"Introduction" to "Amplitudes"

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including work in collaboration with
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Introduction

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As I hope to emphasize, this research program has both practical and "experimental" components.

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experimental → often we undertake irrelevant calculations, not to make busywork, but to try to identify new methods as well as previously hidden "principles"

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To some extent we have seen one revolution (tree level) come to fruition, but we are still in desperate need of another.

$$\mathcal{A} = \int d^4p_1 \cdots d^4p_L \underbrace{\sum \text{Feynman diagrams}}$$

$$= \int d^4p_1 \cdots d^4p_L \text{ (relatively simple integrand)}$$

Methods have been developed for efficiently processing the integrand, ($\Rightarrow$ Britto's talk) but there is no practical, general algorithm for writing down the results for such integrals.

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My examples are in $\mathcal{N}=4$ supersymmetric Yang–Mills theory, which is a favorite playground of amplitudeologists.

($\Rightarrow$ Korchemsky's talk)

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and, most honestly, hubris.

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what are they functions <u>of</u>?

i.e. what is the Kinematic domain for SYM theory?

Somewhat surprisingly, this question was fully settled only quite recently (2007 – 2009), even though SYM theory itself is over 35 years old!

What Variables Do They Depend On?

An n-particle amplitude depends on n 4-component vectors

$$p_i = (\underset{\text{energy}}{p_i^0}, \underbrace{p_i^1, p_i^2, p_i^3}_{\text{momentum}})$$

subject to

- $p_i^2 = 0 \quad \forall i$ for massless particles (with respect to Minkowski metric)
- $\sum_{i=1}^{n} p_i = 0$ energy/momentum conservation

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Soon we'll run into (generalized) polylogarithm functions whose arguments are algebraic functions of the p's, so our first task will be to help ease our burden by finding a trivial parameterization of this submanifold.

Dual conformal symmetry

(Drummond, Henn, Korchemsky, Sokatchev)

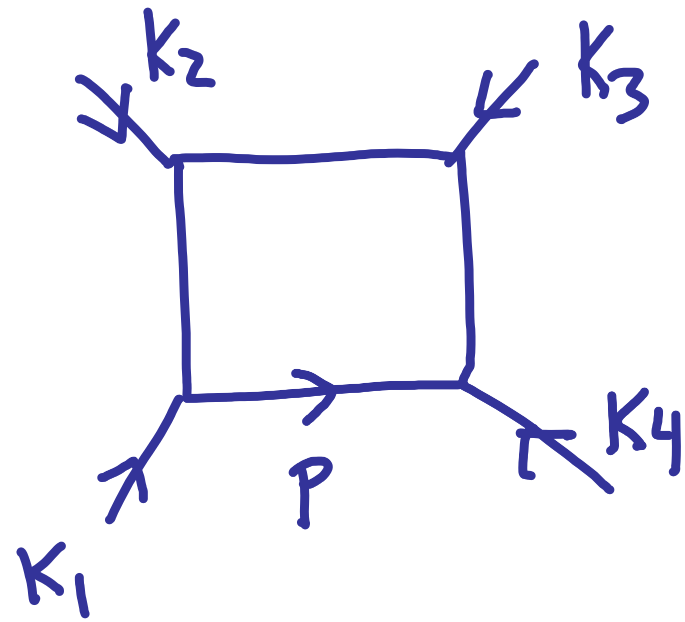
Parameterize

$$\left.\begin{aligned} k_1 &= x_1 - x_2 \\ k_2 &= x_2 - x_3 \\ &\vdots \\ k_n &= x_n - x_1 \end{aligned}\right\} \sum k_i = 0$$

$\Rightarrow$ Then amplitudes are invariant under conformal transformations on the x_i.

Example: 1-loop supersymmetric YM

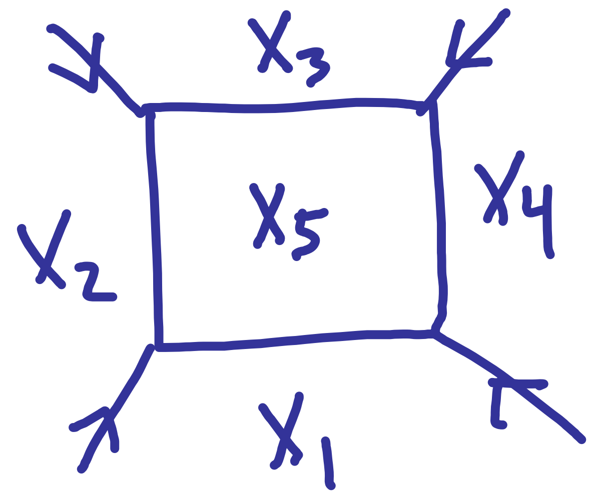
$$\frac{A_4^{\text{1-loop}}}{A_4^{\text{tree}}} = -\frac{1}{2} st \int d^4p$$



