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Dissipative magnetogasdynamic flow

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An analysis of the structure of the wakes and waves in steady compressible magnetohydrodynamics is presented. No restriction is made on the equation of state of the gas or on the ratios of the various dissipative parameters. An asymptotic solution is obtained which furnishes directly the flow far from a body and which may be used in the construction of the entire flow field. The non-dissipative solutions are obtained as a non-uniform limit for vanishing dissipation; no matter how small the dissipation, one can go far enough from the origin that the flow is essentially dissipative. For non-aligned fields the wave pattern consists of a downstream wake and either two or four standing waves, depending on the flow regime. For aligned fields, two of these waves become wakes, so that the wake is a superposition of three structured layers, with either all downstream or two downstream and one up-stream. It is found that the non-dissipative limit of the wake is non-unique for the aligned fields case. Different limits are obtained depending on how the various dissipative parameters vanish.

1. Introduction

In this paper we consider steady magnetohydrodynamic flow in the Oseen approximation. No restriction is placed on the equation of state of the gas or on the various dissipative parameters (viscosity, thermal and electrical conductivity). The method of approach is to obtain the fundamental solution. It was shown in an earlier paper (Salathe & Sirovich 1967) how this could be used to obtain the solution for arbitrary boundary-value problems. In § 2 of the present paper we demonstrate that the fundamental solutions themselves provide the far field flow past a finite body. This is obtained simply in terms of such quantities as the total force on the body, heat added, etc.

In § 3 we obtain the fundamental solutions for the non-aligned fields case. An asymptotic solution is obtained applicable for distance from the origin large compared to the mean free path. It is well known that there exist two distinct flow regimes in magnetohydrodynamics, the doubly hyperbolic and the hyperliptic. In each of these we obtain a downstream wake which is a pure entropy wake. That is, it carries only density and temperature disturbances and is structured by thermal conductivity. In the doubly hyperbolic regime, the flow exhibits, in addition, four structured waves, while in the hyperliptic case,

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two structured waves are superposed on the elliptic flow of the non-dissipative theory. In both cases, some of the waves may be inclined upstream. The flow in the waves is isentropic, and the waves are structured by all three dissipative parameters.

In the limit of vanishing dissipation, the non-dissipative solution (Salathe & Sirovich 1967) is recovered. However, the limit is non-uniform—no matter how small the dissipation, the flow is essentially dissipative at large distances.

When the angle between the applied magnetic field and the free-stream velocity vanishes, two of the waves collapse on the axis and become part of the wake They may both collapse downstream, in which case the wake consists of three downstream structured layers. On the other hand, one may collapse upstream, and the wake then consists of two structured layers downstream and one upstream.

The compressible aligned fields wake was studied previously by Fan (1964), who considered a number of special cases. Unlike the gasdynamic case, each of the layers which make up the wake are, in general, structured by all dissipative mechanisms. Each carries entropy, vorticity and current, and the sum of the fluid plus magnetic pressure is constant across the wake.

In the limit of vanishing dissipation, the non-dissipative solution (Salathe & Sirovich 1967) is obtained (again as a non-uniform limit) provided that all three wakes lie downstream. However, when one of the wakes lies upstream, the non-dissipative solution obtained depends on the way in which the dissipative parameters vanish. In this regime, therefore, the correct non-dissipative flow can be obtained only by considering the underlying dissipative problem. The case of an upstream wake in non-dissipative magnetohydrodynamics is well known to be non-unique and a variety of different solutions have been suggested (Sears & Resler 1959; Stewartson 1960; Leibovich & Ludford 1966; Salathe & Sirovich 1967). All of these are, of course, correct within the framework in which they have been obtained. However, when a non-dissipative fluid is regarded as a limiting form of a real fluid, we see that none of the aforementioned papers can furnish the proper solution.

