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Exact and Approximate Gas Dynamics Using the Tangent Gas

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Steady, inviscid, irrotational flow of a perfect gas in two dimensions is considered in the tangent gas approximation. A fast and accurate method of solution is proposed and solved numerically. Comparison of tangent gas and exact flows are presented. Tangent gas solutions when used as the first step in the iterative solution of the exact flowfield are shown to give substantial reduction in computational time.

1. INTRODUCTION

The computation of steady flow past an airfoil is crucial to the determination of aerodynamic characteristics such as lift, drag and moment coefficients. In many instances potential theory suffices. Neglecting viscosity it is exact for shockless flow and is a satisfactory approximation for transonic flow with weak shocks. For two dimensions the calculations are usually carried out in a conformally mapped plane, an approach used by Sells [1], Garabedian and Korn [2], and Jameson [3]. Similar techniques have been used for multi-element airfoils [4, 5] and nacelles [6]. Three dimensional potential theory has been treated by Caughey [7].

Since the equations are nonlinear, the potential equation is usually solved iteratively. In some instances the potential equation does not admit unique solutions [8–10] and in addition becomes a poor approximation for increasingly strong shock strengths. As a result more recent investigations treat the full Euler equations. Finite difference and finite volume methods have been successfully implemented by Jameson [11] and Lerat and Sides' [12]. Because of slow rates of convergence considerable effort has been directed towards accelerating these methods [13]. Convergence rates depend on factors such as the grid, initial guess, time stepping scheme and method of solution.

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In this paper we present a set of flow dependent grid systems and initial flowfield guesses which substantially improve convergence rates when applied to the Euler equations for flows past an airfoil. These are based on solution of the tangent gas equations introduced by Chaplygin [14] and further developed by von Kármán and Tsien [15, 16].

Woods [17], who extensively studied these equations, proposed certain iterative methods for solving both the analysis and design problems for flows past an airfoil. The methods developed in this paper are substantially different and offer a method for a fast and accurate solution to a problem. (We have also addressed the inverse problem and presented an exact method for its solution [18].)

As will be seen the tangent gas solution lies close to the Euler solution even for high subcritical flows. This is used as a basis for iterative solution of Euler equation for flows past an airfoil by means of FLO52S (written by A. Jameson, E. Turkel and M. Salas). The grid used is the natural one generated by the tangent gas equations and the starting guess is the tangent gas solution. As will be seen this results in substantial computational reduction even for supercritical flows.

2. Basic Equations

Consider steady, inviscid, irrotational flow of a perfect gas in two dimensions, then in the usual notation

$$\nabla \cdot (\overrightarrow{\rho q}) = 0, \qquad \nabla \times \vec{q} = 0, \qquad p/\rho^{\gamma} = 1. \tag{1}$$

The variables are normalized by their free stream values and linear dimensions by an appropriate lengthscale.

The stream function ψ and potential ϕ are introduced in the usual way

$$\overrightarrow{\rho q} = c\,\nabla \times (\psi \mathbf{k}), \qquad \vec{q} = \nabla\phi, \tag{2}$$

where $\mathbf{k}$ denotes a vector perpendicular to the plane of motion. The constant c has been introduced for later purposes.

If s and n are local distances along streamlines and potential lines, respectively, (2) can be written as

$$ds + i\,dn = \frac{1}{q}\left(d\phi + i\,\frac{c}{\rho}\,d\psi\right). \tag{3}$$

If equations can be derived that map the space of ϕ, ψ on to the space of the velocity magnitude and direction (q, θ), then one can take advantage of the fact that the tangent of the flow direction, $\tan\theta$, is the same as the slope of the airfoil surface where $\psi = 0$. Then if q vs. θ can be found corresponding to $\psi = 0$ on ϕ, ψ

plane, the state of flow on the airfoil surface will be known. Toward this end, we write Eq. (3) alternatively as

$$dz = dx + i\,dy = \frac{e^{i\theta}}{q}\left(d\phi + i\,\frac{c}{\rho}\,d\psi\right), \tag{4}$$

where x, y are cartesian coordinates and θ flow direction angle. With q and θ as independent variables, it is easy to derive from (4)

$$\phi_\theta = \frac{q}{\rho}\,c\psi_q, \qquad \phi_q = -\frac{1-M^2}{\rho q}\,c\psi_\theta. \tag{5}$$

