the dispersion relation that they obtain applies to free waves in a half-space. The relevant parameter in this case is the mean free path between intermolecular collisions $\Lambda = c_m/f_c$, c_m being the most probable speed of the molecules. For their analysis to be meaningful, the sound wave must propagate over several of these mean free paths.

In practice, experiments involve a vibrating surface (transmitter) and a receiver separated by a finite distance, say d. This separation distance is ultimately limited by either the size of the vacuum chamber or the distance over which a measurable signal can be propagated. In the pressure range such that $f/f_c \ll 1$, there is no difficulty in measuring propagation over many mean intermolecular free paths. Data obtained in this manner can be used to test Sirovich and Thurber's theoretical predictions in this range. However, as the pressure is reduced (or the sound frequency increased) such that f/f_c becomes greater than unity, it is practically impossible, with present-day techniques, to measure sound propagation over several mean intermolecular free paths. All known measurements[2–4] in this range of f/f_c have been carried out in the régime where the mean free paths of the molecules are limited by the separation d. Indeed, Meyer and Sessler,[3] whose data are used by Sirovich and Thurber, make a strong point of this in constructing their Fig. 13. Greenspan, whose data are also used by the authors, has informed us that data relating to propagation over distances less than the gas mean free path were purposely excluded from his 1956 paper.[2] Data in this range were discussed and appropriately interpreted by Greenspan and coworkers in earlier presentations.[5,6]

Sherman and Talbot[7] and Maidanik, Fox, and Heckl[8] also discuss this problem; one may define a new frequency $f_s = c_m/d$ and consider the dispersion parameters in this region as functions of f/f_s. That the experimental data produce attenuation and phase parameters that are constants with respect to changes in Λ when $\Lambda > d$ is thus not relevant in testing theories that by definition required propagation of sound over distances of at least several intermolecular mean free paths. Thus, the agreement that is obtained by Sirovich and Thurber in the régime defined by $f/f_s \geq 1$ must be viewed as fortuitous and their theory in this régime remains as yet unverified.

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Comparison of Theory and Experiment for Forced Sound-Wave Propagation in Rarefied Gasdynamics: Reply to "Comments on 'Propagation of Forced Sound Waves in Rarefied Gasdynamics'" [G. Maidanik and H. L. Fox, J. Acoust. Soc. Am. 38, 477–478(L) (1965)]

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The comparison of theory with experiment for rarefied-gas sound propagation is discussed. It is shown that the salient comments of Maidanik and Fox [J. Acoust. Soc. Am. 38, 477–478(L) (1965)] in connection with our paper [J. Acoust. Soc. Am. 37, 329–339 (1965)] are either incorrect or unfounded. It is shown by means of estimates on the percentage of intermolecular collisions occurring in present-day high-Knudsen-number experiments that the plane-wave propagation described in our paper is still a significant effect.

THE COMMENTS OF MAIDANIK AND FOX[1] RAISE SEVERAL INTERESTING questions in connection with the comparison of theory and experiment for forced sound-wave propagation in a rarefied medium. Before entering into a detailed discussion of these points, we first clarify two items brought up by Maidanik and Fox.

- In reference to our work,[1] they state that "the theory does not describe the experimental situation." This is incorrect; the theory presented in Eq. 1 applies only to sound-waves (which we are careful to define as certain plane-wave solutions). As such, they are independent of geometry and are only a property of the equations. The pressure field (which presumably is measured in an experiment), on the other hand, does depend on geometry as well as the other modes of propagation. If in an experiment the pressure field departs from the sound field, it states simply that sound is not the major contribution. As yet, no experiment has this for a conclusion.
- It is stated in Ref. 1 that our "theory is [only] appropriate to free waves,[3] propagating a distance that is large compared with the mean-free-path . . ." Our previous comment refutes this. However, by this remark Maidanik and Fox may mean that the sound-pressure field is dominant only at large distances (with respect to the mean free path) from the sound source. This too is without foundation. There are no analytical results to support this statement, nor do the experiments support it (if anything, they indicate the contrary).

Before amplifying these, as well as other points raised, we briefly review the problem at hand.

