1/x_+}{1-1/x_-}\right) - (x_+ \leftrightarrow x_-)}{\sqrt{\det x_{ij}^2}}$$

n=5 - pentagon

???

n=6 - hexagon

Schlafli's formula and pentagon

$$\mathrm{d}V_k = \frac{-1}{2i(k-1)} \sum_{i<j}^{n} V_{(k-2)}^{(ij)} (-1)^{i+j}\, \mathrm{d}\log \left(\frac{W_i \cdot W_j + \sqrt{(W_i \cdot W_j)^2 - W_i^2 W_j^2}}{W_i \cdot W_j - \sqrt{(W_i \cdot W_j)^2 - W_i^2 W_j^2}} \right)$$

- For n=5, lower dimensional volume is constant! Easy to compute:

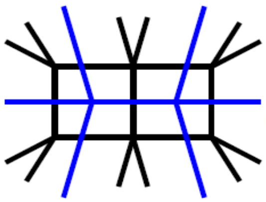
$$\tilde{I}^{(5)} = \frac{\pi^{\frac{3}{2}}}{2\sqrt{-\Delta^{(5)}}}(1+g+g^2+g^3+g^4)\left\{\log\left|\left(\frac{r-\sqrt{-\Delta^{(5)}}}{r+\sqrt{-\Delta^{(5)}}}\right)\left(\frac{s-\sqrt{-\Delta^{(5)}}}{s+\sqrt{-\Delta^{(5)}}}\right)\right|\right\}$$

$$\begin{aligned}\Delta^{(5)} &= \frac{1}{2}\frac{\det P_{ij}}{P_{13}P_{14}P_{24}P_{25}P_{35}} \\ &= 1-[u_1(1-u_3(1+u_4)+u_2u_4^2)+\text{cyclic}]-u_1u_2u_3u_4u_5\end{aligned}$$

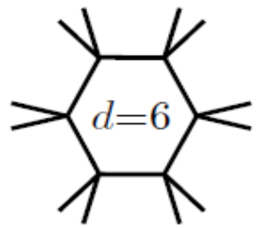
$$\begin{aligned} r &= \frac{(1-u_2)(1-u_5)-u_1(2-u_3-u_4-u_3u_5-u_2u_4+u_1u_3u_4)}{2}, \\ s &= \frac{(1-u_5)(1-u_2u_5)-u_1\,(1+u_5-2\,u_3u_5+u_4+u_2u_4u_5+u_1u_4)}{2\,\sqrt{u_1u_5}}\end{aligned}$$

What about hexagon?

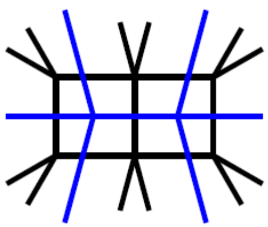
- Hexagon interesting since it's related to double box.



$$(u,\ldots) = -\frac{1}{2}\int_u^{+\infty}\frac{du'}{u'}$$



$$(u',\ldots)$$



$$= \int_{u_8}^{+\infty}\frac{du'_8}{u'_8}\,\frac{\mathrm{Li}_3(\ldots)+\ldots}{\sqrt{\Delta^{(6)}}}$$

Hyperbolic volume

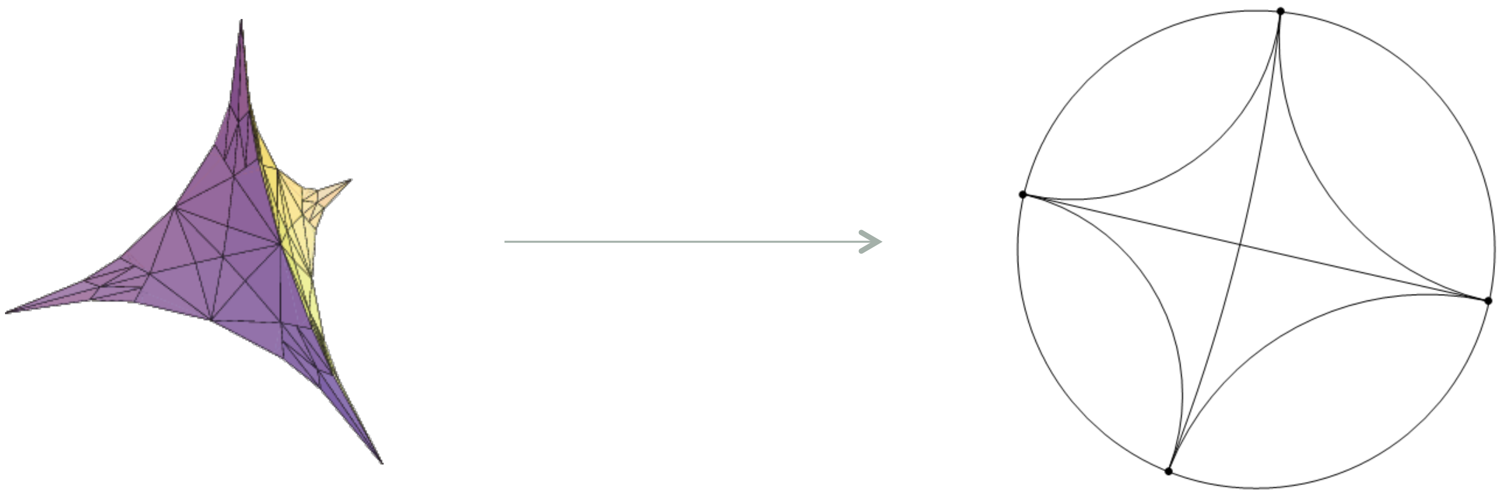
- Schlafli's formula translates into a formula for the hexagon symbol.
- Unfortunately, seems hard to integrate it...