$$= -\frac{1}{2} st \int \frac{d^4p}{p^2(p+k_1)^2(p+k_1+k_2)^2(p-k_4)^2}$$

Example: 1-loop supersymmetric YM

$$\frac{A_4^{1\text{-loop}}}{A_4^{tree}} = -\frac{1}{2} x_{13}^2 x_{24}^2 \int d^4 x_5$$



$$= -\frac{1}{2} \int d^4 x_5 \, \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

where $x_{ij}^2 = (x_i - x_j)^2$

Example: 1-loop supersymmetric YM

$$\frac{A_4^{1\text{-loop}}}{A_4^{\text{tree}}} = -\frac{1}{2}\int d^4x_5 \, \frac{x_{13}^2 \, x_{24}^2}{x_{15}^2 \, x_{25}^2 \, x_{35}^2 \, x_{45}^2}$$

which has manifest conformal symmetry, in particular under inversion $x_i^\mu \to x_i^\mu / x_i^2$.

$$x_{ij}^2 \longrightarrow \frac{x_{ij}^2}{x_i^2 \, x_j^2} \qquad\qquad d^4x_5 \to \frac{d^4x_5}{(x_5^2)^4}$$

The New, "Experimental" S-Matrix Theory

Dual conformal symmetry is a purely "experimental" discovery which highlights the difference in approach between "new" and "old" S-matrix theory.

A Significantly Different Approach

Then, the goal was to first enumerate the principles (locality, unitarity, analyticity) and then construct an (the?) S-matrix.

Now, the approach is to first write down the solution, and then deduce the principles (some of which may never have occurred to us) from its structure.

The Kinematic Configuration Space

Amplitudes in SYM theory are multi-valued functions on the $3(n-5)$ dimensional domain

$$\{ p_i \in \mathbb{C}^{4n} : \ p_i^2 = 0, \ \sum p_i = 0 \} / GL(4, \mathbb{C})$$

$$= Gr(4,n) / (\mathbb{C}^*)^{n-1}$$

(not explained here)

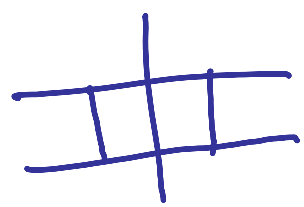
$$\equiv \mathrm{Conf}_n(\mathbb{P}^3)$$

Simplest Nontrivial Example

$n<6$ particle amplitudes in SYM "$=0$"

— no cross-ratios they can depend on!

The $n=6$ particle

$L=2$ loop ( + many more)

MHV (four $+$ and two $-$ particles)

Heroically computed by Del Duca, Duhr, Smirnov.

Simplest Nontrivial Example

two-loop,
n=6 MHV

[Goncharov, MS, Vergu, Volovich]

$$\sum_{\text{dihedral}} \mathrm{Li}_4\left(\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 6123\rangle\langle 2345\rangle}\right) - \frac{1}{4}\mathrm{Li}_4\left(\frac{\langle 6124\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$

$+$ ☆-conjugate

$+$ similar "simpler" terms

where $\langle abcd\rangle = \det(Z_a Z_b Z_c Z_d)$

Polylogarithms and their Variants

It is well-known that polylogarithms appear frequently in loop computations, $\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(t)\, d\log t$, as do their generalizations, $G(a_1, \ldots, a_k, z) = \int_0^z G(a_1, \ldots, a_{k-1})\, d\log(t - a_k)$

These are believed to be sufficient to express all SYM amplitudes for $n<8$

(though it is known that beyond, there be dragons!)

($\Rightarrow$ Vanhove's talk)

What about their arguments?

The argument of each polylogarithm is some simple conformal cross-ratio like

$$\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 6123\rangle\langle 2345\rangle}$$

When expressed in terms of Mandelstam variables, these are algebraic functions like

$$\frac{s_{12}s_{45}+\cdots+\sqrt{(s_{12}s_{45}+\cdots)^2-s_{123}s_{345}}}{s_{12}s_{45}+\cdots-\sqrt{(s_{12}s_{45}+\cdots)^2-s_{123}s_{345}}}$$

In other amplitudes even more complicated arguments appear,

eg $$\frac{\langle 1278\rangle\langle 1235\rangle\langle 2456\rangle\langle 5678\rangle}{\langle 1256\rangle\langle 2578\rangle\left(\langle 1237\rangle\langle 4568\rangle-\langle 1238\rangle\langle 4567\rangle\right)}$$

Is there any pattern?