For the analytical treatment of a sound-propagation problem, it is convenient to regard a geometry in which the gas is restricted to a half-space and which is driven by a sinusoidally oscillating infinite plane. One then sees[2] that the solution of this problem depends on the dimensionless parameter $r = p/\omega\mu$ (p the pressure, ω the frequency, and μ the viscosity) introduced by Greenspan.[4] In particular, the sound characteristics, i.e., the dimensionless attenuation rate and sound speed, are functions of r only.[5] The related experimental geometry consists of an emitter and receiver of sound waves separated by a distance d, say. A comparable theoretical problem is the study of a gas contained between two infinite parallel planes separated by a distance d, one plane of which is sinusoidally oscillating. A solution to the latter problem depends on the ratio of the mean free path l to d—i.e., the Knudsen number $\mathrm{Kn} = l/d$ (Ref. 6), as well as the frequency ratio r. However the sound characteristics still remain functions of r alone. However, one measures pressure (more exactly, normal stress) at the receiving wall, and hence this measurement can depend on Kn. For small values of Kn, we can expect this dependence to be lost. On the other hand, for very large values of Kn it might be supposed that the mean-free-path dependence is lost (i.e., μ is no longer a parameter of the problem). Under this assumption, only the dimensionless parameter $R_d = d/(2RT)^{\frac{1}{2}}$ [$(2RT)^{\frac{1}{2}}$ a mean molecular speed] enters. The comments of Maidanik and Fox[2] pertain to these points and, in particular, question the comparison of our results for the sound characteristics with experiment. This is certainly of importance in as much as the experiments of Maidanik and of Heckl,[7] Meyer and Sessler,[8] and Greenspan[4] are, for small values of r, in the Kn>1 range. (Greenspan, however, did not plot his Kn>1 results.)

Basing the mean free path on the hard-sphere definition for mean free path,

$$l = 2.5/\rho(RT)^{\frac{1}{2}}\mu \tag{1}$$

one can show that the Knudsen number is given by,

$$\mathrm{Kn} = 1.8/R_d r. \tag{2}$$

From the data furnished by Meyer and Sessler,[8] we have computed their value for R_d as being $\cong 8.25$. (This is based on a value of $\omega d = 3\times 10^5$ cm/sec for argon, which they give.) This leads to a $\mathrm{Kn} \cong 15$. (In the work of Ref. 7, Kn was as high as 30.)

In an attempt to depict the Kn>1 range, Maidanik, Fox, and Heckl[9] consider the free-flow equation for the distribution function

$$(\partial f/\partial t) + \xi(\partial f/\partial x) = 0. \tag{3}$$

(Earlier treatments using the same type of approach were given by Meyer and Sessler[8] and Greenspan *et al.*[10]) It is clearly seen that for Eq. 3 the only plane-wave solutions now are particle paths. In the above technique, for the purpose of making an analogy with sound waves, the logarithmic derivative of p the pressure is taken

$$-(1/\beta_0 p)(\partial p/\partial x)=(\alpha-i\beta)/\beta_0, \tag{4}$$

(where β_0 is the value of β at adiabatic speed) and the real and imaginary parts of Eq. 4 are identified with the dimensionless attenuation rate and speed of sound, respectively. These quantities are functions of $R_d=\omega d/(2RT)^{\frac{1}{2}}$. One then shows that for R_d large (which is usually the case in an experiment)

$$\alpha-i\beta/\beta_0\sim\left(\frac{5}{24}\right)^{\frac{1}{2}}\left(\frac{2}{R_d}\right)^{\frac{1}{2}}(1-i\sqrt{3}). \tag{5}$$

(Maidanik, Fox, and Heckl[1,7,9] suggest that Eq. 5. depicts $\mathrm{Kn}>1$ region more appropriately than do our results.[2]) More-precise results based on a numerical integration are given in Fig. 3 of Ref. 9. Using the value of $R_d=8.25$ for the Meyer and Sessler experiment, we find using the Maidanik, Fox, Heckl[9] results that

$$\text{Meyer and Sessler}\begin{cases}\dfrac{\alpha}{\beta_0}\simeq 0.45.\\[2mm] \dfrac{\beta}{\beta_0}\simeq 0.2\end{cases} \tag{6}$$

[The value of β/β_0 stays roughly constant for $3\leq R_d\leq 13$.] These values are constants for an experimental run and one would expect to realize such values for relatively low values of r in any run. The value of α/β_0 is in good agreement with the Meyer and Sessler results for small r (Ref. 11). The value of $\beta/\beta_0\simeq 0.2$, on the other hand, is lower than any experimental value found in the noble-gas experiments of Greenspan[4] and Meyer and Sessler.[8] It is also roughly 20% below the value found by our analysis (in the lower range), which does go through the experimental points.