If dependent and independent variables are interchanged and the Prandtl Meyer function

$$v = \int_1^q \sqrt{|1-M^2|}\,\frac{dq}{q} \tag{6}$$

is introduced in place of q, then

$$\theta_\phi - \frac{1}{K(v)}\,v_\psi = 0, \; \theta_\psi \pm K(v)\,v_\phi = 0. \tag{7}$$

The $\pm$ sign refers to subsonic and supersonic conditions, respectively and

$$K(v) = \beta\,\frac{c}{\rho(q(M))}, \tag{8}$$

where

$$\beta^2 = |1-M^2|. \tag{9}$$

Typical physical $z(=x+iy)$ and potential $w(=\phi+i\psi)$ planes are shown in Fig. 1. The airfoil maps into a slit in the w-plane. The gap BB' in the potential plane corresponds to Γ, where circulation about the airfoil is $-\Gamma$.

The system (7) should be solved subject to the density speed relation obtained from (1) and Bernoulli's relation

$$\frac{q^2}{2} + \frac{1}{\gamma M_\infty^2}\int \frac{dp}{\rho} = \text{constant}. \tag{10}$$

3. Tangent Gas Approximation

Equations (7) are nonlinear and are therefore difficult to solve. A good approximation to those equations under certain conditions can be obtained by

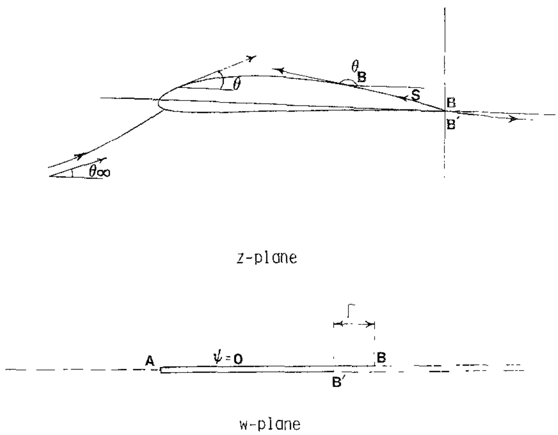

FIG. 1. Airfoil in physical z-plane and potential w-plane.

introducing the so-called "tangent gas approximation" [17], in which the isentropic relation between p and ρ given in Eq. (1) is replaced by a tangent to the curve of p vs. $1/\rho$. This approximation is then given by

$$(p-1)=\gamma\left(1-\frac{1}{\rho}\right). \tag{11}$$

From (10) we obtain

$$\rho=\beta/\beta_{\infty}. \tag{12}$$

With the constant c in (8) taken as

$$c=1/\beta_{\infty}, \tag{13}$$

we obtain from (8)

$$K(v)=1. \tag{14}$$

Then for subsonic flow (7) becomes the Cauchy Riemann equations

$$\theta_{\phi}-v_{\psi}=0, \qquad \theta_{\psi}+v_{\phi}=0. \tag{15}$$

Equations (15) are exact for the tangent gas and also for incompressible flow ($M=0$). In addition, it will be seen that it can be a very good approximation to the original equations. In the above formulation the tangency point has been taken to the freestream

$$p_{\infty}=1, \qquad \rho_{\infty}=1. \tag{16}$$

With this selection of tangency point the following relations hold for the tangent gas [17]

$$q = \sinh v^* \operatorname{cosech}(v^* - v), \qquad \beta = \tanh(v^* - v), \qquad c_p = \frac{2}{1 - \beta_\infty \coth v}, \tag{17}$$