- Why do only some algebraic functions of Mandelstam variables appear, and not others?
- Why do only "simple" cross-ratios appear, as opposed to (in principle) arbitrary algebraic functions of cross-ratios?
- Why, in the $n=6$ result, do only 9 out of 45 possible cross-ratios of the form $\langle\ \rangle\langle\ \rangle/\langle\ \rangle\langle\ \rangle$ appear?

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e.g. the Dixon-Drummond-Henn computation of the 3-loop of the 3-loop $n=6$ MHV amplitude

[Note to experts: constraints on the set of arguments of a function are much stronger than constraints on the symbol entries of a function.]

Cluster Coordinates

There are special collections of coordinates on $\mathrm{Conf}_n(\mathbb{P}^3)$ which are well-known in the mathematics literature, known as $\mathcal{X}$-coordinates (discovered by Fock & Goncharov) which have started creeping up in various physics recently.

Cluster Coordinates

- Each cluster has $3(n-5)$ cluster coordinates

$x_1''', x_2''', \ldots$

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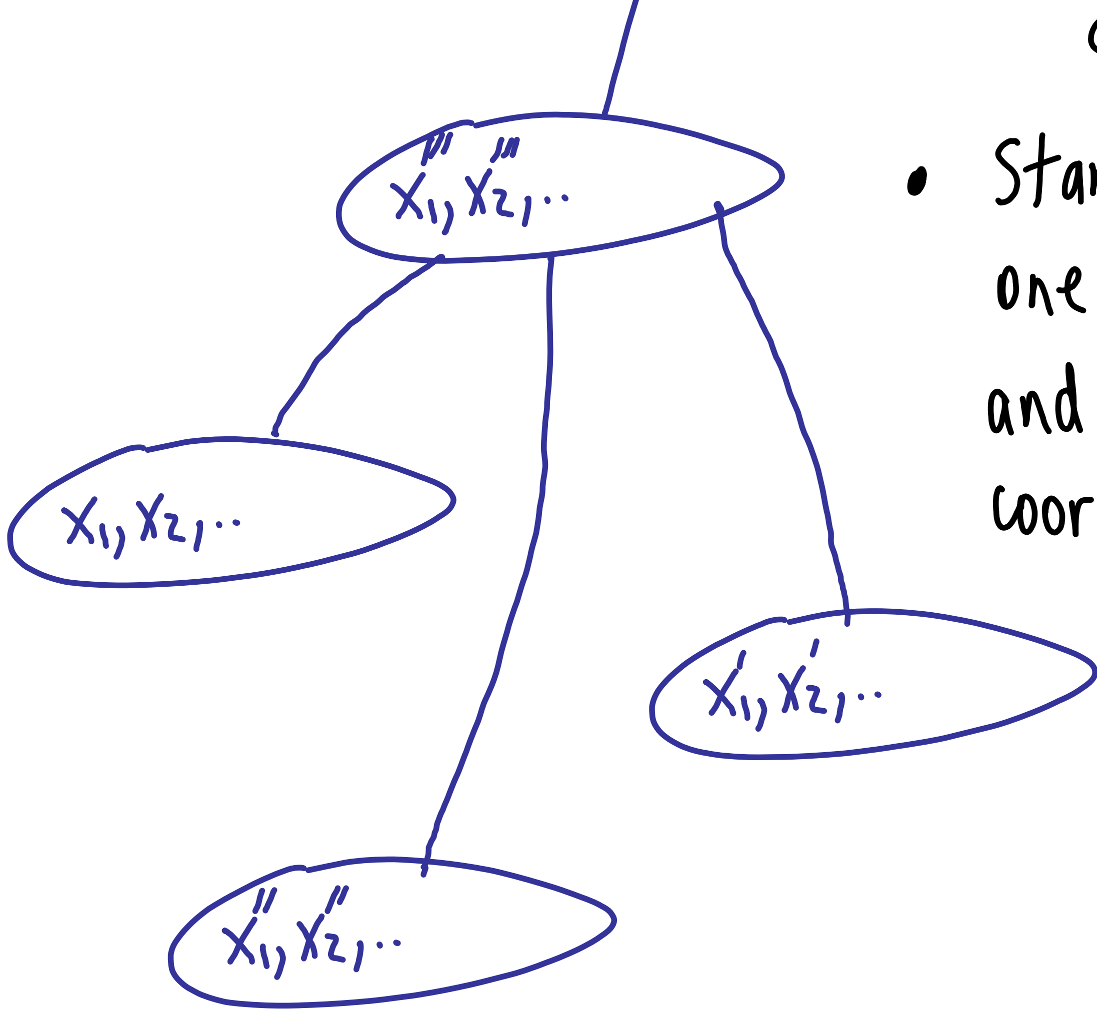
Cluster Coordinates