2. Far field flow and fundamental solutions

We consider magnetohydrodynamic flow past a finite body. In previous papers (Sirovich 1967*a*; Salathe & Sirovich 1967) it was shown that the body, having a surface $S = 0$, could be replaced by a surface singularity in the flow field on which sources are distributed. The equations for an impermeable body are:

$$\partial\rho'/\partial t + \nabla'.\rho'\mathbf{u}' = 0, \tag{2.1}$$

$$(\partial/\partial t)\,\rho' u_i' + (\partial/\partial x_j')\,\rho' u_i' u_j' + (\partial p'/\partial x_i') - (\partial/\partial x_j')\,P_{ij} - (\mathbf{J}\times\mathbf{B}')_i = [p' n_i - P_{ij} n_j]\,\delta(S), \tag{2.2}$$

$$\partial/\partial t\rho'(\epsilon' \tfrac{1}{2}u'^2) + \partial/\partial x_j'\{\rho(\epsilon' + \tfrac{1}{2}u'^2)\,u_j' + p' u_j' - P_{ij} u_i' + Q_j\} - J_i E_i = [Q_j n_j]\,\delta(S), \tag{2.3}$$

$$P_{ij} = \overline{\mu}(\mu_{i,j}' + u_{j,i}') - \tfrac{2}{3}\overline{\mu}\nabla'.\mathbf{u}'\delta_{ij}, \tag{2.4}$$

$$\mathbf{Q} = -\kappa\boldsymbol{\nabla}'T' + [\kappa T'\mathbf{n}]\,\delta(S), \tag{2.5}$$

$$\mathbf{J} - \frac{1}{\mu}\boldsymbol{\nabla}'\times\mathbf{B}' = \left[\frac{1}{\mu}\mathbf{B}'\times\mathbf{n}\right]\delta(S), \tag{2.6}$$

$$\boldsymbol{\nabla}'.\mathbf{B}' = -[\mathbf{B}'.\mathbf{n}]\,\delta(S), \tag{2.7}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u}'\times\mathbf{B}'). \tag{2.8}$$

Here ρ', ϵ', p', $\mathbf{u}'$ denote the mass density, internal energy, fluid pressure and velocity, and $\mathbf{E}$, $\mathbf{J}$, $\mathbf{B}'$ denote the electric field, total current density and magnetic induction. P_{ij} is the viscous stress tensor and $\mathbf{Q}$ the heat flow vector; $\bar{\mu}$ is the viscosity, κ the thermal conductivity, σ the electrical conductivity and μ the permeability. $\delta(S)$ is a one-dimensional delta function having the property

$$\int F(x)\,\delta(S)\,dx = \int_S F\,dS. \tag{2.9}$$

The bracket in each of the source terms denotes the jump of the quantity in brackets across S, $\mathbf{n}$ denoting the outward normal to S. For our purposes it is convenient to take as the internal flow ρ_0', T_0', p_0', $u' = 0$, where zero subscripts denote upstream values, and the magnetic field is determined by $[\mathbf{B}.\mathbf{n}] = 0$ (which of course is the natural condition).

Far field

The problem of finding the far field may be considerably reduced. To see this, we normalize $\mathbf{x}'$ with respect to distance R from the body, then for R large compared to a characteristic body dimension l, we can expand the right-hand sides of the equations. Using the relation (2.9) the sources can be written in the form (Sirovich 1967*b*)

$$\begin{aligned} F\delta(S) &= \int_S F(\mathbf{x}_0)\,\delta(\mathbf{x}-\mathbf{x}_0)\,dS_0 \\ &= \delta(\mathbf{x})\int_S F\,dS_0 - \frac{\partial}{\partial x_k}\delta(\mathbf{x})\int_S x_{0k}F\,dS_0 + \ldots, \end{aligned} \tag{2.10}$$

which represents a multipole expansion, obtained by expanding $\delta(\mathbf{x}-\mathbf{x}_0)$. Keeping only the leading term,

$$F\,\delta(S) = \delta(\mathbf{x})\int_S F\,dS_0 + O\left(\frac{l}{R}\right).$$

Consequently, the right-hand sides of (2.1)–(2.6) reduce to $\delta(\mathbf{x})$ times the quantity in brackets integrated over the body surface. They therefore represent the equations for the fundamental solution; however, the coefficients of the delta function, representing the strength of the source at the origin, have all been determined. For example, the source in the momentum equation is the total pressure and viscous force on the body, the source in the energy equation is the total heat added by the body, the source in Ampère's law is the total current induced by the body. The source term in the heat conduction equation (2.5) is a result of temperature change across the body due to finite thermal conductivity, and will yield an effective heat source term in the energy equation. When the thermal conductivity of the body is infinite, the temperature on S is constant, and this term vanishes. Equation (2.7) reduces to $\nabla'.B = 0$, by our choice of the boundary condition on $B'.n$.

