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MICROECONOMETRIC DEMAND SYSTEMS WITH BINDING NONNEGATIVITY CONSTRAINTS: THE DUAL APPROACH

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This paper considers the problem of specifying and estimating demand systems for samples which contain a significant proportion of observations with zero consumption of one or more goods. Our approach uses virtual prices, which are dual to the Kuhn-Tucker conditions, to select the set of goods consumed—the demand regime—and to transform binding nonnegativity constraints into nonbinding constraints. It has the advantage of permitting the use of indirect cost and utility functions such as the translog, and the analytic decomposition of demand effects for goods at the nonnegativity limit.

KEYWORDS: Demand system, nonnegativity constraints, limited dependent variables.

1. INTRODUCTION

RECENTLY, Wales and Woodland (1983) have considered the problem of estimating consumer demand systems for samples which contain a significant proportion of observations with zero consumption of one or more goods. Their econometric model is derived by maximizing a random direct utility function subject to budget and nonnegativity constraints. The Kuhn-Tucker conditions determine the set of nonconsumed goods. In this paper, we propose an alternative approach to the zero corner solution problem based upon the use of virtual prices. This approach, which is dual to that of Wales and Woodland, allows the use of indirect utility and cost functions such as the translog.

2. THE FRAMEWORK

Let $H(v;\, \theta, \varepsilon)$ be an indirect utility function defined as

$$(1) \qquad H(v;\, \theta, \varepsilon) = \max_q \{U(q;\, \theta, \varepsilon) \mid vq = 1\}$$

where $U(\ ;\, \theta, \varepsilon)$ is a strictly quasi-concave utility function defined on K commodities, v is a vector of normalized market prices, θ a vector of unknown parameters, and ε a vector of random components. Applying Roy's Identity, the notional demand equations $D(v;\, \theta, \varepsilon)$ for a set of K goods are

$$(2) \qquad q_i = \frac{\partial H(v;\, \theta, \varepsilon)}{\partial v_i} \Bigg/ \sum_{j=1}^{K} v_j \frac{\partial H(v;\, \theta, \varepsilon)}{\partial v_j} \qquad (i = 1, \ldots, K).$$

These demand equations are deemed notional because they may take negative values since the problem (1) does not include nonnegativity constraints. The notional demands q_i are thus latent variables which correspond to a vector of nonnegative observed demands (x_i) as follows. There exist vectors of positive virtual prices π which can exactly support these zero demands (or any other

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allocation) as long as the preference function is strictly quasi-concave, continuous, and strictly monotonic (Neary and Roberts (1980)). Virtual prices have heretofore been used mostly in analyzing the effects of rationing on consumer behavior. Although analytically deriving virtual price functions when demands are rationed has been shown to be difficult for many popular functional forms (Deaton and Muellbauer (1981)), the problem is enormously simplified when the "ration" is zero, in which case the denominator in Roy's Identity (2) drops out of the virtual price function. If demands for the first l goods are zero, the virtual prices $\pi_i(v_{l+1}, \ldots, v_K)$ are solved from the equations

$$0 = \partial H(\pi_1(\bar{v}), \ldots, \pi_l(\bar{v}), \bar{v}; \theta, \varepsilon)/\partial v_i \qquad (i = 1, \ldots, l) \tag{3}$$

where $\pi_i(\bar{v})$ is the virtual price of the ith good and $\bar{v}$ is the set of market prices of the positively consumed goods $l+1$ to K. The market prices $\bar{v}$ are also virtual prices as they exactly support the observed positive demands of goods $l+1$ to K. The remaining (positive) demands are

$$x_i = \frac{\partial H(\pi_1(\bar{v}), \ldots, \pi_l(\bar{v}), \bar{v}; \theta, \varepsilon)}{\partial v_i} \Bigg/ \sum_{j=1}^{K} v_j \frac{\partial H(\pi_1(\bar{v}), \ldots, \pi_l(\bar{v}), \bar{v}; \theta, \varepsilon)}{\partial v_j} \qquad (i = l+1, \ldots, K). \tag{4}$$

The equations (4) are estimable and the parameters of the notional demand equations (2) can be identified by estimating this conditional demand system.[2]