In the experiments of Maidanik and Heckl,[7] values of $\beta/\beta_0=0.2$ (and below) were found. However, their experiments were carried out in air. The air experiments of Greenspan[12] and Meyer and Sessler[8] also show such low values for the attenuation constant. Although we can expect high-frequency sound propagation to behave qualitatively, as it would for a monatomic gas, the very fact that it leads to experimental values differing from those produced in like noble-gas situations disqualifies it from comparison with the theory under discussion. Both the free-flow calculations of Maidanik, Fox, and Heckl[9] and the theory presented by us apply to a simple monatomic gas. Air, being a mixture of gases with internal degrees of freedom, has a succession of mean free paths associated with it. These naturally complicate the transition régime.

We have made several calculations that we believe explain, at least partially, the abovementioned failure of the free-flow calculation in régimes for which the Knudsen number is relatively high. We consider the problem of determining the percentage of particles that leave a wall with a Maxwellian distribution and that undergo a collision within a distance d of the emitting wall. For the sake of brevity, we outline only the method and give the results. We denote the collision frequency of the gas by ν. This is, in general, taken to be velocity-dependent,

$$\nu=\nu(|\xi|), \tag{7}$$

where $|\xi|$ is the molecule speed. For a particle of speed $|\xi|$, we assume that it undergoes a collision in a distance $|\xi|/\nu$. For Maxwell molecules, ν is a constant and the fraction f of particles undergoing a collision in a distance d is given by

$$f\cong\mathrm{Erf}[(9/4)(d/l)]. \tag{8}$$

Therefore, even if the Knudsen number $\mathrm{Kn}=l/d$ is as great as 10, 24% of the particles experience a collision in traveling to d. One must exceed a Knudsen number of 20 to achieve a value of f as small as $\frac{1}{10}$. At the other extreme of collision-frequency functions is the case of rigid spheres, for which we took

$$\nu\cong 4[(RT)/\pi]^{\frac{1}{2}}[1+(|\xi|/4)[\pi/(2RT)]^{\frac{1}{2}}]. \tag{9}$$

This is a patched collision frequency, which agrees with the correct hard-sphere collision frequency at high and low molecular speeds. In this case, we find that

$$f=\frac{d}{\sqrt{2}l}+\frac{8d}{\sqrt{2\pi}l}e^{-a^2}+\left(1-\frac{d}{l\sqrt{2}}\right)\left[\frac{-2a}{\sqrt{\pi}}e^{-a^2}+\mathrm{Erf}(a)\right], \tag{10}$$

where

$$a=(4/\sqrt{\pi})\left[\left(\sqrt{2}\frac{l}{d}-1\right)^{-1}\right]. \tag{11}$$

The numerical values for f are essentially the same as given above in the Maxwell molecule case. We mention in passing that Eq. 10 predicts that all hard spheres undergo a collision in a distance $d=\sqrt{2}l$. This is in contrast to the Maxwell molecule case where $d=\infty$ for the same result.

These calculations point out clearly that the size of the mean free paths should not be used cavalierly as a basis upon which to judge a flow as being collisionless. For as we have seen even at substantially high Knudsen numbers, a relatively large number of particles undergo collisions. This certainly offers an explanation of the failure of Eq. 5 to describe the experimental values. The large number of collisions therefore gives a possible explanation of why our forced-sound-wave analysis has such good agreement with experiment. In future sound experiments, it certainly would be of great value to vary R_d while holding r fixed. This would then give us a clear picture of the Knudsen effect. Some indication of this is already to be seen in the experiments of Maidanik and Heckl,[7] where two values of R_d are used.

It is of some interest to explain why the free-flow calculation Eq. 5 and the calculation based on our approximate Boltzmann equations produce qualitatively similar results. Some reflection on the mechanism of sound propagation indicates the reason for the increase of sound speed with increase in frequency. Slow-moving molecules, having a relatively small collision frequency, are unable to transmit a relatively high-frequency signal. Hence only the fastest-moving molecules, having a relatively high collision frequency, transmit the signal. In the case of free flow, the same effect takes place, since the slow-moving molecules undergo phase mixing. Loosely speaking, the effect of phase mixing in free flow mimics the collision process in a gas. From the experimental view, it is of course unfortunate that this occurs since the two effects become difficult to distinguish.

To conclude, we wish to point out that our study[2] makes no claim at furnishing a solution of a flow problem. It provides just the sound characteristics of a gas. A solution to the flow problem appropriate to the experimental situation must, especially in view of our above estimates, be based on a solution of the Boltzmann equations (or some model of it). Some progress in this connection has been made recently,[13–15] but no results appropriate to the experimental situation are yet available. We believe that, in view of our close agreement with experiment and the above estimates on the number of collisions in the Knudsen region, that such studies will show that plane-wave propagation is a significant effect even at the relatively high Knudsen numbers thus far achieved in experiment.