Simplices in 2d kinematics

- A tractable case is when kinematics are 2d.

- For n>4, (n-1) hyperbolic simplex cannot fit into an AdS_3 - > $V_H = 0$
- However, star integrals are *ratios* of volumes, both going to zero – leads to finite answer.
- (n-1) simplex ``shatters'' into several hyperbolic tetrahedra.

- 1d kinematics analog: tetrahedron going to triangles

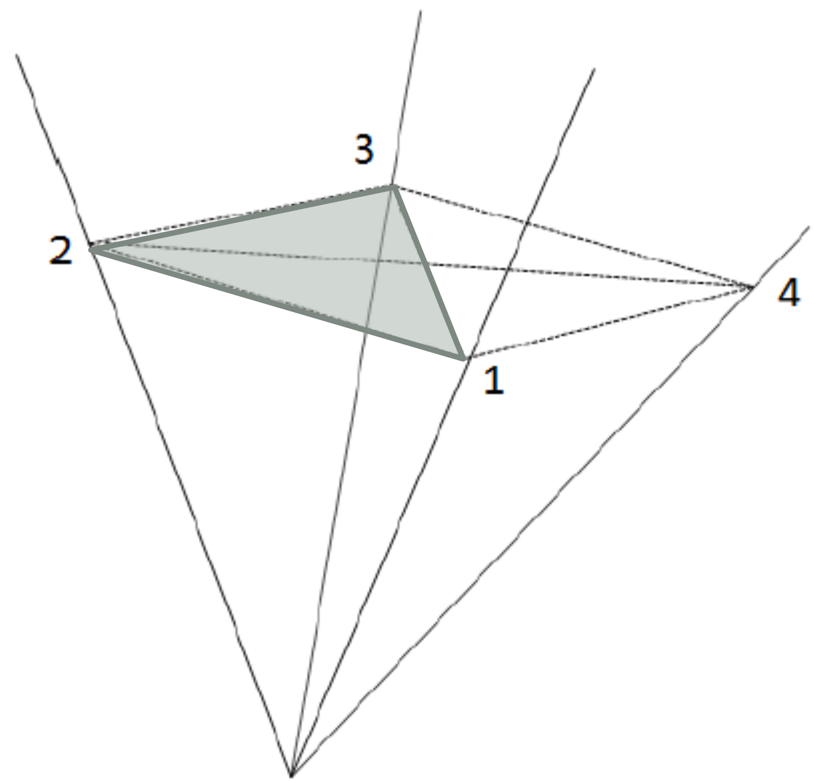


Spline computation

- We are in a situation where effectively, n>D=d+2 (d=2)
- Spline is computed as before: via partial fractions.
- For vectors in general position, the final result is

$$T(X;\{P_i\}) = \sum_{\{b\}} \frac{(W_1^{(b)} \cdot X)^2}{(W_1^{(b)} \cdot \hat{P}_1^{(b)})(W_1^{(b)} \cdot \hat{P}_2^{(b)})} \frac{\chi_{(b)}}{\sqrt{\det b^T b}}$$