Comparisons of virtual and market prices can select among demand regimes, defined as the set of positively consumed goods at the optimum. The regime in which the first l goods are not consumed is characterized by the conditions

$$\pi_i(\bar{v}) \leqslant v_i \qquad (i = 1, \ldots, l). \tag{5}$$

This characterization follows directly from the Kuhn-Tucker conditions and the concept of virtual prices. The Lagrangean function with utility maximization subject to nonnegativity constraints is

$$L = U(q) + \lambda(1 - vq) + \psi q$$

where λ and ψ are Lagrange multipliers and the parameters θ and ε are suppressed for notational simplicity. The Kuhn-Tucker conditions that characterize the demands (x_i) with $x_i = 0$ $(i = 1, \ldots, l)$ and $x_i > 0$ $(i = l+1, \ldots, K)$ are

$$\begin{aligned} &\frac{\partial U(x)}{\partial q_i} - \lambda v_i + \psi_i = 0, \quad \psi_i \geqslant 0 && (i = 1, \ldots, l), \\ &\frac{\partial U(x)}{\partial q_j} - \lambda v_j = 0 && (j = l+1, \ldots, m), \\ &\sum_{j=l+1}^{m} v_j x_j = 1, \qquad \lambda > 0. \end{aligned} \tag{6}$$

[2] Browning (1983) has shown that the unconditional cost function can theoretically be recovered from a conditional cost function. The necessary conditions for the conditional cost function are also sufficient for the recovery of the unconditional cost function when the rationed quantities are positive. Our approach starts with the unconditional functions. Identification in this paper refers to parameter identification given functional forms for the unconditional functions.

The virtual price for good i $(i=1,\ldots,l)$ at x is simply

$$\pi_i(\bar{v})=\frac{\partial U(x)}{\partial q_i}\bigg/\lambda$$
$$=v_m\frac{\partial U(x)}{\partial q_i}\bigg/\frac{\partial U(x)}{\partial q_m}.$$

Hence the Kuhn-Tucker conditions are equivalent to the conditions (5) and $x_i>0$ $(i=l+1,\ldots,m)$. The regime switching conditions (5) are intuitively appealing. Virtual prices can be thought of as reservation or shadow prices. Goods are not consumed unless their reservation price exceeds their market price.

3. APPLICATION TO THE TRANSLOG INDIRECT UTILITY FUNCTION

An advantage of this dual approach over the primal approach of Wales and Woodland is that it is possible to explicitly state demand equations for each possible demand regime. This is particularly useful if one is interested in studying the effects of nonconsumed goods—the demand regime—on the demand for consumed goods, or how switching occurs in response to changes in prices, income, or household characteristics. For example, consider the effects of a change in the price of consumed good i on its own demand:

$$\frac{dx_i}{dv_i}=\frac{\partial D_i(\bar{\pi},\bar{v};\theta,\varepsilon)}{\partial v_i}+\sum_{j=1}^{l}\frac{\partial\pi_j(\bar{v};\theta,\varepsilon)}{\partial v_i}\frac{\partial D_i(\bar{\pi},\bar{v};\theta,\varepsilon)}{\partial\pi_j},$$

where $\bar{\pi}$ is the vector of virtual prices of nonconsumed goods. Note that each nonconsumed good adds a term to adjust the derivative of the unconditional demand equation for its absence from the demand regime. These terms, which are readily decomposable analytically, allow one to determine the levels of prices, income, or household characteristics which just induce the consumption of goods of interest. In addition it is easier to specify demand, cost, or indirect utility functions than direct utility functions. In particular, our approach allows for the use of certain popular flexible functional forms. The advantages of the dual approach for the analysis of quantity rationing have been emphasized by Deaton (1981).

To illustrate this approach, consider the translog indirect utility function of Christensen, Jorgenson, and Lau (1975):

$$H(v;\theta,\varepsilon)=\sum_{i=1}^{K}\alpha_i\ln v_i+\tfrac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K}\beta_{ij}\ln v_i\ln v_j+\sum_{i=1}^{K}\varepsilon_i\ln v_i, \tag{7}$$

where ε is a K-dimensional vector of normal variables $N(0,\Sigma)$.[3] A convenient normalization is $\sum_{i=1}^{K}\alpha_i=-1$ and $\sum_{i=1}^{K}\varepsilon_i=0$.[4] The notional share equations

[3] One can also specify other distributions if they are of interest. Normality is attractive because of its additive property.