[1] G. Maidanik and H. L. Fox, J. Acoust. Soc. Am. **38**, 477–478(L) (1965).
[2] L. Sirovich and J. K. Thurber, J. Acoust. Soc. Am. **37**, 329–339 (1965).
[3] A minor issue is the question of semantics. We use free waves to describe the waves occurring in an initial-value problem. See L. Sirovich and J. K. Thurber, in *Rarefied Gas Dynamics*, J. H. deLeeuw, Ed. [Academic Press Inc., New York, 1965 (to be published)]. Forced waves are reserved for the waves occurring in a boundary-value problem.
[4] M. Greenspan, J. Acoust. Soc. Am. **22**, 568–571 (1950); **28**, 644–648 (1956).
[5] Actually, the intermolecular-force law enters but the results are relatively insensitive to this (see Eq. 1). Interpreted on a macroscopic level, this states that all monatomic gases have a Prandtl number very close to $\frac{2}{3}$.
[6] Not to be confused with the Knudsen number based on wavelength ($\simeq 1/r$) in Ref. 1.

[7] G. Maidanik and M. Heckl, Phys. Fluids 8, 266–272 (1965). From the data furnished by Maidanik and Heckl, we computed that the width of the receiver (1 in.) is roughly the same magnitude as d. This may possibly introduce 3-dimensional effects.
[8] E. Meyer and G. Sessler, Z. Physik 149, 15–39 (1957).
[9] G. Maidanik, H. Fox, and M. Heckl, Phys. Fluids 8, 259–265 (1965).
[10] M. Greenspan and M. C. Thompson, Jr., J. Acoust. Soc. Am. 25, 92 (1953); R. K. Cook, M. Greenspan, and M. C. Thompson, Jr., *ibid.* 25, 192(A) (1953).
[11] Our rigid-sphere curve runs through this point when $r \simeq 0.02$.
[12] M. Greenspan, J. Acoust. Soc. Am. 31, 155–160 (1959).
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Nonlinear Interaction of Two Sound Beams

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Extension of previous contributions by Westervelt [J. Acoust. Soc. Am. 32, 934(A) (1960); 35, 535–537 (1963)] and by Bellin and Beyer [J. Acoust. Soc. Am. 34, 1051–1053 (1962)] concerning the nonlinear interaction of two coincident sound beams to allow for cylindrical and spherical spreading is discussed; conclusions of Naze and Tjøtta [J. Acoust. Soc. Am. 37, 174–175(L) (1965)] are reviewed and commented upon.

In their recent communication,[1] Naze and Tjøtta discuss the effects of the finite aperture formed by a cophasal cross section of the virtual endfire array on the problem of scattering of sound by sound as originally studied by Westervelt[2,3] and Bellin and Beyer.[4] A similar approach has been developed quite independently, but with experimental support, in the Department of Electronic and Electrical Engineering at The University of Birmingham. The experimental results obtained show good agreement with theory.[5]

Naze and Tjøtta also extend the theory to the case where interaction between the primary beams occurs in the Fraunhofer zones—i.e., when the primary beams are spreading spherically, with directivity patterns of the form $2J_1(x)/x$. The result, which is expressed in a closed form, gives an expression for the directivity of the scattered waves in terms of the quantity $|\rho_s(R,\xi)/\rho_s(R,0)|$, where $\rho_s(R,\xi)$ is the density due to the scattered waves at a point (R,ξ) in the far field, while $\rho_s(R,0)$ is that at a point along the axis of symmetry of the configuration; $\rho_s(R,0)$ is said to have the same value as that predicted for the case of collimated primary beams.

The conclusions were that (i) for $ka \lesssim 1$, one still obtains Rutherford scattering; (ii) for $ka > 1$, a slightly sharper directivity is predicted, as compared with that obtained in the case where the primary waves are collimated. Here k is the wavenumber of the scattered sound and a is the radius of the circular sound source; (iii) the experimental results obtained by Bellin and Beyer lie between the directivity curves obtained for the two cases compared in (ii) above. It is my object in this Letter to extend these conclusions to more-realistic practical conditions.

When considering possible exploitation of interaction effects in acoustic waves, it is found necessary to study interaction between primary beams that spread cylindrically or spherically. The difficulties encountered in finding a complete solution can be overcome by considering the primary beams to be uniformly distributed within a cylindrical sector (i.e., a fan beam) and a conical pencil beam, respectively.[6]

In the case of cylindrically spreading primary beams, the directivity pattern and the axial variation of pressure amplitude at the difference frequency were studied using the virtual-sources concept put forward by Westervelt. The results obtained by considering the acoustic power in the primary beams to remain constant while changing the beam angles are shown in Fig. 1. The approximations used in the calculations make these results valid for small values of $|\xi-\psi_1|$, where ψ_1 is as defined below, while ξ is the angular position of the observer with respect to the axis of the primary beams.

