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On Non-Trivial Spectra of Trivial, Two Dimensional Gauge Theories

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- **Nontrivial spectra of two dimensional gauge theories**
- **Lattice: partition function and its continuum limit**
- **Adding external charges:**
  Lattice: transfer matrix and spectrum
  continuum limit
  Feynman kernel
  reduced system, hamiltonian and wave functions
  theta states
  screening and effective fractional charge
- **Fractional charges on a lattice and the continuum limit**
- **Nonabelian case**

I. Nontrivial spectra of trivial gauge theories

- **Two dimensional gauge theories are trivial - no transverse degrees of freedom.**
- **True only if we neglect boundary conditions.**

Quantum Maxwell Dynamics in 1+1 dimensions (QMD_2) on a circle

$$E_n = \frac{e^2}{2} L n^2, \qquad n = 0, \pm 1, \pm 2, ... \qquad [Manton,'84]$$

An effective 1DOF hamiltonian

$$H = -\frac{e^2}{2L}\frac{d^2}{dA^2}, \qquad 0 \leq A \ < \ L_A = \frac{2\pi}{L} \tag{1}$$

The spectrum

$$\psi_n(A) = e^{inAL} = e^{ip_n A}, \qquad p_n = n\frac{2\pi}{L_A} = nL, \qquad E_n = \frac{e^2}{2} L n^2 \tag{2}$$

What is A ?

$$A_x(x,t) = A(x,t), \qquad \xrightarrow{\partial_x A(x,t)=0} A(x,t) = A(t) \neq 0$$

In a periodic (in x) world one cannot set a constant A to 0 by a gauge transformation – 1 DOF left

Why periodicity in A ?

If space is periodic, gauge transformations also have to be periodic

$$g(x) = e^{i\Lambda(x)} = g(x+L), \quad \longrightarrow \quad \Lambda(x+L) = \Lambda(x) + 2\pi n$$

Take $\Lambda(x) = 2\pi \frac{x}{L}$, **then**

$$A \longrightarrow A + \partial_x \Lambda(x) = A + \frac{2\pi}{L}, \quad are \;\; gauge \;\; equivalent \quad \Longrightarrow \quad A \in (0, \frac{2\pi}{L}]$$

Interpretation

- **a string with n units of electric flux winding around a circle**
- **Gauss's law satisfied thanks to the nontrivial topology - topological strings**
- **electric charge even without electrons/sources !**

A generalization: Θ parameter

a)

$$H = -\frac{e^2}{2L}\left(\frac{d}{dA} + i\Theta L\right)^2,$$

$$E_n = \frac{e^2}{2}L(n+\Theta)^2, \quad \psi_n(A) = e^{inAL}$$

b)

$$\tilde{H} = -\frac{e^2}{2L}\frac{d^2}{dA^2},$$

$$E_n = \frac{e^2}{2}L(n+\Theta)^2, \quad \tilde{\psi}_n(A) = e^{i(n+\Theta)AL},$$

$$\tilde{\psi}_n(A) = e^{i\Theta AL}\psi_n(A)$$

Interpretation: $e\Theta$ – classic, constant electric field

II. QMD_2 on a lattice

Partition function on a 2x2 lattice

$$
\begin{aligned}
Z \;=\; & \int_0^{2\pi} B(\theta_{12}+\vartheta_{22}-\theta_{11}-\vartheta_{12})B(\theta_{22}+\vartheta_{12}-\theta_{21}-\upsilon_{22}) \\
& \quad B(\theta_{11}+\vartheta_{21}-\theta_{12}-\vartheta_{11})B(\theta_{21}+\vartheta_{11}-\theta_{22}-\vartheta_{21}) \\
& d(links)
\end{aligned}
$$

$$B(\phi_P) = e^{\beta\cos(\phi_P)}, \quad d(links) = \Pi_l \frac{d\alpha_l}{2\pi}$$

Change variables from links to plaquettes ϕ_P

- **One constraint between plaquette angles (PBC)**

$$\sum_P \phi_P = 0$$

$$Z = \int_0^{2\pi} d\phi_1 d\phi_2 d\phi_3 B(\phi_1)B(\phi_2)B(\phi_3)B(\phi_1+\phi_2+\phi_3).$$