where the contant v^* is given by

$$v^* = \ln\left(\frac{M_\infty}{1 - \beta_\infty}\right). \tag{18}$$

From (6) it is seen that $v_\infty = 0$ and at stagnation points (denoted by zero subscript)

$$v_0 = -\infty, \qquad c_{p,0} = \frac{2}{1 + \beta_\infty}. \tag{19}$$

4. Solution Procedure

It follows from (15) that

$$\tau = -v + i\theta \tag{20}$$

is an analytic function of w. It will be useful to map the $w(=\phi + i\psi)$ plane onto the plane of a new variable $\sigma = |\sigma| \, e^{i\alpha}$ such that the body in the w-plane which is a slit (a part of the line $\psi = 0$) maps onto the unit circle $\sigma = e^{i\alpha}$; $0 \leqslant \alpha \leqslant 2\pi$ and the rest of the w-plane maps onto the exterior of the unit circle. This is accomplished by

$$w = a(\sigma e^{-i\alpha_0} + \sigma^{-1} e^{i\alpha_0}) + i2a \sin \alpha_0 \ln(\sigma e^{-i\alpha_0}) \tag{21}$$

which allows for angle of attack and circulation about an airfoil surface, to be related to $|\sigma| = 1$. Circulation $-\Gamma$ is related to the constant a by

$$\Gamma = 4\pi a \sin \alpha_0. \tag{22}$$

Here constants a and α_0 are as yet unknowns.

From (21) one obtains

$$\frac{dw}{d\sigma} = -ae^{+i\alpha_0}(1 - \sigma^{-1})(e^{-i\alpha_s} - \sigma^{-1}). \tag{23}$$

On the body $\sigma = e^{i\alpha}$; $0 \leqslant \alpha \leqslant 2\pi$, ϕ and ψ are given by

$$\phi(\alpha) = 2a[\cos(\alpha - \alpha_0) - (\alpha - \alpha_0) \sin \alpha_0], \; \psi(\alpha) = 0. \tag{24}$$

α_s in (23) is given by

$$\alpha_s = \pi + 2\alpha_0. \tag{25}$$

Thus the rear and front stagnation points map into $\sigma = 1$ and $\sigma = e^{i\alpha_s}$, respectively.

Since τ is an analytic function of σ, a convenient representation of $\tau(\sigma)$ is given by (see also Ref. [19])

$$\exp(\tau(\sigma)) = (1-\sigma^{-1})^{-\delta}\,(e^{-i\alpha_s}-\sigma^{-1})^{-1}\exp\left(\sum_{n=0}^{\infty} c_n\sigma^{-n}\right), \tag{26}$$

where $\delta = \theta_t/\pi$, θ_t the trailing edge angle. The complex constants c_n are represented by,

$$c_n = A_n + iB_n. \tag{27}$$

Note that (26) contains the Kutta condition. Two Schwarz–Christoffel factors appear in (26) because of the discontinuity in θ at the two stagnation points.

From (26) the relationship between upstream flow direction θ_∞ and α_0 is given by

$$\theta_\infty = B_0 + \pi + 2\alpha_0. \tag{28}$$

The free stream condition is given by

$$A_0 = 0. \tag{29}$$

On the unit circle, (26) reduces to

$$\exp(\tau(e^{i\alpha})) = G(\alpha)\,e^{i\eta(\alpha)}\exp\left(\sum_{n=0}^{\infty} c_n e^{-in\alpha}\right), \tag{30}$$

where

$$G(\alpha) = \left|2\sin\frac{\alpha}{2}\right|^{1-\delta}|2(\sin\alpha_0+\sin(\alpha-\alpha_0))|^{-1}, \tag{31}$$

$$\eta(\alpha) = \frac{1}{2}(1-\delta)(\pi-\alpha)+\left(\alpha+\frac{\pi}{2}\right)-\pi U(\alpha-\alpha_s)+\alpha_0. \tag{32}$$