- The sum runs over the set of unbroken basis ***b***; the corresponding simplicial cones pave the full polyhedral cone.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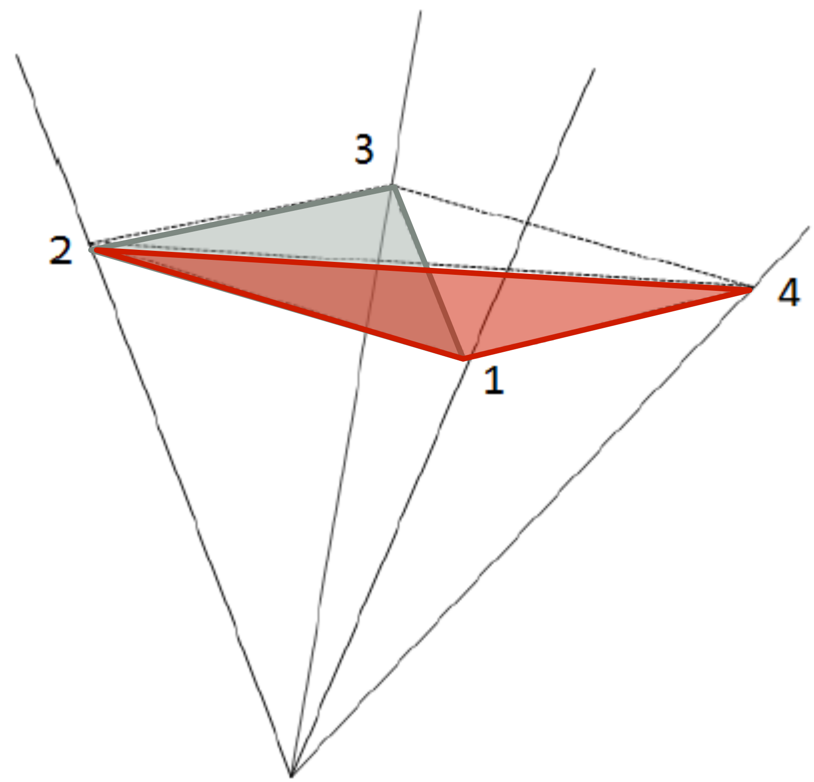
2
4
1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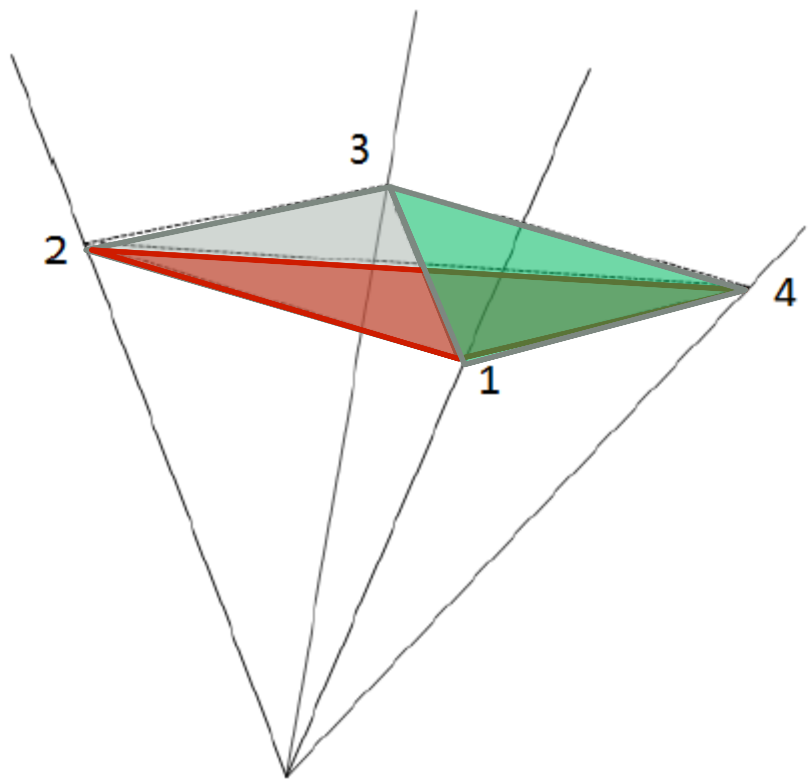
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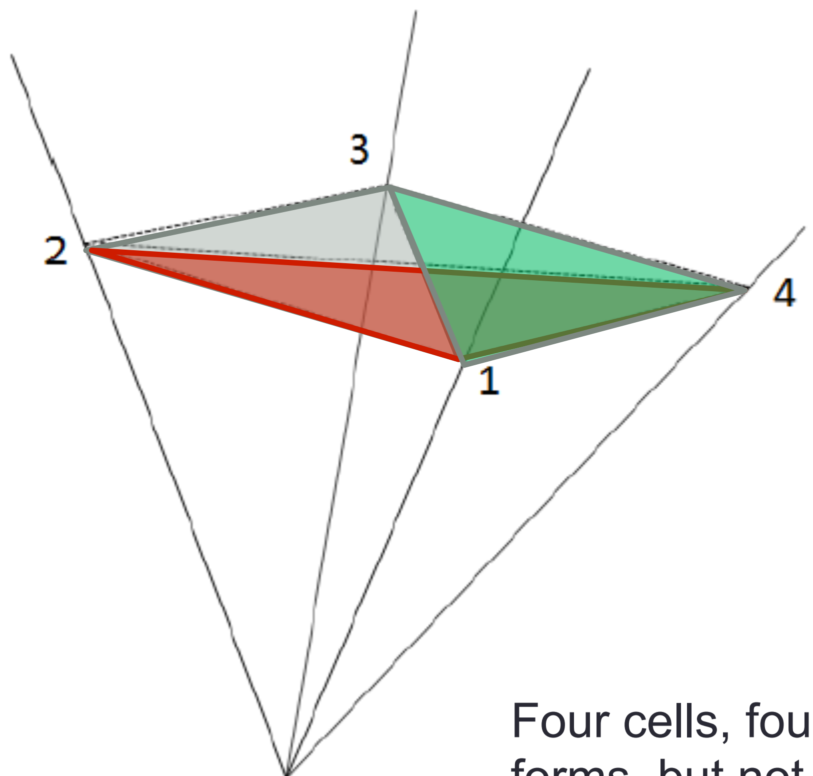