A character expansion (Fourier analysis on a group)

$$B(\phi) = \Sigma_{n=-\infty}^{\infty} I_n(\beta)\exp{(in\phi)},$$

The partition function "almost" factorizes

$$Z = \Sigma_n I_n(\beta)^4$$

For N_xxN_t lattice

$$Z \;=\; \int d^{N_V-1}\phi_P \left(\Pi_P^{N_V-1} B(\phi_P)\right) B\left(\Sigma_P^{N_V-1}\phi_P\right) = \Sigma_n I_n(\beta)^{N_V}, \quad N_V = N_t * N_x. \quad (3)$$

The continuum limit

$$Z \;=\; \#\, \Sigma_n \left(\frac{I_n(\beta)}{I_0(\beta)}\right)^{N_x * N_t},$$

$$aN_t = T, \quad aN_x = L \quad \beta = \frac{1}{e^2 a^2}, \quad a \to 0.$$

Asymptotic expansion of modified Bessel function

$$I_n(\beta) \to \frac{e^\beta}{\sqrt{2\pi\beta}}\left(1 - \frac{4n^2 - 1}{8\beta} + ...\right)$$

gives

$$Z_{LQMD_2} \;\to\; \#\;\Sigma_n \left(1 - \frac{e^2}{2} n^2 a^2\right)^{N_x N_t} = \Sigma_n e^{-E_n T}, \qquad E_n = \frac{1}{2} e^2 n^2 L,$$

$\longrightarrow$ **Manton fluxes result in the continuum limit of lattice** QMD_2

Emergence of a constant mode - Coulomb gauge on a lattice

A single row of $N_x = 3$ horizontal links $\theta_1, \theta_2, \theta_3$

A local gauge transformation specified by $\alpha_1, \alpha_2, \alpha_3$

$$\begin{aligned}
\theta_1 &\rightarrow {}^g\theta_1 = \theta_1 + \alpha_1 - \alpha_2 \\
\theta_2 &\rightarrow {}^g\theta_2 = \theta_2 + \alpha_2 - \alpha_3 \\
\theta_3 &\rightarrow {}^g\theta_3 = \theta_3 + \alpha_3 - \alpha_1
\end{aligned}$$

or

$${}^g\theta_i = \theta_i + \beta_i, \quad \Sigma_{i=1}^{3}\beta_i = 0$$

If we choose

$$\begin{aligned}
\beta_1 &= \frac{1}{3}(\theta_1 + \theta_2 + \theta_3) - \theta_1 \\
\beta_2 &= \frac{1}{3}(\theta_1 + \theta_2 + \theta_3) - \theta_2 \\
\beta_3 &= \frac{1}{3}(\theta_1 + \theta_2 + \theta_3) - \theta_3
\end{aligned}$$

then all new link angles are equal

$$ {}^{g}\theta_1 =^{g} \theta_2 =^{g} \theta_3 = \frac{1}{3}(\theta_1 + \theta_2 + \theta_3) \equiv \theta_{row}. $$

$\Longrightarrow$**Only one degree of freedom remains**

A transfer matrix

• partition function

$$ Z = \int d4\ d3\ d2\ d1\ \ < 4|\Pi|3 >< 3|\Pi|2 >< 2|\Pi|1 >< 1|\Pi|4 >= Tr\left(\Pi^4\right), \tag{4} $$

where $di = d\alpha_i d\beta_i$ and the states $|i >= |\alpha_i, \beta_i >$.

• elements of transfer matrix, in the angular representation, are

$$ < \alpha', \beta'|\Pi|\alpha, \beta >= \int d\vartheta_1 d\vartheta_2 B(\alpha + \vartheta_2 - \alpha' - \vartheta_1)B(\beta + \vartheta_1 - \beta' - \vartheta_2) \tag{5} $$

for N_x sites, and in the Coulomb gauge, they can be rewritten as

$$< \theta|\Pi|\theta' >= \Sigma_n I_n(\beta)^{N_x} e^{inN_x(\theta-\theta')}$$

with θ being now a single, common coordinate of all N_x horizontal links.

Continuum limit $N_x\theta \rightarrow LA$

$$\beta = \frac{1}{e^2a^2}, \quad aN_x = L, \quad \theta = aA$$

repeating earlier steps gives

$$< \theta|\Pi|\theta' > \longrightarrow \Sigma_n e^{-E_n a} e^{inL(A-A')} = K(A, A', \epsilon = a)$$

which is nothing but a spectral representation of the Feynman kernel propagating the system (1-2) through a time lapse $\epsilon = a$.