Oseen equations

We consider steady flows linearized about an undisturbed flow field consisting of a uniform free-stream velocity $\mathbf{U}_0$, a uniform applied magnetic field $\mathbf{B}_0$, and an electric field $\mathbf{E}_0 = -\mathbf{U}_0 \times \mathbf{B}_0$ (this assures zero current in the undisturbed flow). The resulting equations will be referred to as the Oseen form of the magneto-hydrodynamic equations.

We introduce the following normalization:

$$\mathbf{x} = \frac{\mathbf{x}'}{L}, \quad \mathbf{u} = \frac{\mathbf{u}' - \mathbf{U}_0}{(\partial p_0/\partial \rho_0)^{\frac{1}{2}}_{T_0}}, \quad \rho = \frac{\rho' - \rho_0}{\rho_0}, \tag{2.11}$$

$$T = \frac{(\partial p_0/\partial T_0)_{\rho_0}}{\rho_0(\partial p_0/\partial \rho_0)_{T_0}}(T' - T_0), \quad \mathbf{b} = \frac{\mathbf{B}' - \mathbf{B}_0}{|\mathbf{B}_0|}, \quad \mathbf{U} = \frac{U_0}{(\partial p_0/\partial \rho_0)^{\frac{1}{2}}_{T_0}},$$

$$\mathbf{B} = \frac{\mathbf{B}_0}{|\mathbf{B}_0|}, \quad A^2 = \frac{B_0^2}{\mu\rho_0(\partial p_0/\partial \rho_0)_{T_0}}, \quad \mathbf{e} = \frac{\mathbf{E}' - \mathbf{E}_0}{|\mathbf{B}_0|(\partial p_0/\partial \rho_0)^{\frac{1}{2}}_{T_0}}.$$

Fundamental solution

In this notation the equations for the fundamental solutions are:

$$\nabla.\mathbf{u} + \mathbf{U}.\nabla\rho = 0, \tag{2.12}$$

$$\mathbf{U}.\nabla u + \nabla\rho + \nabla T - A^2(\nabla \times \mathbf{b}) \times \mathbf{B} - \frac{1}{Re}(\nabla^2\mathbf{u} + \tfrac{1}{3}\nabla\nabla.u) = \mathbf{M}\delta(\mathbf{x}), \tag{2.13}$$

$$\mathbf{U}.\nabla T + (\gamma - 1)\nabla.\mathbf{u} - \frac{\gamma}{Pr\,Re}\nabla^2 T = H\delta(x) + \hat{\mathbf{H}}.\nabla\delta(\mathbf{x}), \tag{2.14}$$

$$\mathbf{U} \times \mathbf{b} + \mathbf{u} \times \mathbf{B} + \mathbf{e} - \frac{1}{Rm}\nabla \times \mathbf{b} = \mathbf{F}\delta(\mathbf{x}), \tag{2.15}$$

where $Re = (L/\nu)(\partial p_0/\partial \rho_0)^{\frac{1}{2}}_{T_0}$ is the Reynolds number, $Rm = \sigma\mu L(\partial p_0/\partial \rho_0)^{\frac{1}{2}}_{T_0}$ is the magnetic Reynolds number and $Pr = \nu[\rho_0 c_p]/\kappa$ is the Prandtl number; $\gamma = c_p/c_v$, $\nu = \bar{\mu}/\rho_0$ and $A^2 = B_0^2/\mu\rho_0(\partial p_0/\partial \rho_0)_{T_0}$. We have assumed the gas state to be specified by equations of the form

$$\left.\begin{aligned} \epsilon' &= \epsilon'(\rho', T'), \\ p' &= p'(\rho', T'). \end{aligned}\right\} \tag{2.16}$$

We seek solutions which depend only on x and z, and so take $\mathbf{x} = (x, z)$.