Four cells, four local homogeneous forms, but not all independent – smaller set suffices to encode the full spline. Such a set labelled by unbroken basis.

The integrals

- The integral becomes sum of terms of the form

$$\int \mathrm{d}X\, e^{X^2} \frac{(W_1^{(b)} \cdot X)^2}{(W_1^{(b)} \cdot \hat{P}_1^{(b)})(W_1^{(b)} \cdot \hat{P}_2^{(b)})} \frac{\chi_{(b)}}{\sqrt{\det b^T b}}$$

- Integrate by parts! Two type of terms, boxes and lower dim simplices
- Experimentally, coefficients of lower dim simplices vanish (implies constant transcendentality!)

- Result: hexagon in 2d is sum of boxes with well defined coefficients,

$$\frac{(W_1^{(b)})^2}{(W_1^{(b)} \cdot \hat{P}_1^{(b)})(W_1^{(b)} \cdot \hat{P}_2^{(b)})} \int \mathrm{d}X\, e^{X^2} \frac{\chi_{(b)}}{\sqrt{\det b^T b}}$$

Box integral

2d Hexagon

$$I_6 = \frac{(2n-4)!!}{2^{n-2}} \frac{\chi_1^- \chi_2^- \chi_1^+}{\left(\left(\chi_1^- - \chi_2^-\right)\chi_1^+ + \left(\chi_1^- + 1\right)\chi_2^- \chi_3^+\right)\left(-\chi_1^+ + \chi_3^+ + \chi_1^-\left(\chi_3^+ + 1\right)\right)} \times$$

$$\times \frac{\left(\chi_1^- + 1\right)\left(\chi_1^+ - \chi_3^+\right)\left(\chi_3^+ + 1\right)^2}{\left(\chi_3^+\left(\chi_3^+ - \chi_1^+\right) + \chi_1^-\left((\chi^+)_3^2 + \chi_1^+ - \chi_2^+\left(\chi_3^+ + 1\right)\right)\right)} \times B + \ldots,$$

$$B = 2\,\mathrm{Li}_2\left(\frac{\chi_1^+ - \chi_3^+}{\chi_2^+ - \chi_3^+}\right) + 2\,\mathrm{Li}_2\left(\frac{\chi_1^- - \chi_3^+}{\chi_3^+\chi_1^- + \chi_1^-}\right) +$$