- Volume reduction

III. Adding external charges

Wilson loops - a tailing trick

$$W[\Gamma] = \Pi_{l\in\Gamma} e^{i\theta_l} = \Pi_{p\in in(\Gamma)} e^{i\phi_p}$$

$$\begin{aligned} Z\langle W\rangle &= \int d^{N_V-1}\phi_p \left(\Pi_{p\in in(\Gamma)} e^{i\phi_p} B(\phi_p)\right)\left(\Pi_{p\in out(\Gamma)} B(\phi_p)\right) B\left(\Sigma_p^{N_V-1}\phi_p\right) \\ &= \Sigma_n I_n(\beta)\left(\Pi_{in(\Gamma)}\int_{\phi_p} e^{i(n+1)\phi_p} B(\phi_p)\right)\left(\Pi_{out(\Gamma)}\int_{\phi_{p'}} e^{in\phi_{p'}} B(\phi_{p'})\right) \\ &= \Sigma_n I_n(\beta)^{N_x * N_t - n_x * n_t} I_{n+1}(\beta)^{n_x * n_t}. \end{aligned} \tag{6}$$

Time like Polyakov loops

As before

$$Z < P^{\dagger}(1)P(n_x+1) >= \Sigma_n I_n(\beta)^{N_t*(N_x-n_x)} I_{n+1}(\beta)^{N_t*n_x}, \tag{7}$$

Continuum limit

As earlier, introduce the dimensionful lattice constant, use the asymptotic form of Bessel functions and express (7) in terms of physical distances (in particular the distance between sources, $an_x = R$) to obtain

$$Z < P(0)^{\dagger}P(R) >= \Sigma_n e^{-E_n^{PP}T}, \tag{8}$$

with

$$E_n^{PP} = \frac{e^2}{2}\left(n^2(L-R)+(n+1)^2R\right), \quad n = 0, \pm 1, \pm 2, ... \tag{9}$$

A straightforward interpretation:

$$E_n^{PP} = \frac{e^2}{2}\left(n^2(L-R) + (n+1)^2 R\right), \quad n = 0, \pm 1, \pm 2, ... \tag{10}$$

- Time like Polyakov lines modify Gauss's low at spatial points 0 and R - they introduce external unit charges at these positions.
- Such charges cause additional unit of flux extending over distance R.
- Hence the two contributions to the eigenenergies: an "old" flux over the distance $L-R$ and the new one, bigger by one unit (fluxes are additive !) , over R.
- Interesting special cases:

  → at large T the lowest, $n = 0$ and $n = -1$, states dominate. Then we just have standard (unit flux) strings of length R and L-R ,

  → $R = 0$ – old topological flux with charge n.

  → $R = L$ – when external charges meet at the "end point" of a circle, they annihilate ($e^+\delta_P(0)+e^-\delta_P(L) = 0$)and leave behind a topological string with length L and charge bigger by one unit.
- Varying R interpolates between integer valued topological fluxes.

Equivalent form

$$E_n^{PP} = \frac{e^2}{2} L (n+\rho)^2 + const.(L,R), \quad \rho = \frac{R}{L}, \quad const. = \frac{e^2}{2} L\rho(1-\rho) \tag{11}$$

- Indeed $e\frac{R}{L}$ is the electric field, generated by two sources, *averaged* over the whole volume.
- The system does not see any distances, $A_x(x) = const.$, hence averaging over the volume.
- Changing R allows to mimic arbitrary real charge $q = e(n+\rho)$.
- Only $[\rho]$ is relevant.

Hamiltonian and wave functions

Transfer matrix: repeat previous steps with two Polyakov lines

$$< \theta|\Pi^{PP}|\theta' > \;=\; \Sigma_n I_n(\beta)^{N_x-n_x} I_{n+1}(\beta)^{n_x} e^{in(N_x-n_x)(\theta-\theta')} e^{i(n+1)n_x(\theta-\theta')} \quad (12)$$

$$\equiv K_L^{PP}(\theta,\theta') \;=\; \Sigma_n I_n(\beta)^{N_x-n_x} I_{n+1}(\beta)^{n_x} e^{inN_x(\theta-\theta')} e^{in_x(\theta-\theta')} \quad (13)$$