$\mathbf{M}$, H, $\hat{\mathbf{H}}$, and $\mathbf{F}$ are constants normalized according to

$$\mathbf{M} = \frac{\mathbf{M}'}{\rho_0(\partial p_0/\partial \rho_0)_{T_0}}, \quad \begin{Bmatrix} H \\ \hat{\mathbf{H}} \end{Bmatrix} = \begin{Bmatrix} H' \\ \hat{\mathbf{H}}' \end{Bmatrix} \frac{(\partial p_0/\partial T_0)_{\rho_0}}{\rho_0^2 c_v (\partial p_0/\partial \rho_0)^{\frac{3}{2}}_{T_0}}, \tag{2.17}$$

$$\mathbf{F} = \frac{\mathbf{F}'}{|\mathbf{B}_0|\,L\,(\partial p_0/\partial \rho_0)^{\frac{1}{2}}_{T_0}},$$

where $\mathbf{M}'$, H, $\mathbf{H}'$, $\mathbf{F}'$ are the corresponding dimensional source strengths, given by

$$\left.\begin{aligned}\mathbf{M}' &= \int_S (p\mathbf{n}-\mathbf{P}.\mathbf{n})\,dS+\left\{\int_S \mathbf{J}_s dS+\int_v \mathbf{J}\,dv\right\}\times\mathbf{B}_0,\\ H' &= \int_S \mathbf{Q}.\mathbf{n}dS-\mathbf{U}.\mathbf{M}',\\ \hat{\mathbf{H}}' &= -\int_S \kappa T\mathbf{n}dS,\\ \mathbf{F}' &= \int_S \mathbf{J}_s dS+\int_v \mathbf{J}\,dv.\end{aligned}\right\} \quad (2.18)$$

The currents appear in the momentum source as a result of substituting $\mathbf{J}$ from Ampère's law, and represent the additional force on the body due to surface currents at the body and current carried by the body. The $\mathbf{U}.\mathbf{M}'$ term in the H' expression is a result of subtracting the momentum equation from the energy equation, and the $\hat{\mathbf{H}}'$ term is a result of the source in the Fourier law, representing the temperature jump across the body. This term is kept since the temperature jump can be large, for example, by maintaining a large temperature gradient in the body.

The dissipative parameters of the problem occur in the three dimensionless combinations Re, Rm, $PrRe$. Re and Rm are not, strictly speaking, Reynolds numbers unless L is based on a characteristic dimension of the problem. In fact, in seeking the fundamental solution, no outside length scale appears. We find it convenient to fix L by the condition

$$\max(Re^{-1}, Rm^{-1}, [PrRe]^{-1}) = O(1). \quad (2.19)$$

Defined in this way, L is a characteristic or intrinsic length-scale of the fluid itself. (For simple gases, kinetic theory shows that L may be identified with the mean-free-path.) The condition (2.19) is purposely left vague, since in later sections various limits will be carried out.

The condition $\nabla.\mathbf{b} = 0$ is automatically satisfied on introducing the function ϕ by

$$b_x = \frac{\partial\phi}{\partial z}, \quad b_z = -\frac{\partial\phi}{\partial x}. \quad (2.20)$$

We now Fourier transform the equations, using the same symbol for the transformed variable as for the untransformed variable, e.g.

$$\rho(\mathbf{k}) = \int_{-\infty}^{+\infty} \rho(\mathbf{x})\, e^{-i\mathbf{k}.\mathbf{x}}\,d\mathbf{x}, \quad (2.21)$$

where $\mathbf{k} = (k_1, k_3)$. In component form the transformed equations are

$$\begin{bmatrix} ik_1U_x & ik_1 & ik_3 & 0 & 0\\ ik_1 & ik_1U_x & 0 & ik_1 & (k_1^2+k_3^2)\,A^2B_z\\ ik_3 & 0 & ik_1U_x & ik_3 & -(k_1^2+k_3^2)\,A^2B_x\\ 0 & (\gamma-1)\,ik_1 & (\gamma-1)\,ik_3 & ik_1U_x & 0\\ 0 & -B_z & B_x & 0 & ik_1U_x \end{bmatrix}\begin{bmatrix}\rho\\ u_x\\ u_z\\ T\\ \phi\end{bmatrix}$$