[4] It is necessary to specify $\sum_{i=1}^{K}\varepsilon_i=0$, since, for the homogenous case, $\sum_{i=1}^{K}\beta_{ij}=0$ so that $D=-1$ in the share equations (8) and the sum of the shares is unity.

derived from Roy's Identity are

$$(8) \qquad v_i q_i = \frac{\alpha_i + \sum_{j=1}^{K} \beta_{ij} \ln v_j + \varepsilon_i}{D} \qquad (i = 1, \ldots K),$$

where $D = -1 + \sum_{i=1}^{K} \sum_{j=1}^{K} \beta_{ij} \ln v_j$. For the regime for which the quantity demanded for one of the goods is zero and positive for all others, i.e., $x_1 = 0$, $x_2 > 0, \ldots, x_K > 0$, the virtual price π_1 as a function of $v_2, \ldots, v_K$, is

$$\ln \pi_1 = -\left(\alpha_1 + \sum_{j=2}^{K} \beta_{1j} \ln v_j + \varepsilon_1\right) \Big/ \beta_{11}.$$

The remaining positive share equations are

$$(9) \qquad v_i x_i = \frac{\alpha_i - \alpha_1 \dfrac{\beta_{i1}}{\beta_{11}} + \sum_{j=2}^{K}\left(\beta_{ij} - \beta_{1j}\dfrac{\beta_{i1}}{\beta_{11}}\right) \ln v_j + \varepsilon_i - \dfrac{\beta_{i1}}{\beta_{11}} \varepsilon_1}{\sum_{j=2}^{K}\left(\beta_{\cdot j} - \beta_{\cdot 1}\dfrac{\beta_{1j}}{\beta_{11}}\right) \ln v_j - \left(1 + \dfrac{\alpha_1}{\beta_{11}} \beta_{\cdot 1}\right) - \dfrac{\beta_{\cdot 1}}{\beta_{11}} \varepsilon_1} \qquad (i = 2, \ldots, K),$$

where $\beta_{\cdot j} = \sum_{i=1}^{K} \beta_{ij}$. Note from the above equations that ε_i can be expressed as functions of x_i and ε_1. The switching conditions for this demand regime are

$$\varepsilon_1 \geqslant -\left(\alpha_1 + \sum_{j=1}^{K} \beta_{1j} \ln v_j\right)$$

and $x_i > 0$ $(i = 2, \ldots, K)$.

Let $f(\varepsilon_1)$ be the density function of ε_1 and $g(\varepsilon_2, \ldots, \varepsilon_{K-1} | \varepsilon_1)$ be the conditional density function, conditional on ε_1. The Jacobian transformation $J_1(x, \varepsilon_1)$ from $(\varepsilon_2, \ldots, \varepsilon_{K-1})$ to $(x_2, \ldots, x_{K-1})$, which can be derived from (9), is a function of x and ε_1. The likelihood function for this demand regime for one observation is

$$\int_{-(\alpha_1 + \sum_{j=1}^{K} \beta_{1j} \ln v_j)}^{\infty} J_1(x, \varepsilon_1) g(\varepsilon_2, \ldots, \varepsilon_{K-1} | \varepsilon_1) f(\varepsilon_1) \, d\,\varepsilon_1$$

where ε_i $(i = 2, \ldots, K-1)$, are functions of x and ε_1 from (9). For the demand regime in which the demands for the first two commodities are zero and all remaining demands are positive, the virtual prices π_1 and π_2, functions of $v_3, \ldots, v_K$, can simply be obtained by matrix inversion as

$$\begin{pmatrix} \ln \pi_1 \\ \ln \pi_2 \end{pmatrix} = -B^{-1} \begin{pmatrix} \alpha_1 + \sum_{j=3}^{K} \beta_{1j} \ln v_j \\ \alpha_2 + \sum_{j=3}^{K} \beta_{2j} \ln v_j \end{pmatrix} - B^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

where

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}.$$

The remaining positive shares are

$$(10) \qquad v_i x_i = \frac{\alpha_i + \beta_{i1} \ln \pi_1 + \beta_{i2} \ln \pi_2 + \sum_{j=3}^{K} \beta_{ij} \ln v_j + \varepsilon_i}{-1 + \beta_{\cdot 1} \ln \pi_1 + \beta_{\cdot 2} \ln \pi_2 + \sum_{j=3}^{K} \beta_{\cdot j} \ln v_j} \qquad (i = 3, \ldots, K).$$