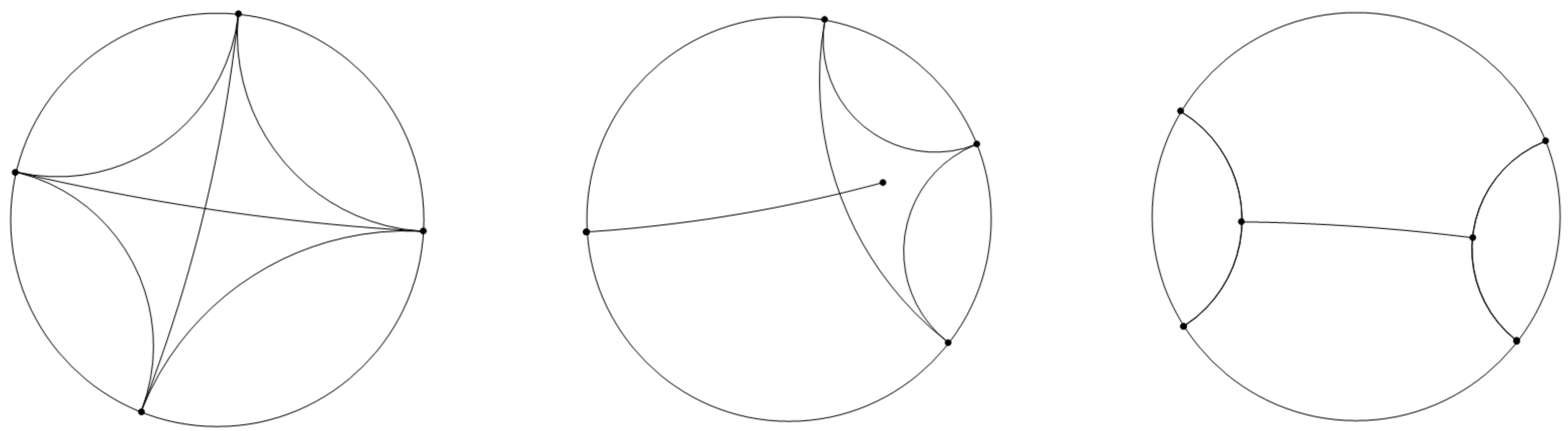
$$\log\left(\frac{\chi_1^-\left(\chi_1^+ - \chi_3^+\right)\left(\chi_3^+ + 1\right)}{\left(\chi_1^- - \chi_3^+\right)\left(\chi_2^+ - \chi_3^+\right)}\right)\log\left(-\frac{\chi_1^-\left(\chi_1^+ - \chi_2^+\right)\left(\chi_3^+ + 1\right)}{\left(\chi_1^- + 1\right)\left(\chi_2^+ - \chi_3^+\right)\chi_3^+}\right) +$$

$$\log\left(\frac{\chi_3^+ - \chi_1^-}{\chi_1^-\left(\chi_3^+ + 1\right)}\right)\log\left(\frac{\chi_3^+ - \chi_1^+}{\chi_2^+ - \chi_3^+}\right) + \frac{\pi^2}{3}$$

The χ variables encode the 6 independent cross-ratios for 6 pts in 2d.

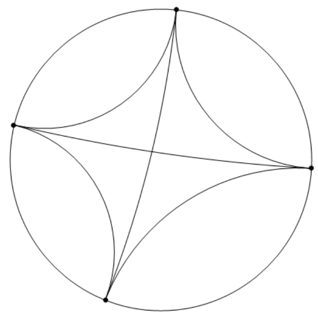
Convolutions

- Remarkably, single and higher loop integrals can be written in terms of spline convolutions.