$$
+\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{Re}(k_1^2+k_3^2)+\frac{1}{3Re}k_1^2 & \frac{1}{3Re}k_1k_3 & 0 & 0 \\ 0 & \frac{1}{3Re}k_1k_3 & \frac{1}{Re}(k_1^2+k_3^2)+\frac{1}{3Re}k_3^2 & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma}{PrRe}(k_1^2+k_3^2) & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{Rm}(k_1^2+k_3^2) \end{bmatrix}\begin{bmatrix} \rho \\ u_x \\ u_z \\ T \\ \phi \end{bmatrix}=\begin{bmatrix} m \\ M_x \\ M_z \\ H \\ F_y \end{bmatrix} \tag{2.22}
$$

or, in matrix notation,

$$
\mathbf{M}\mathbf{v} = \mathbf{M}_{ND}\mathbf{v}+\mathbf{M}_D\mathbf{v} = \mathbf{S}, \tag{2.23}
$$

where $\mathbf{M}_D$ contains only dissipative terms,

We have included the source m in the continuity equation for generality, and have eliminated the $\hat{\mathbf{H}}$ source in the energy, since its effect can be obtained by suitable differentiation of the resulting solution.

In the above equations the free stream velocity $\mathbf{U}$ has, without loss of generality, been taken in the (x,y)-plane, and it is assumed that the applied field $\mathbf{B}$ lies in the (x,z)-plane, i.e. $\mathbf{U} = (U_x, U_y)$, $\mathbf{B} = (B_x, B_z)$. This results in a decoupling of the equations for the y components of velocity and magnetic field and the two systems can be considered separately.

The last of the equations in (2.22) is the y component of (2.15).

The equations for u_y and b_y are

$$
\begin{bmatrix} ik_1U_x & -ik_1A^2B_x-ik_3A^2B_z \\ -ik_1B_x-ik_3B_z & ik_1U_x \end{bmatrix}\begin{bmatrix} u_y \\ b_y \end{bmatrix} + \begin{bmatrix} \frac{1}{Re}(k_1^2+k_3^2) & 0 \\ 0 & \frac{1}{Rm}(k_1^2+k_3^2) \end{bmatrix}\begin{bmatrix} u_y \\ b_y \end{bmatrix} = \begin{bmatrix} M_y \\ ik_1F_z-ik_3F_x \end{bmatrix},
$$

The second equation is a combination of the x and z components of (2.15).

3. The dissipative solution

The solution to equations (2.11) is

$$
\mathbf{v}(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{e^{i\mathbf{k}.\mathbf{x}}\mathbf{M}^*\mathbf{S}\,d\mathbf{k}}{D}, \tag{3.1}
$$

where $\mathbf{M}^*$ is the signed transpose of the cofactor matrix of $\mathbf{M}$ (i.e. the classical adjoint) and D is the determinant of $\mathbf{M}$.

The direct evaluation of (3.1) is not feasible. Instead, we follow an approach introduced in a similar analysis of the gas-dynamic case by Sirovich (1961, 1967*b*). We start from a knowledge of the discontinuities† of (3.1) in the non-dissipative case (Salathe & Sirovich 1967). Such discontinuities imply that the higher-order dissipative operators become important in that region and that a boundary-layer analysis is required. In regions removed from discontinuities the non-dissipative solutions are regarded as correct. This method may be given a mathematically rigorous basis. For a discussion of this point see Sirovich (1967*b*).

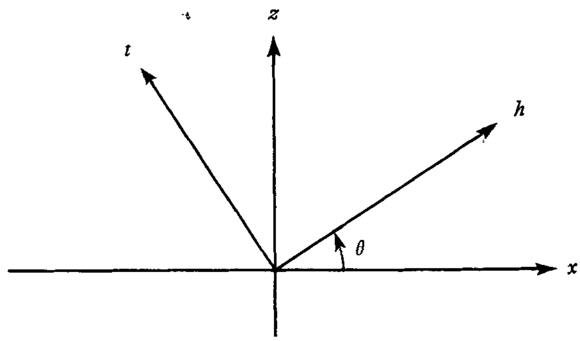

FIGURE 1. (h, t) co-ordinates, measuring distance along and normal to a given wave.