$U(\alpha-\alpha_s)$ in (32) is the unit step function. The tangent angle θ_B of the body is related to θ by

$$\theta(\alpha) = \theta_B(\alpha)-\pi-\pi U(\alpha-\alpha_s). \tag{33}$$

Separation of (30) into real and imaginary parts leads to

$$\tilde{v}(\alpha) = \sum_{n=0}^{\infty}(A_n\cos n\alpha + B_n\sin n\alpha) \tag{34}$$

$$\tilde{\theta}(\alpha) = \sum_{n=0}^{\infty}(B_n\cos n\alpha - A_n\sin n\alpha)+\pi+\alpha_0 \tag{35}$$

where

$$\tilde{v}(\alpha) = -v(\alpha) - \ln G(\alpha), \tag{36}$$

and

$$\tilde{\theta}(\alpha) = \theta_B(\alpha) - \frac{1}{2}(1-\delta)(\pi-\alpha) - \left(\alpha + \frac{\pi}{2}\right). \tag{37}$$

The closure condition of the airfoil is related to the leading terms of the series by (Appendix A).

$$A_1 = (1-\delta) - (1-\beta_\infty)\, 2\sin^2\alpha_0 \tag{38}$$

$$B_1 = (1-\beta_\infty)\sin 2\alpha_0. \tag{39}$$

5. Analysis (Direct) Problem

Here the flow past an airfoil is sought. An iterative method of solution similar to the one for incompressible flow (15) is found to converge with good accuracy. The method of solution goes as follows.

An initial estimate of arclength as a function of circle angle, $s(\alpha)$, (e.g., of a flat plate in incompressible flow) is made. From the given contour $\theta_B(s)$, $\theta_B(\alpha)$ is estimated and $\tilde{\theta}(\alpha)$ is calculated from (37). α_0 is obtained from (28). After the closure conditions (38) and (39) are imposed, a new form of $\tilde{\theta}(\alpha)$ is generated and then its conjugate $\tilde{v}(\alpha)$ is obtained from (34). $v(\alpha)$ is then obtained from (36) and speed $q(\alpha)$ is obtained from (17). The updated value of $s(\alpha)$ is now obtained from $q(\alpha)$ using the relation

$$\begin{aligned} s(\alpha) &= \int_0^\alpha \frac{1}{q} \frac{|d\phi|}{d\alpha}\, d\alpha \\ &= 2a \int_0^\alpha \frac{|\sin\alpha_0 + \sin(\alpha-\alpha_0)|}{q}\, d\alpha, \end{aligned} \tag{40}$$

where the constant a is now given by

$$s(2\pi) = 1. \tag{41}$$

The above procedure is repeated until convergence is obtained. The criterion for convergence was taken to be that maximum difference in arc-length between successive iterations be $O(10^{-6})$. Typically the number of iterations required was no more than eight and the computation time was roughly one second on an IBM 3081 with 128 points taken on the unit circle. The actual numerical calculation is facilitated through the use of the fast fourier transform (FFT) and the fact that (34) and (35) are conjugate fourier series. The fourier constants, c_n, are also obtained easily during FFT which are used for generating grids.

6. Grid Generation

The physical plane is related to the circle plane through [17]

$$dz = \frac{1}{2\beta_\infty} \left\{ (1+\beta_\infty)\, e^{\tau} \frac{dw}{d\sigma}\, d\sigma - (1-\beta_\infty)\, \overline{e^{-\tau} \frac{dw}{d\sigma}\, d\sigma} \right\}. \tag{42}$$

Here an overbar denotes complex conjugate. Note that for incompressible flow z is an analytic function of σ, as it should be.