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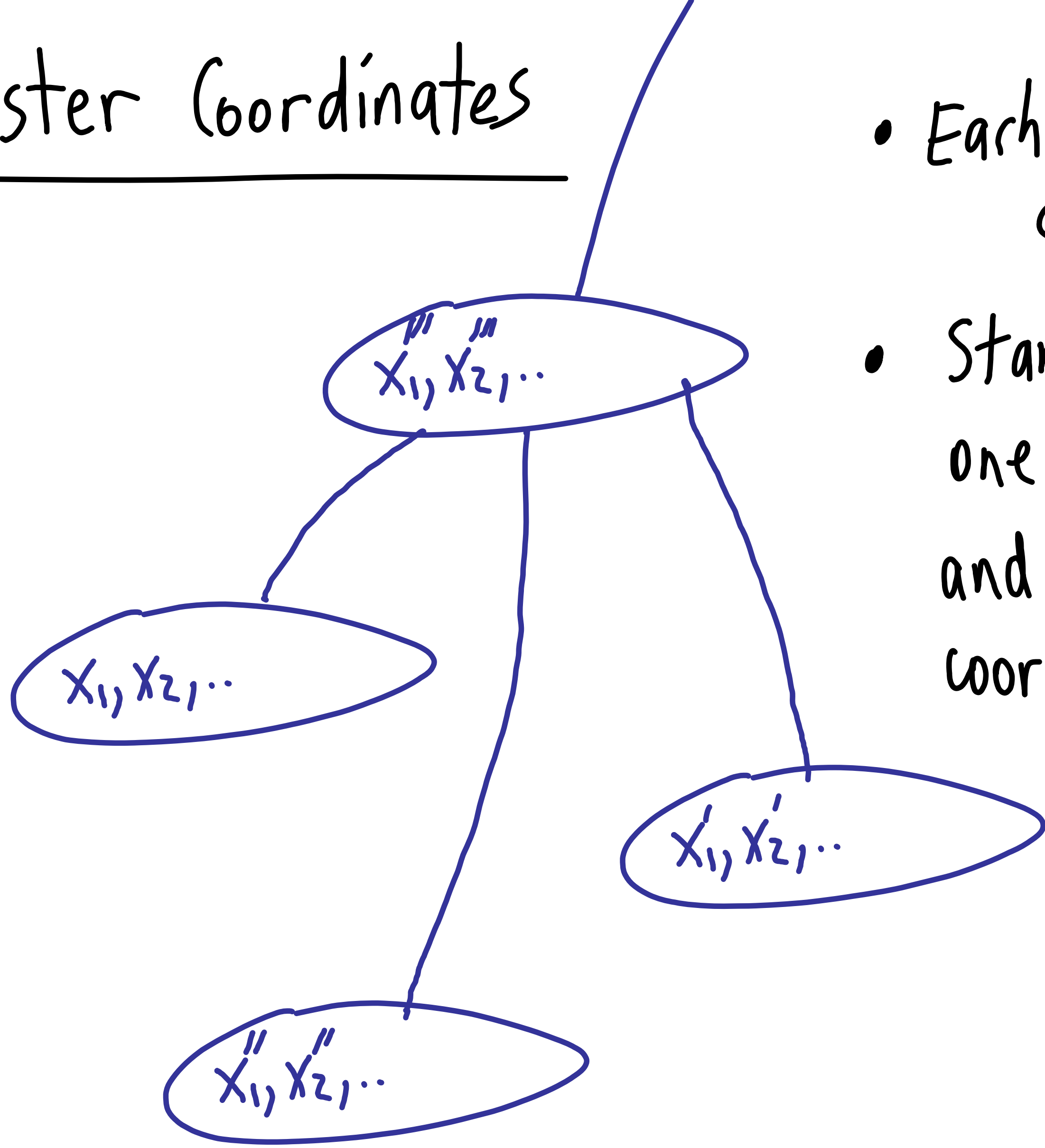