To centre attention on a particular wave, we transform from (x, y) co-ordinates to (n, t) co-ordinates, where h denotes distance along the wave and t distance across the wave (figure 1). θ is the angle the wave makes with the x-axis. Then the transformation is given by

$$\begin{pmatrix} h \\ t \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \tag{3.2}$$

or, in wave-number space, by

$$\begin{pmatrix} k_1 \\ k_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \tag{3.3}$$

where m is the wave-number across the layer and n is the wave-number along the layer. The angle θ corresponds to a root d of the equation $D_{ND}(k_1/k_3) = 0$ through the relation $\theta = -\tan^{-1} d$, where D_{ND} is the determinant of the matrix $\mathbf{M}_{ND}$.

The boundary-layer condition across a wave is

$$\frac{\partial}{\partial t} \gg \frac{\partial}{\partial n}, \tag{3.4}$$

which in wave-number space becomes $|m| \gg |n|$. From the normalization (2.19) this becomes

$$|n| \ll |m| \ll 1. \tag{3.5}$$

This is true since we are only interested in scale variations which are large compared to the 'mean free path'.

† The discontinuities are in the form of MHD Mach lines and wakes. We shall use the term 'wave' generically to refer to all such discontinuities.

The boundary-layer analysis is accomplished by taking the lowest-order non-dissipative and dissipative terms of D in (3.1) as being of the same order. We shall assume the applied magnetic field and free stream velocity are not aligned. The aligned fields case will be discussed in the next section.

We begin by considering the wake. Since it lies along the x axis the transformation (3.2), (3.3) is not necessary. It is easily found that the leading terms in D are

$$D \sim ik_1 k_3^4[A^2B_z^2\gamma U_x] + k_3^6\frac{1}{PrRe}[A^2B_z^2] \tag{3.6}$$

provided B_z and A are bounded away from zero. Since we require these terms to be of the same order, we obtain

$$k_1 \sim k_3^2, \tag{3.7}$$

which furnishes us with the relative order of k_1, k_3. With (3.6) we can easily obtain the lowest-order form of M^*, and from this the leading order in the wake is

$$\left.\begin{aligned} \rho &= \{(\gamma-1)m-E\}\left(\frac{PrRe}{4\gamma U_x x}\right)^{\frac{1}{2}}\exp\left(-\frac{z^2PrRe\gamma U_x}{4x}\right),\\ T &= -\rho. \end{aligned}\right\} \tag{3.8}$$

The magnetic field and velocity perturbations vanish to this order.

In the limit $1/PrRe \to 0$ this becomes

$$\left.\begin{aligned} \rho &= \frac{1}{\gamma U_x}\{(\gamma-1)m-E\}\,\delta(x),\\ T &= -\rho. \end{aligned}\right\} \tag{3.9}$$

This is the non-dissipative wake obtained by Salathe & Sirovich (1967).

The approach to the non-dissipative solution is not uniform. The non-dissipative solution is obtained in the limit $\epsilon = x/(PrRe) = 0$. Therefore, no matter how small $1/PrRe$ is, it is possible to choose x large enough that ϵ is no longer small. This effect has been noted previously by Sirovich (1961, 1967*b*) for gas dynamics.

We now turn to the analysis of the waves. Imposing the transformation (3.3) on D, we obtain

$$\begin{aligned} D \sim\; & inm^4\{4\cos\theta\sin^4\theta U_x^5 + \sin\theta(\sin^2\theta-\cos^2\theta)\,2\gamma U_x A^2B_xB_z \\ & -2\sin^2\theta\cos\theta U_x^3(\gamma+A^2) + 2\sin^2\theta\cos\theta\gamma U_x A^2(B_x^2-B_z^2)\} \\ & + in^2m^3\{-10\cos^2\theta\sin^3\theta U_x^5 + (\cos^4\theta\sin\theta - 4\cos^2\theta\sin\theta - \sin^5\theta) \\ & \times[U_x\gamma A^2 - U_x^3(\gamma+A^2)] + 4\sin\theta\cos^2\theta A^2B_z^2\gamma U_x \\ & + (\cos^5\theta - \sin^4\cos\theta)\,2U_x\gamma A^2B_xB_z\} + m^6\{(1/Re)\,U_x^2\sin^2\theta \\ & \times[\tfrac{7}{3}U_x^2\sin^2\theta - \gamma - A^2 - \tfrac{1}{3}A^2\sin^2(\theta-\psi)] + (\gamma/PrRe)\,[\sin^4\theta\,U_x^4 \\ & + A^2\sin^2(\theta-\psi) - \sin^2\theta U_x^2(1+A^2)] + (1/Rm)\sin^2\theta U_x^2[\sin^2\theta U_x^2-\gamma]\}. \end{aligned} \tag{3.10}$$