From (21) and (26) it is easily seen that

$$e^{\tau} \frac{dw}{d\sigma} = -ae^{+i\alpha_0}(1-\sigma^{-1})^{1-\delta} \exp\left(\sum_{n=0}^{\infty} c_n \sigma^{-n} \right), \tag{43}$$

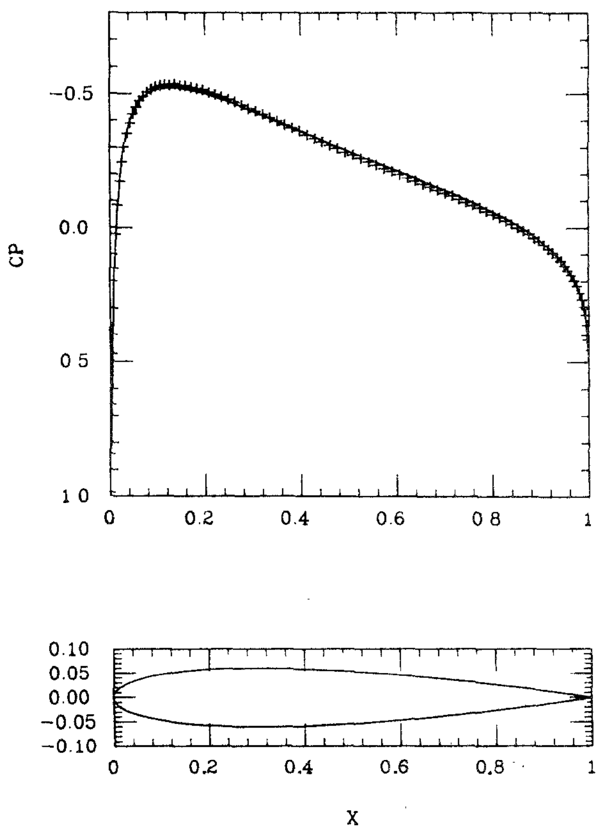

Fig. 2. Comparison of tangent gas solution and Euler solution over NACA 0012 Airfoil at Mach = 0.6 and angle of attack = 0.0. ——, tangent gas solution; + + +, Euler solution.

and

$$e^{-\tau}\frac{dw}{d\sigma} = -ae^{+i\alpha_0}(1-\sigma^{-1})^{1+\delta}(e^{-i\alpha_s}-\sigma^{-1})^2 \exp\left(-\sum_{n=0}^{\infty} c_n\sigma^{-n}\right). \tag{44}$$

Equations (42), (43) and (44) are used to map the circle plane into physical plane and the flowfield variables are obtained from (26), (17), and (18).

Observe that the grid generated is flow dependent. Since the mapping from σ plane to z-plane is not conformal except when $M=0$, the grid generated in physical plane is not in general orthogonal. The grid produced by this method appears to be more natural than the incompressible conformal grid.