In the continuum limit , $N_x\theta = LA, n_x\theta = RA$**, we get**

$$K_L^{PP}(\theta,\theta') \longrightarrow K^{PP}(A,A',\epsilon) = \Sigma_n e^{-\frac{e^2L}{2}\left((n+\rho)^2+\rho(1-\rho)\right)\epsilon} e^{i(n+\rho)L(A-A')}. \quad (14)$$

which is the momentum expansion of the Feynman kernel describing 1DOF QM with above spectrum. Now we can identify eigenfunctions and the hamiltonian

$$H = -\frac{e^2L}{2}\frac{d^2}{d\chi^2} + \frac{e^2L}{2}\rho(1-\rho), \quad \psi_n(\chi) = e^{i(n+\rho)\chi}. \quad (15)$$

Or, in another basis

$$\bar{K}^{PP}(A, A', \epsilon) \equiv e^{-i\rho(A-A')L} K^{PP}(A, A', \rho).$$

$$\bar{H} = -\frac{e^2 L}{2}\left(\frac{d}{d\chi} + i\rho\right)^2 + \frac{e^2 L}{2}\rho(1-\rho), \quad \chi = LA, \quad \bar{\psi}_n(\chi) = e^{in\chi},$$

with the spectrum (11) and corresponding, *periodic* eigenfunctions.

- **Θ parameter acquires now a straightforward interpretation**

$$\Theta_{Manton} = \rho = \frac{R}{L},$$

- **A new constant term.**

Θ-vacua

- The transformation $A \longrightarrow A + \frac{2\pi}{L}$ is a large gauge transformation, $\Lambda(x) = \frac{2\pi x}{L}$, $\Lambda(x + L) = \Lambda(x) + 2\pi$
- Full analogy 4D YM and/or the crystal : many classical configurations around which we can quantize
- Θ vacua: $|\Theta\rangle = \Sigma_n e^{i\Theta n}|n\rangle$
- The wave function of a Θ-state $\psi_\Theta(x) = \langle x|\Theta\rangle$ satisfies $\psi_\Theta(x - d) = e^{i\Theta}\psi_\Theta(x)$
- The solution (Bloch theorem) : $\psi_\Theta(x) = e^{i\Theta x/d}u_\Theta(x)$, with periodic $u_\Theta(x)$
- Our case: $\psi_n(A) = e^{i(n+\rho)AL} = e^{i\rho AL}e^{inAL}$ is exactly of Bloch type upon identification $x \to A$, $d \to 2\pi/L$, $\Theta \to 2\pi\rho$

- Introducing external charges fixes the Θ-vacuum in QMD_2.

- D=4 : in a Θ-vacuum some field configurations acquire electric charge [Witten '76].

More, different charges

R_2 - distance between doubly charged sources
R_1 - distance between singly charged ones

$$Z < P(i)^{\dagger} P(j)^{2\dagger} P^2(j+n_2) P(i+n_1) >=$$

$$\Sigma_n I_n(\beta)^{N_t(N_x-n_1)} I_{n+1}(\beta)^{N_t(n_1-n_2)} I_{n+3}(\beta)^{N_t n_2},$$

- eigenenergies in the continuum limit

$$\begin{aligned} E_n^{PPPP} &= \frac{e^2}{2}\left(n^2(L-R_1)+(n+1)^2(R_1-R_2)+(n+3)^2 R_2\right) \\ &= \frac{e^2}{2} L\left((n+\rho_1+2\rho_2)^2+\rho_1(1-\rho_1)+4\rho_2(2-\rho_1-\rho_2)\right) \end{aligned}$$

etc. 1 DOF quantum mechanical systems can be also readily constructed.

- This time $\Theta = (R_1 + 2R_2)/L$, i.e. it is again equal to the external field averaged over the whole volume.

IV. Arbitrary charges on a lattice

Why? To learn about screening

Massive Schwinger model

$$\sigma_q = m\, e \left(1 - \cos\left(2\pi \frac{q}{e}\right)\right) \qquad m/e << 1, \qquad [Coleman\ et\ al.,\ '75]$$

$\Rightarrow$ generalizations for large N QCD_2.

$\Rightarrow$ How to put arbitrary (noncongruent with e) charges on a lattice?