Although the nm^4 term is of lower order than the n^2m^3 term, the latter has been retained since the coefficient of the former vanishes at what we shall refer to as magnetosonic conditions (i.e. the transition between doubly hyperbolic and hyperliptic). As we shall see, a solution valid through magnetosonic conditions will be obtained.

Matching the leading dissipative and non-dissipative terms of D gives

$$n \sim m^2 \tag{3.11}$$

and at magnetosonic conditions

$$n^2 \sim m^3, \tag{3.12}$$

which furnishes us with the relative ordering of n and m. With this ordering we can obtain the lowest order form of M^*. As in a previous paper (Salathe & Sirovich 1967) the following simplified representation may be found:

$$M^*_{ij} \sim X_i Y_j,$$

where

$$\left.\begin{aligned}
Y_1 &= Y_4 = m^2 \frac{U_x^2}{\gamma}[(U_x^2-\gamma)\sin^4\theta - A^2\sin^2\theta\cos^2\theta],\\
Y_2 &= m^2[-U_x^3\sin^4\theta + U_x A^2 B_x^2 \sin^2\theta - U_x B_x B_z A^2 \sin\theta\cos\theta],\\
Y_3 &= m^2[U_x^3\sin^3\theta\cos\theta + U_x A^2 B_x B_z \sin^2\theta - U_x A^2 B_z^2 \sin\theta\cos\theta],\\
Y_5 &= im^3 U_x^2 A^2 \sin^2\theta[B_z\sin\theta + B_x\cos\theta],\\
\mathbf{X} &= \left[m^2, m^2\frac{Y_2}{Y_1}, m^2\frac{Y_3}{Y_1}, (\gamma-1)m^2, \frac{Y_5}{A^2Y_1}\right].
\end{aligned}\right\} \tag{3.13}$$

In each of the above elements of $\mathbf{M}^*$ the leading dissipative term is of higher order than the leading non-dissipative term, and so the dissipative terms have not been included.

Equation (3.10) for the denominator, and the expressions for the M^*_{ij}, can now be substituted into (3.1) to obtain the solution. The matrix elements M^*_{ij} contain only m^4 or m^5. Consequently, the solution can be expressed in terms of the integral

$$I = \frac{1}{(2\pi)^2}\iint \frac{e^{imt+inh}\, m\, dm\, dn}{n^2\Phi + nm\Theta - im^3\Psi} \tag{3.14}$$

and its derivative with respect to t. The definitions of Θ, Φ and Ψ are obvious from (3.10). This integral has been carried out previously by Sirovich (1961) and can be expressed in terms of Airy functions. It should be emphasized that in this form the solutions are uniformly valid through magnetosonic conditions.

If the integration is first carried out in the complex n plane, the poles can be found by factoring the denominator into the form

$$\Phi(n-n^+)(n-n^-), \tag{3.15}$$

where

$$n^{\pm} = \frac{m\Theta}{2\Phi}[1 - \pm(1 + 4i\Phi\Psi m/\Theta^2)^{\frac{1}{2}}]. \tag{3.16}$$

It can be shown that if $\Theta\Psi > 0$ the n^+ pole is in the upper half plane and the n^- pole is in the lower half plane, and that if $\Theta\Psi < 0$ the reverse is true. The pole corresponding to the wave being studied is the one which goes to zero as $\Psi \to 0$; this is the n^+ pole. Therefore, the contour should be closed in the half plane in which this pole lies. If the contour is completed in the upper half plane, $h > 0$; if the contour is completed in the lower half plane, $h < 0$. The sign of h, together with the orientation of the h, t co-ordinates, determines whether the wave extends upstream or downstream.