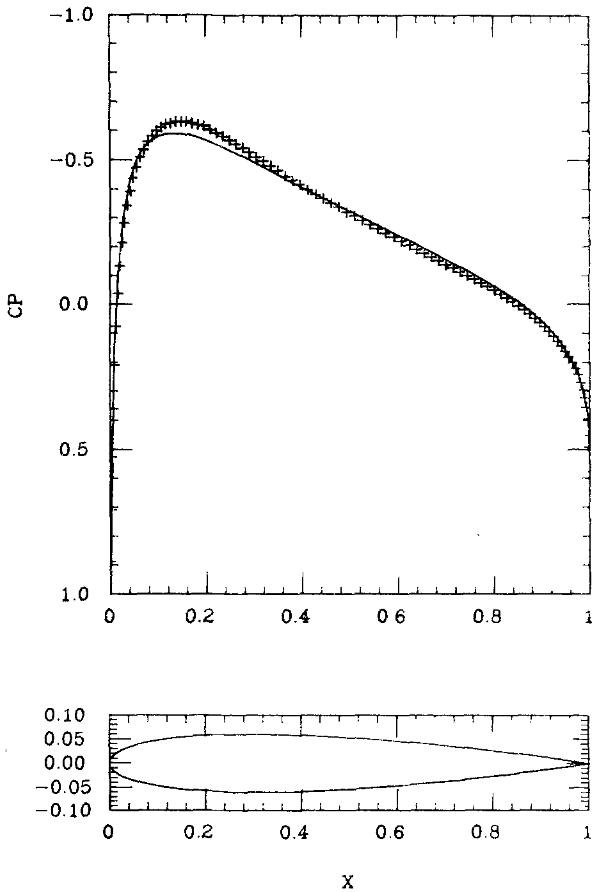

FIG. 3. Comparison of tangent gas solution and Euler solution over NACA 0012 Airfoil at Mach 0.7 and angle of attack = 0.0. ——, tangent gas solution; + + +, Euler solution.

7. Results

Figures 2–5 compare the tangent gas solution with the converged Euler solution (as calculated by FLO52S). The tangent gas solution is seen to be remarkably accurate even at the near critical case depicted in Fig. 3 and the slightly critical case shown in Fig. 4. Even when a clear shock is present as in Fig. 5, the tangent gas solution only fails in a relatively small neighborhood of the shock.

Figures 6 and 7 indicate for two typical cases the number of iterative cycles to achieve a convergence criterion. The criterion used is the enthalpy error introduced by Jameson [20]. In each figure we indicate the number iterations required to reach the indicated criterion. The first column of each figure refers to use of the tangent gas grid and the tangent gas solution as a starting flow. The second column

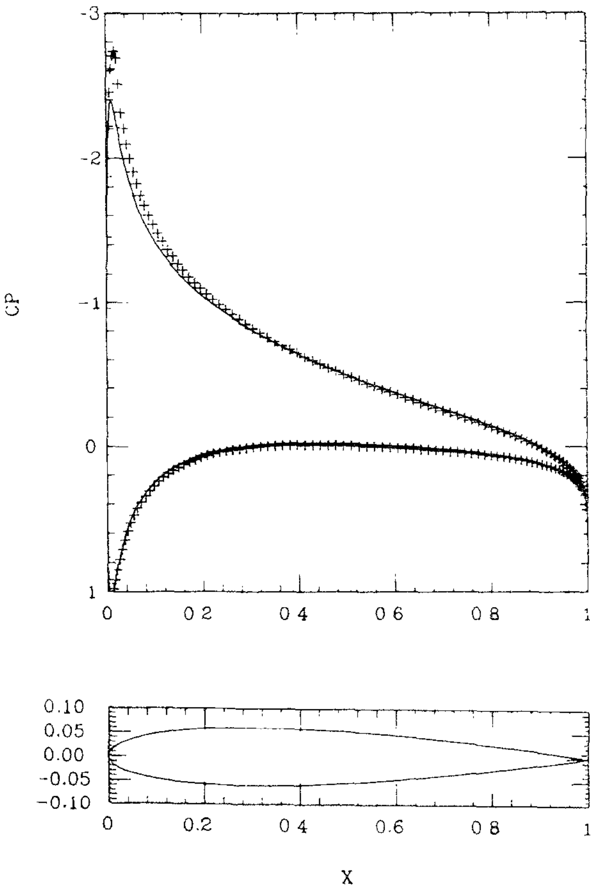

FIG. 4. Comparison of tangent gas solution and Euler solution over NACA 0012 Airfoil at Mach = 0.50 and angle of attack = 5.0 degrees. ——, tangent gas solution; + + +, Euler solution.

gives the analogous values using the conventional grid, viz., that generated by conformal mapping and a uniform flowfield as the starting guess. (Little change in convergence was observed if incompressible flow was taken as the initial guess.) As is seen the reduction in cycles is substantial. In this same vein if the convergence criterion is reduced by a factor of 10 the comparison becomes more dramatic—the tangent gas approach leads to a 10-fold reduction in cycles over the usual approach.