- One way: as above $q = e(n + R/L)$
- Another way: new observables

Wilson loops with arbitrary charge

$$Z\langle W_Q\rangle = \int (W[\Gamma])^Q \, e^{-S}, \qquad Q = q/e$$

Contras:

gauge invariance – not if you carefully/consistently deal with multivaluedness

dependence on the boundaries in angular variables – not if you do loops

Pros:

Results are consistent ($MC \leftrightarrow TH$)

New structure appears QMD_2

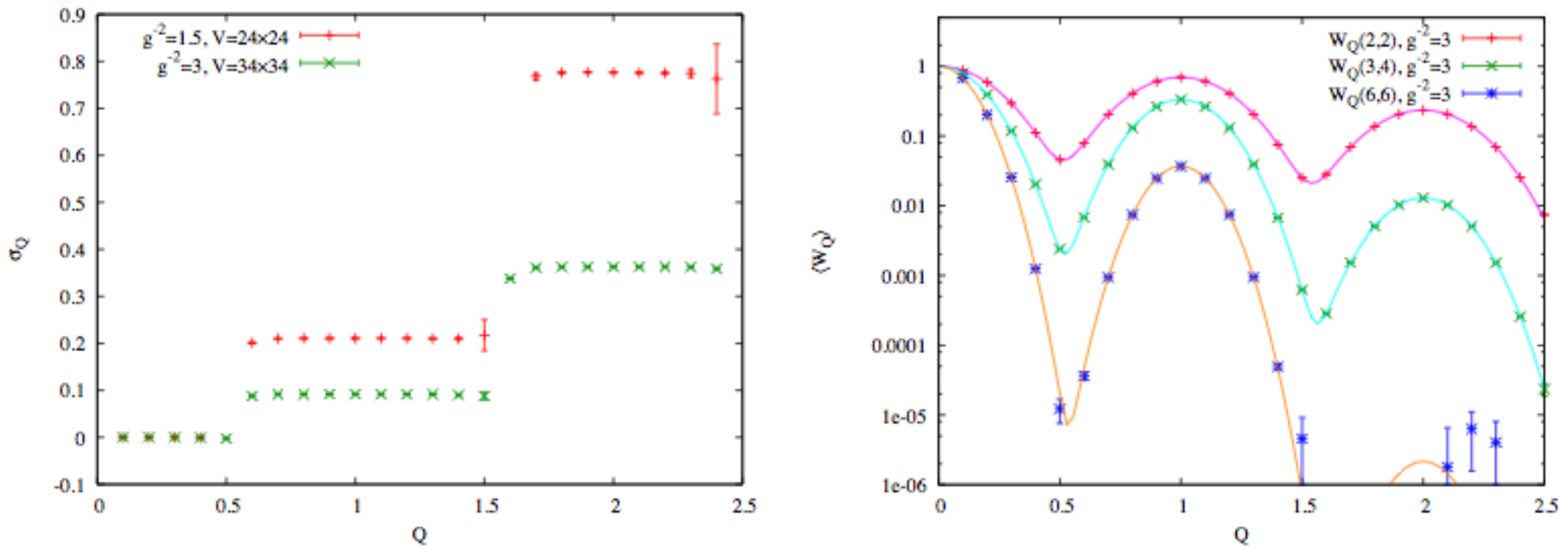
Why not !

Q-loops theoretically

$$
\begin{aligned}
Z\langle W_Q \rangle &= \int_0^{2\pi} d(links) \left(\Pi_{l\in\Gamma} e^{iQ\theta_l}\right) \left(\Pi_p^{N_V} B(\phi_p)\right) \\
&= \Sigma_{m_1,m_2,\ldots,m_{N_V}} I_{m_1} \ldots I_{m_{N_V}} \int_{links} \left(\Pi_{l\in\Gamma} e^{iQ\theta_l}\right) \left(\Pi_p^{N_V} e^{im_p\phi_p}\right) \\
&= \Sigma_{m_1,m_2,\ldots,m_{N_V}} I_{m_1} \ldots I_{m_{N_V}} \left(\Pi_{l\notin\Gamma} \delta_{m_L(l),m_R(l)}\right) \left(\Pi_{l\in\Gamma} \bar{S}(Q - m_L(l) + m_R(l))\right) \\
&= \Sigma_{m,n} I_n^{N_x N_t - n_x n_t} I_m^{n_x n_t} S(Q - m + n)^{n_x + n_t},
\end{aligned}
$$

$$
\bar{S}(x) = \frac{\sin \pi x}{\pi x}, \qquad S(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2
$$

and "experimentally" [P. Korcyl, M. Koren]



(a) String tension as a function of Q

(b) Ansatz vs. data (not a fit)