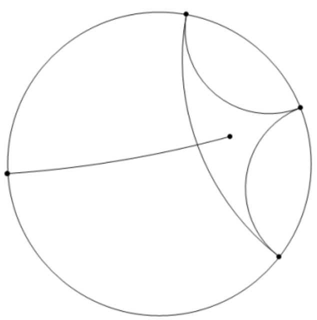


Volume = Area x Length = Length x Length x Length

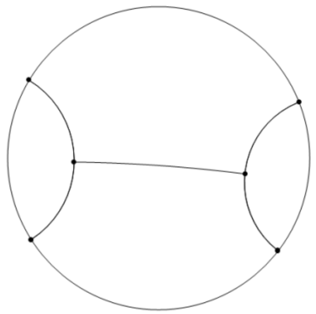
Convolutions



$$\int_{\mathrm{AdS}} \mathrm{d}X\, \mathcal{T}(X, \{P_1, P_2, P_3, P_4\})$$

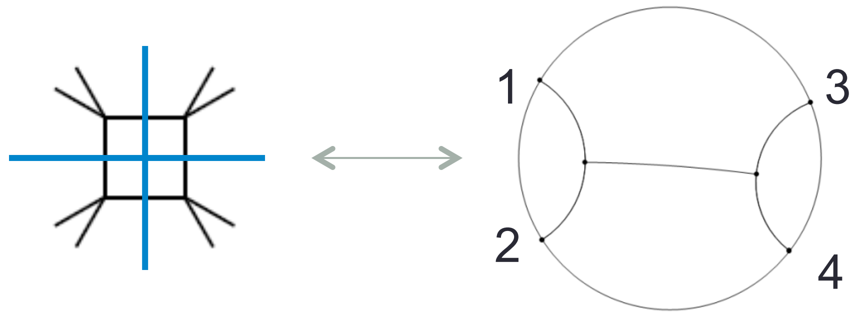


$$\int_{\mathrm{AdS}} \mathrm{d}X \mathrm{d}X'\, \mathcal{T}_{1,3}(X, \{P_4, X'\})\, \mathcal{T}(X', \{P_1, P_2, P_3\})$$

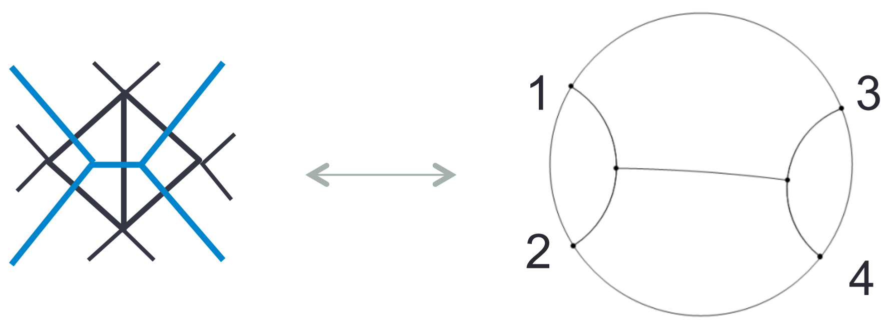


$$\int_{\mathrm{AdS}} \mathrm{d}X \mathrm{d}X_1 \mathrm{d}X_2\, \mathcal{T}_{2,2}(X, \{X_1, X_2\})\, \mathcal{T}(X_1, \{P_1, P_2\}) \mathcal{T}(X_2, \{P_3, P_4\})$$