In order to distinguish whether the grid or the tangent gas approximation was more significant in speeding convergence, we also ran the programs using the tangent gas grid with a uniform first guess. Although some improvement resulted, the clear implication from this was that the tangent gas solution as a first guess was the most important factor.

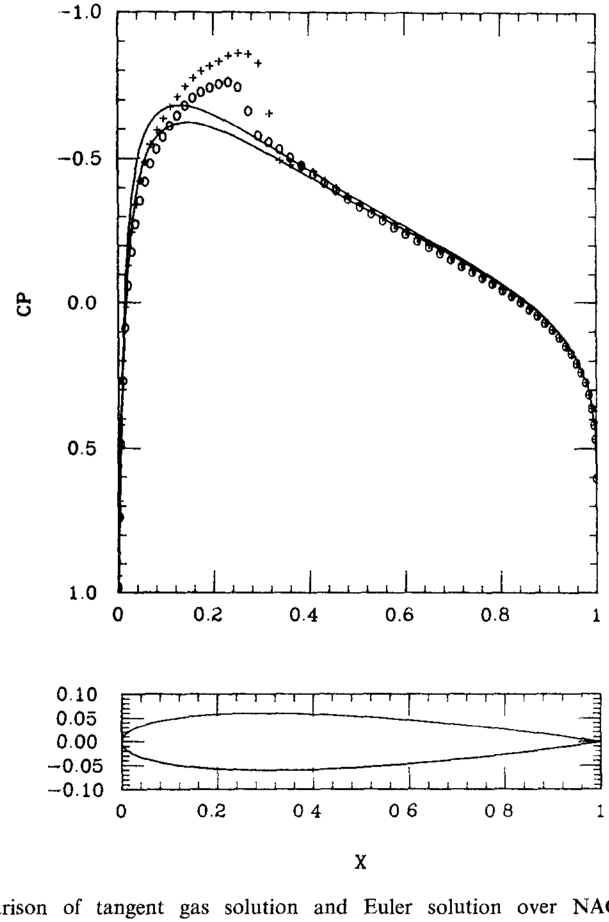

FIG. 5. Comparison of tangent gas solution and Euler solution over NACA 0012 Airfoil at Mach = 0.758 and angle of attack = 0.14 degrees. ——, tangent gas solution; (+, ○) Euler solution; +, upper surface; ○, lower surface.

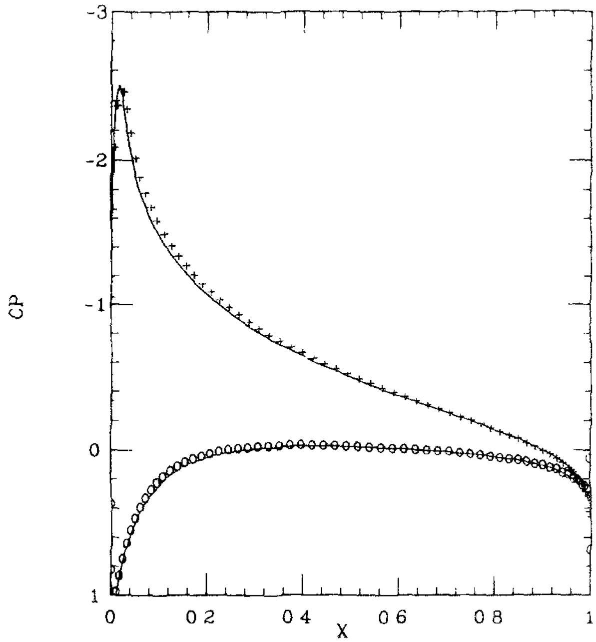

FIG. 6. Euler solution (FLO52S) for near critical flow past an NACA 0012 Airfoil at Mach 0.50 and angle of attack = 5.0 degrees. (+, ○): grid, 64∗32; grid type, tangent; initial guess, tangent; number of cyclces, 344. (——): grid, 64∗32; grid type, conformal; initial guess, uniform; number of cycles 913. Average error in enthalpy, 0.1385E-03. +, upper surface; ○, lower surface.