The $\varepsilon_i (i=3,\ldots,K)$, can be expressed (from (10)) as functions of x, ε, and ε_2. The regime switching conditions are

$$B^{-1}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix} \geqslant -\begin{pmatrix}\ln v_1\\ \ln v_2\end{pmatrix} - B^{-1}\begin{pmatrix}\alpha_1+\sum_{j=3}^{K}\beta_{1j}\ln v_j\\ \alpha_2+\sum_{j=3}^{K}\beta_{2j}\ln v_j\end{pmatrix}.$$

Let

$$\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = B^{-1}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}.$$

Furthermore, let $g(\varepsilon_3,\ldots,\varepsilon_{K-1}|\,\eta_1,\eta_2)$ be the conditional density function of $(\varepsilon_3,\ldots,\varepsilon_{K-1})$, conditional on η_1 and η_2, and $f(\eta_1,\eta_2)$ be the marginal density of η_1 and η_2. The Jacobian transformation $J_2(x,\eta_1,\eta_2)$ from $(\varepsilon_3,\ldots,\varepsilon_{K-1})$ to $(x_3,\ldots,x_{K-1})$ can be derived from (10) and is a function of x and η_1, η_2. The likelihood function for this regime for one observation is

$$\int_{s_2}^{\infty}\int_{s_1}^{\infty} J_2(x,\eta_1,\eta_2)g(\varepsilon_3,\ldots,\varepsilon_{K-1}|\,\eta_1,\eta_2)f(\eta_1,\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2$$

where

$$\begin{pmatrix}s_1\\ s_2\end{pmatrix} = -\begin{pmatrix}\ln v_1\\ \ln v_2\end{pmatrix} - B^{-1}\begin{pmatrix}\alpha_1+\sum_{j=3}^{K}\beta_{1j}\ln v_j\\ \alpha_2+\sum_{j=3}^{K}\beta_{2j}\ln v_j\end{pmatrix}$$

and ε's are functions of x and η_1, η_2 derived from (10). The likelihood function for other regimes can similarly be derived.

Let $I_i(c)$ be a dichotomous indicator such that $I_i(c)=1$ if the observed consumption pattern for individual i is the demand regime c, zero otherwise. Let $l_c(x_i;\theta)$ denote the likelihood function for regime c for sample i. The likelihood function for an independent sample with N observations is

$$L=\prod_{i=1}^{N}\prod_c[l_c(x_i;\theta)]^{I_i(c)}.$$

The method of maximum likelihood can be applied to the estimation of this model.

4. SOME EXTENSIONS AND PROBLEMS

The analysis above has focused on the nonnegativity constraints of the consumer's problem, but can readily be extended to a wide class of problems involving kink points in the choice set of consumers or producers. Binding nonnegativity constraints are just a special case of kink points on the boundary choice set. Kink points that arise from quantity rationing or increasing block pricing can be analyzed within the same framework (Lee and Pitt (1984), Deaton (1981), Burtless and Hausman (1978)).

The empirical implementation of this approach however is troubled by the computational complexity of maximizing the likelihood function. For example, in the case of the translog demand system discussed above, estimation would require numerical integration involving multiple probability distributions. The problem is somewhat simpler for the case of production. With a translog (or other) cost function, the linearity of the derived demand equations allows for additive and normal errors. Estimation of a translog cost function with three inputs has been accomplished in Lee and Pitt (1984). Evaluation of multiple integrals even in the normal case has currently been accomplished only for small numbers of goods. Computation with many commodities is as difficult as the multivariate probit model. However, improved computational algorithms, supercomputers, and the possibility of devising functional forms for the stochastic model with computational ease as a primary attribute, suggest that these methods may be available for applied research in demand analysis in the future.

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