From convolutions to multi-loops

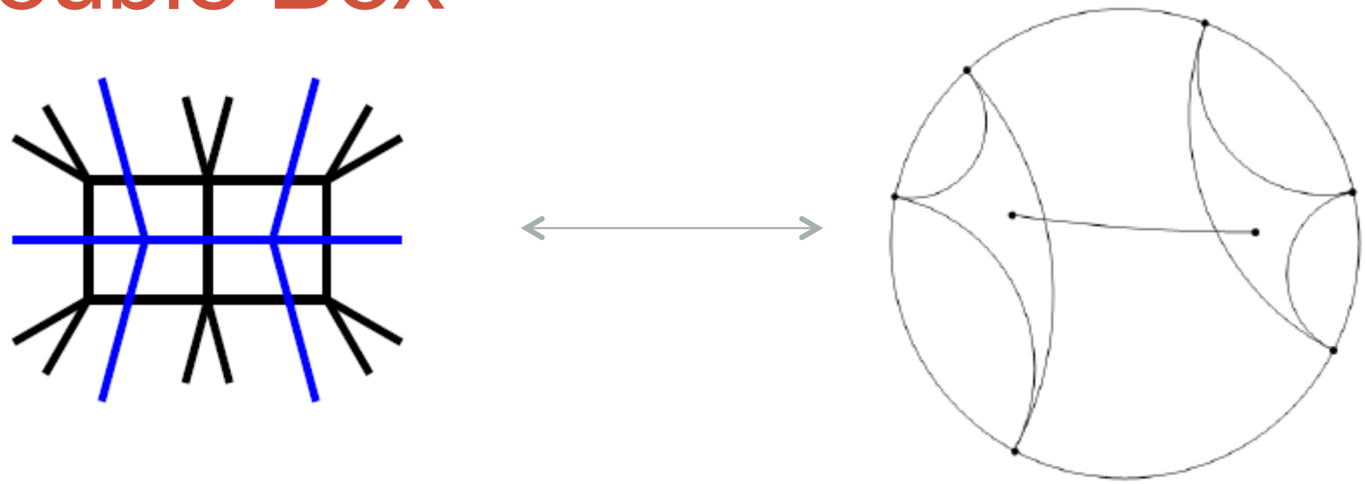


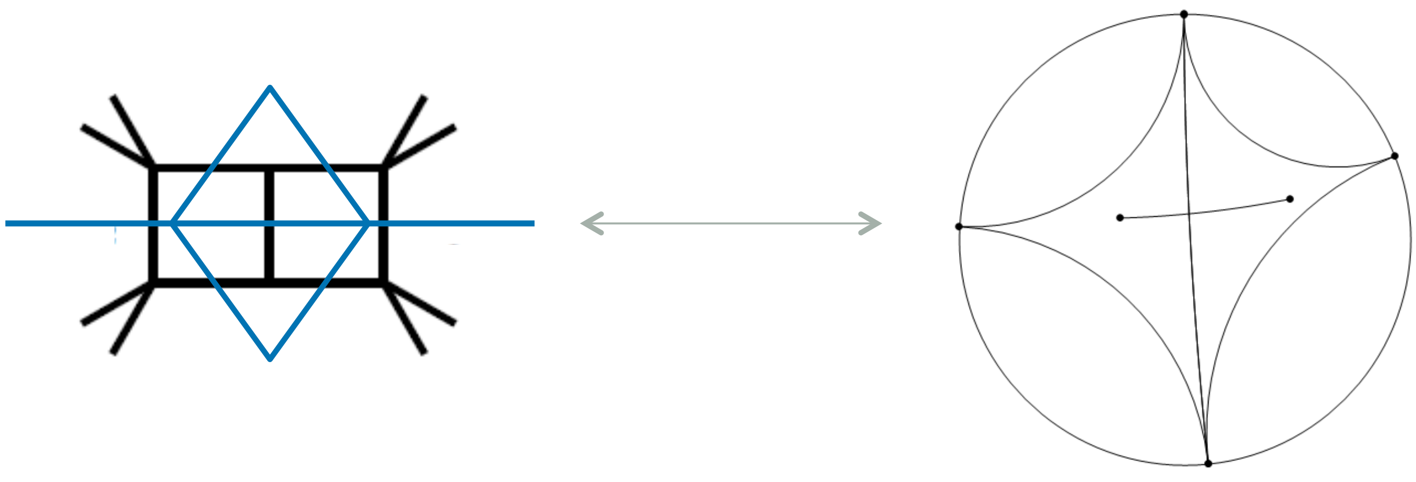
$$\int_{\mathrm{AdS}} \mathrm{d}X \mathrm{d}X_1 \mathrm{d}X_2 \, \mathcal{T}_{2,2}(X, \{X_1, X_2\}) \, \mathcal{T}(X_1, \{P_1, P_2\}) \mathcal{T}(X_2, \{P_3, P_4\})$$



$$\int_{\mathrm{AdS}} \mathrm{d}X \mathrm{d}X_1 \mathrm{d}X_2 \, \mathcal{T}_{1,3}(X \;, \{X_1, X_2\}) \mathcal{T}(X_1, \{P_1, P_2\}) \mathcal{T}(X_2, \{P_3, P_4\})$$

Double Box





Conclusions and Outlook

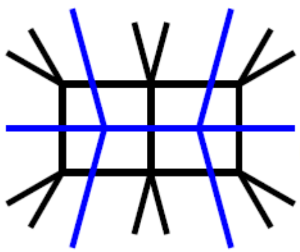
- Splines – geometrization of loop integrals and rational function identities.
- Interesting links to matroid theory and hyperplane arrangements

- Most of the work done for one-loop; how do interesting 2-loop calculations (e.g. 4 pt stress-tensor) look like geometrically?

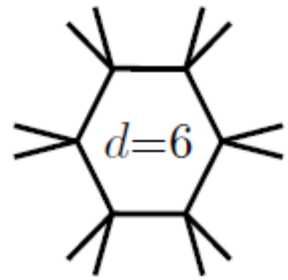
- Connections to Grassmannian story ?
- Splines are continuum limit of what? (**Spoiler**: non-local field theories)

Thank you!

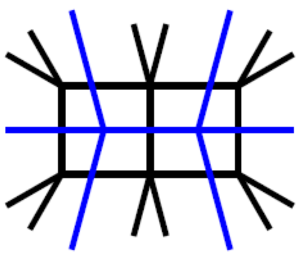
Beyond polylogs



$$(u,\ldots) = -\frac{1}{2}\int_{u}^{+\infty}\frac{du'}{u'}$$



$$(u',\ldots)$$



$$(u,\ldots) = \int_{u_8}^{+\infty}\frac{\mathrm{d}u_8'}{u_8'}\,\frac{\mathrm{Li}_3(\ldots)+\ldots}{\sqrt{\Delta^{(6)}}}$$

$$\Delta^{(6)} = \left[4\,u_1u_2u_5u_6u_7u_9u_8^3 + \text{lower-order terms in } u_8\right]$$

$$\int \frac{\mathrm{d}u_8}{u_8\sqrt{(u_8-a)(u_8-b)(u_8-c)}}$$

- Elliptic functions

Embedding space formalism

- Integrals with numerators can also be addressed – though no Feynman rules for those.