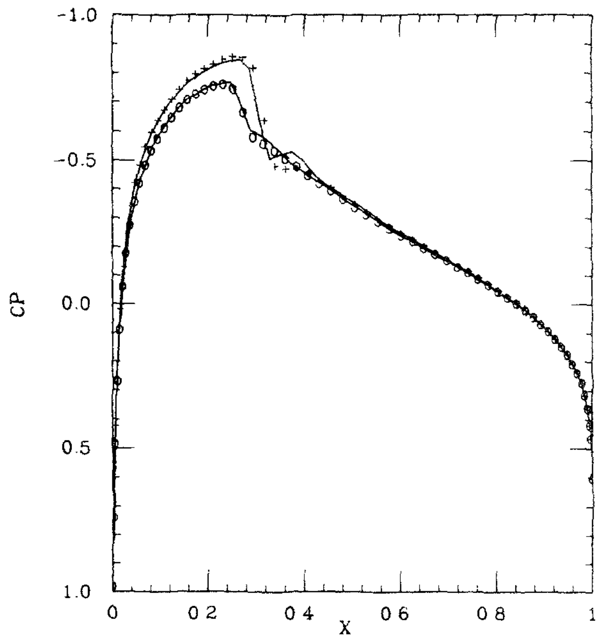

FIG. 7. Euler solution (FLO52S) for supercritical flow past an NACA 0012 Airfoil at Mach 0.758 and angle of attack = 0.14 degrees. (+, ○): grid, 64∗32; grid type, tangent; initial guess, tangent; number of cycles, 381. (——): grid, 64∗32; grid type, conformal; initial guess, uniform; number of cycles, 715. Average error in enthalpy, 0.2454E-03. +, upper surface; ○, lower surface.

Appendix A: Closure Conditions

If C is a closed contour around an airfoil, the the closure condition is

$$\oint_c dz = 0. \tag{A1}$$

Hence from (42) we obtain (see also Ref. [17])

$$(1+\beta_\infty)\oint_c e^\tau \frac{dw}{d\sigma}\,d\sigma = (1-\beta_\infty)\overline{\oint_c e^{-\tau}\frac{dw}{d\sigma}\,d\sigma}. \tag{A2}$$

From (43) and (44) it follows

$$e^\tau \frac{dw}{d\sigma} = -ae^{i(\beta_0+\alpha_0)}\left[1+\frac{K_1}{\sigma}+O(\sigma^{-2})\right] \tag{A3}$$

$$e^{-\tau}\frac{dw}{d\sigma} = +ae^{i(\alpha_0-\beta_0)}\left[-e^{-2i\alpha_s}+\frac{K_2}{\sigma}+O(\sigma^{-2})\right] \tag{A4}$$

where

$$K_1 = c_1+\delta-1, \qquad K_2 = (1+\delta+c_1)\,e^{-2i\alpha_s}+2e^{-i\alpha_s}. \tag{A5}$$

Use of residue theorem, (A3) and (A4) reduces (A2) to

$$(1+\beta_\infty)\,e^{i\alpha_0}K_1 = (1-\beta_\infty)\,e^{-i\alpha_0}\bar{K}_2. \tag{A6}$$

Equating real and imaginary parts we obtain

$$(A_1+\delta-1)\cos\alpha_0 = B_1\sin\alpha_0,\ (A_1+\delta+1-2\beta_\infty)\sin\alpha_0 = B_1\cos\alpha_0. \tag{A7}$$

From (A7) we obtain

$$A_1 = (1-\delta)-(1-\beta_\infty)\,2\sin^2\alpha_0,\ B_1 = (1-\beta_\infty)\sin 2\alpha_0. \tag{A8}$$

For the incompressible case ($\beta_\infty = 1$) this reduces to

$$A_1 = (1-\delta), \qquad B_1 = 0. \tag{A9}$$

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