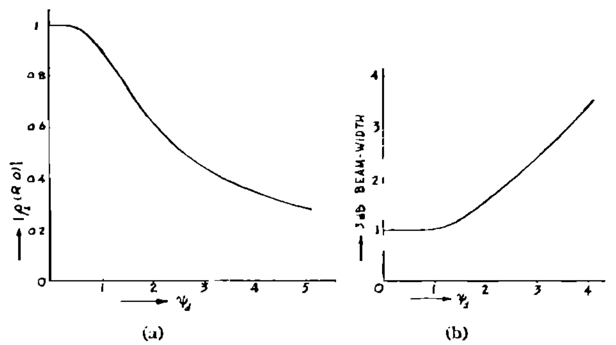

Fig. 1. Interaction between cylindrically spreading primary waves. Variation of (a) the amplitude along the axis of symmetry of the configuration and (b) the 3-dB beam containing the primary waves. Ordinates in relative units. ψ_d defined in text.

For the case of spherically spreading primary beams, only the variation of pressure amplitude along the axis of symmetry was computed. The results, which are shown in Fig. 2, are valid for small values of ψ_1 such that $1-\cos\psi_1 \cong \psi_1^2/2$.

In both the Figures, the abcissa is in terms of the nondimensional variable $\psi_d = \psi_1/\theta_d$, where $2\psi_1$ is the beam angle of the fan beam or of the (conical) pencil beam, respectively, and $2\theta_d$ is the 3-dB beam width at the difference frequency in the case of Rutherford scattering and is given by

$$2\theta_d \cong 4\sqrt{(A/2k)}, \tag{1}$$

where $A \simeq \alpha_1+\alpha_2-\alpha$, and α_1 and α_2 are the absorption coefficients at the primary frequencies, α is the absorption coefficient at the difference frequency, and k is the wavenumber at the difference frequency, as before.

One can conclude from these new results that the simple plane-wave approach used by Westervelt can be extended to the cases of cylindrically and spherically spreading beams *provided* that the primary beam angle is of the same order as that predicted for the scattered component $2\theta_d$ in the case considered in Ref. 2-4. If, however, the primary beam angle is increased, a loss of directivity (and a consequent loss in the magnitude of the effect along the axis of symmetry) results.

I should now like to make the following comments regarding the conclusions reached by Naze and Tjøtta:

(a) In the case of spherically spreading primary beams studied by Naze and Tjøtta, the axial value of the scattered component of density, $\rho_s(R,0)$ would be expected to be, in general, a function of the primary beam angle, and not independent of it as stated by them. But their conclusion can be justified, within limits, by the following physical argument. In the case of interaction between spherically spreading primary waves, the virtual-source function at the intermodulation frequency will have spherical

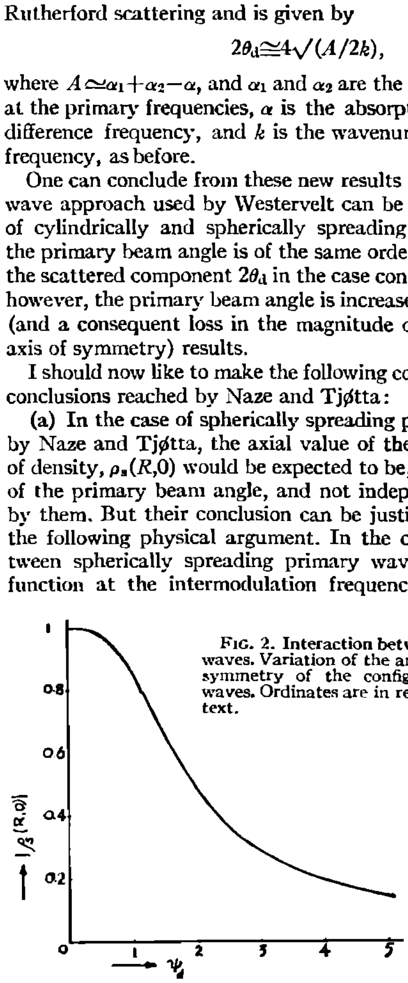

Fig. 2. Interaction between spherically spreading waves. Variation of the amplitude along the axis of symmetry of the configuration of the scattered waves. Ordinates are in relative units. ψ_d defined in text.