- Picture is clearer in embedding space: go to d+2 dimensions to linearize action of the conformal group, SO(d+1,1)

$$P^M P_M = -P^+ P^- + P^\mu P_\mu = 0, \qquad P \simeq \lambda P$$

$$\frac{P^M}{-P \cdot I} = \sqrt{2}\,(1, x^2, x^\mu) \qquad\qquad P_{ij} \equiv \frac{(-P_i \cdot P_j)}{(-P_i \cdot I)(-P_j \cdot I)} = (x_i - x_j)^2$$

- “I” is a fixed reference vector which set the mass scale. It breaks SO(d+1,1) conformal symmetry. In conformal expressions it must always drop out!

Dealing with numerators

- One-loop integral with two numerators (chiral hexagon)

$$I_6^2 = \frac{1}{2\pi^2} \int \mathrm{d}^4 Q \, \frac{(-Q\cdot Y)(-Q\cdot Y')}{\prod_{i=1}^{6}(-P_i \cdot Q)} \qquad \begin{aligned} Y\cdot P_i &= 0, \qquad i = 1,\ldots,4, \\ Y'\cdot P_i &= 0, \qquad i = 3,\ldots,6, \end{aligned}$$

- After a little work,

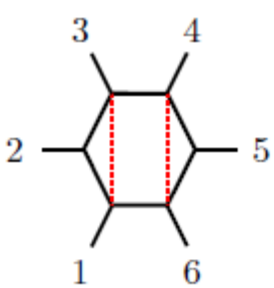
$$= Y_A Y_B' \left(\eta^{AB} - \sum_{i,j} \frac{P_i^A P_j^B}{P_{ij}} \hat{\partial}_{ij} + \sum_i P_i^A P_i^B \hat{S}_i \right) \underbrace{\left[\oint \mathrm{d}\delta_{ij} \prod_{i<j} \Gamma(\delta_{ij})\, P_{ij}^{-\delta_{ij}} \right]}_{\text{M=1, star integral!}}$$

Dealing with numerators

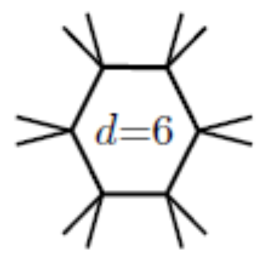
- One-loop integral with two numerators (chiral hexagon)

$$I_6^2 = \frac{1}{2\pi^2}\int \mathrm{d}^4 Q\, \frac{(-Q\cdot Y)(-Q\cdot Y')}{\prod_{i=1}^{6}(-P_i\cdot Q)} \qquad \begin{aligned} Y\cdot P_i &= 0, & i&=1,\ldots,4,\\ Y'\cdot P_i &= 0, & i&=3,\ldots,6, \end{aligned}$$

- In terms of cross-ratios:



$$= \Big(u_2(1-u_1-u_3)+u_2(1-u_1)u_1\hat{\partial}_{u_1} + u_2(1-u_3)\,u_3\partial_{u_3} - (1-u_2)(1-u_1-u_3)\,u_2\partial_{u_2}\Big)$$



- (Result first obtained in ...)

Consequences of Feynman rules

- The factorized form of Mellin amplitudes can be put to good use:

$$M^f(s) = \int_0^{+\infty} \frac{\mathrm{d}x}{x}\, x^s\, f(x), \qquad M^g(s) = \int_0^{+\infty} \frac{\mathrm{d}x}{x}\, x^s\, g(x)$$

$$\begin{aligned} h(x) &= \oint \frac{\mathrm{d}s}{2\pi i}\, M^f(s) M^g(s) x^{-s} = \oint \frac{\mathrm{d}s}{2\pi i} \int_0^{+\infty} \frac{\mathrm{d}y}{y}\, y^s\, f(y) M^g(s) x^{-s} \\ &= \int_0^{+\infty} \frac{\mathrm{d}y}{y}\, f(y) g(x/y). \end{aligned}$$

- Final position space expression is convolution of simpler integrals.