Figure 1:

- Q-loops can be defined on a lattice - MC agrees with TH
- They do not create states with arbitrary charge
  – they excite the only existing quantum states with integer charges

Continuum limit

$$Z\langle W_Q\rangle = \Sigma_{m,n} I_n^{N_x N_t - n_x n_t} I_m^{n_x n_t} S(Q-(n-m))^{n_t+n_x} =$$

$$\Sigma_{m,n} \exp\left(-\frac{e^2}{2} n^2 L(T-t)\right) \exp\left(-\frac{e^2}{2}\left(n^2(L-R)+m^2 R\right) t\right)$$

$$S(Q-(n-m))^{(t+R)/a}$$

does not exist at fixed, not integer Q.

$\Longrightarrow$ However the *classical* limit:

$Q \to \infty$, with $q = Qe - fixed$, on a fixed lattice $(a, N's, const.)$ does exist!

Then $\beta \equiv b^2 = 1/e^2a^2 \to \infty$, but not because $a \to 0$, but because $e \to 0$.
The spectrum of fluxes becomes continuous: $n \to u = n/b, m \to v = n/b$

Therefore $(Q = q/e = \sqrt{\beta/\kappa} = b/g, g = 1/qa)$

$$ZK_{\Pi QQ} = \beta \int dudv \exp\left(-\frac{1}{2}(u^2(N_x - n_x) + v^2 n_x)\right)$$
$$S\left(b(g^{-1} - (u - v))\right)^2 e^{ibu(\Theta_{L-R} - \Theta'_{L-R})} e^{ibv(\Theta_R - \Theta'_R)}$$

using

$$S(b\Delta) \stackrel{b\to\infty}{\longrightarrow} \frac{1}{b}\delta(\Delta)$$

gives

$$ZK_{\Pi QQ} = \sqrt{\beta} \int du \exp\left(-\frac{1}{2}(u^2(N_x - n_x) + (u - g^{-1})^2 n_x)\right)$$
$$e^{ibu(\Theta_{L-R} - \Theta'_{L-R})} e^{ib(u-g^{-1})(\Theta_R - \Theta'_R)}$$

Now, do the gaussian integral, take the continuum limit to obtain

$$Z K_{\Pi QQ} = \sqrt{\beta}\sqrt{\frac{2\pi a}{L}} \exp\left(-\frac{L}{2}\frac{(A-A')^2}{a}\right) \exp\left(-\frac{q^2}{2}\rho(1-\rho)La\right)$$

$\Longrightarrow$ a free particle propagating over a time a, but in a constant background potential

$$V = \frac{q^2}{2}\rho(1-\rho)L$$

with arbitrary, real value of a classical charge q.

- The classical energy with a continuous charge q results from the contribution of many microscopic states with discrete charges.
- the structure (zeroes of the string tension)

V. Nonabelian case: YM_2 on a circle

• Continuum: problem reduces to N constant in space, but constrained, angles θ_i, $\Sigma_i \theta_i = 0$.

Hamiltnian is again quadratic and the spectrum is known explicitly [Hetrick and Hosotani '89]

$$E_{\{n\}} = \frac{g^2 L}{4} \left(\Sigma_i n_i^2 - \frac{1}{N} \left(\Sigma_i n_i \right)^2 \right), \quad i = 1, ..., N-1$$

• Continuum: different spectrum was obtained by Rajeev: $E_R = \frac{g^2 L}{2} C_2(R)$
• Discrepancy comes from the Casimir energy due to the curvature of the group manifold [Hetrick '93, Witten '91,'92]
• Lattice: continuum spectrum $\Longleftarrow$ the large β behaviour of the character expansion of Boltzman factor.

It is given by the Casimir plus, the N dependent, constant curvature correction/Casimir energy, and agrees with Hetrick and Hosotani .
• External charges in YM_2 – studied by many [Semenoff et al. '97] but above connection with Θ-vacuum not.

INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY

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EUROPEAN REGIONAL DEVELOPMENT FUND

EU grant (via Foundation for Polish Science)

Jagellonian University International PhD Studies on Physics of Complex Systems

- 1 M Euro
- 4 years
- 14 PhD students (1/2 - 2 years abroad)
- 9 Local Supervisors
- 17 Foreign Partners: J. Ambjorn, J.P. Blaizot, H. Nicolai, S. Sharpe, ...