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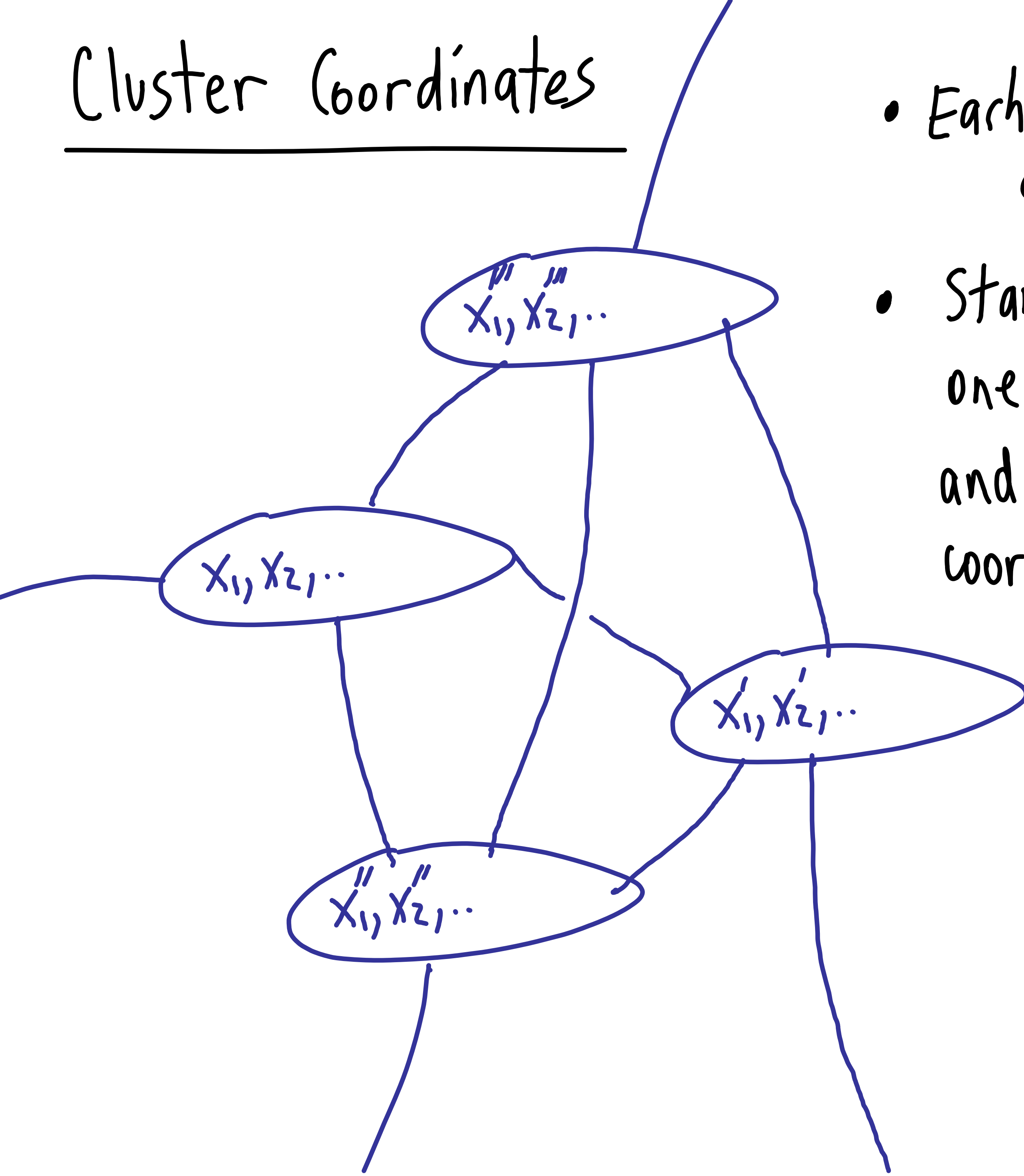

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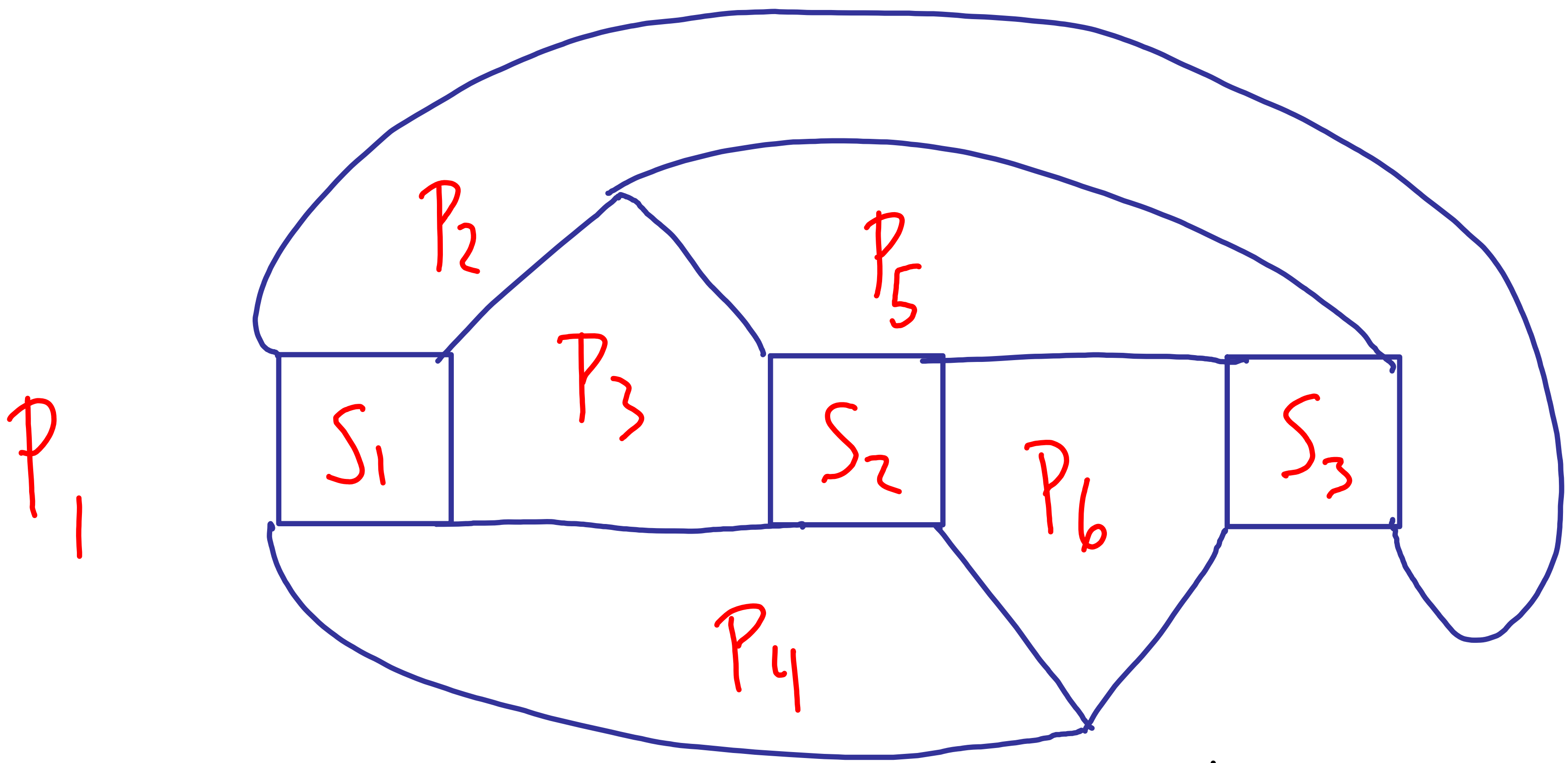
- Starting from some cluster, one can pick any coordinate and "mutate on" that coordinate to generate new clusters
- the coordinates in each cluster are log-canonical
  $\{\log x_i, \log x_j\} = \pm 1$ or 0
- the graph of clusters is called the Stasheff polytope

The Stasheff polytope for $n=6$



There are 14 clusters, with 3 coordinates each, but only 15 distinct coordinates in total.

Faces of the polytope: 3 squares, 6 pentagons

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  (e.g. the 5-term functional equation for $\mathrm{Li}_2 \Longleftrightarrow$ pentagons)
- and, apparently, the structure of SYM amplitudes!

Results

For the two-loop MHV amplitudes, for <u>all</u> n,

we find that

- they only "depend on" $\mathcal{X}$-coordinates (i.e. only $\mathcal{X}$-coordinates appear in their coproduct)
- the classical polylog obstruction ($\Delta_{2,2}$) is naturally measured by square faces of the associahedron

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Of course, having such a principle would greatly help new computations!

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Perhaps cluster algebras will appear on this list.