Although the non-dissipative analysis does not determine which portion of a given wave is the correct one, the choice can be made by other means, without referring to the dissipative theory (see, for example, McCune & Resler 1960; Salathe & Sirovich 1967). Since it is also determined by the sign of $\Theta\Psi$, this sign must be invariant with respect to changes in the relative value of the viscosity, thermal and electrical conductivity. But Ψ is of the form

$$\Psi = \frac{1}{Re}\Psi_1 + \frac{\gamma}{PrRe}\Psi_2 + \frac{1}{Rm}\Psi_3. \tag{3.17}$$

Therefore, for a given wave Ψ_1, Ψ_2, and Ψ_3 must all have the same sign. This statement is proved by Salathe (1965, appendix B).

Evaluating the integral I gives (Sirovich 1961, 1967 *b*).

$$I = \operatorname{sgn}[\Theta\Psi]\, H(h \operatorname{sgn}\Theta\Psi)\, \frac{iA_i(x)}{3^{\frac{1}{3}}\Phi^{\frac{1}{3}}\Psi^{\frac{2}{3}}h^{\frac{1}{3}}} \exp\left\{\frac{2}{27}\frac{\Theta^3 h}{\Phi^2\Psi} - \frac{1}{9}\frac{\Theta^2 t}{\Phi\Psi} + \frac{2}{27}\frac{\Phi t^3}{\Psi h^2} - \frac{1}{9}\frac{\theta t^2}{\Psi h}\right\}, \tag{3.18}$$

where $H(x)$ is the step function, $A_i(x)$ is the Airy integral, given by

$$A_i(x) = \frac{1}{2\pi i}\int_c \exp\left(\frac{\sigma^3}{3} - x\sigma\right) d\sigma, \tag{3.19}$$

and

$$x = \frac{1}{3^{\frac{4}{3}}}\left(\frac{\Phi^{\frac{2}{3}}t^2}{\Psi^{\frac{2}{3}}h^{\frac{4}{3}}} - \frac{\Theta t}{\Phi^{\frac{1}{3}}\Psi^{\frac{2}{3}}h^{\frac{1}{3}}} + \frac{\Theta^2 h^{\frac{2}{3}}}{\Phi^{\frac{4}{3}}\Psi^{\frac{2}{3}}}\right). \tag{3.20}$$

With this expression for the integral, the entire solution can be written down, uniformly valid through sonic conditions. However, because of the complex forms of these solutions, they will be exhibited only for non-magnetosonic conditions; that is, for the case Θ bounded away from zero. Then using the asymptotic expansion of the Airy integral, I is to lowest order (see for example, Miller (1946))

$$I \sim \frac{i}{(4\Theta\Psi\pi h)^{\frac{1}{2}}} \exp\left(-\frac{\Theta t^2}{4\Psi h}\right). \tag{3.21}$$

In a similar manner, its derivative can be found to lowest order

$$\frac{\partial I}{\partial t} \sim \frac{it}{4(\Theta\Psi\pi h)^{\frac{1}{2}} h^{\frac{3}{2}}}\left(\frac{1}{2}\frac{\Phi}{\Theta} - \frac{\Theta}{\Psi}\right) \exp\left(-\frac{\Theta t^2}{4\Psi h}\right). \tag{3.22}$$

With these reduced expressions for I and $\partial I/\partial t$, we obtain, for example,

$$\begin{aligned} u_z = {} & \{[U_y A^2 B_x B_z \sin^2\theta - U_x A^2 B_z^2 \sin\theta\cos\theta + U_x^3 \sin^3\theta\cos\theta]\,(m+E) \\ & -[U_x A^2 B_x B_z \sin^2\theta + \gamma U_x^2 \sin^3\theta\cos\theta]\, M_x \\ & +[(U_x^4 - \gamma U_x^2)\sin^4\theta - U_x^2 A^2 B_z^2 \sin^2\theta]\, M_z\} \frac{\exp(-\Theta t^2/4\Psi h)}{\sqrt{(4\pi\Theta\Psi h)}} \\ & -\{(U_x A^2 B_x - U_x\gamma A^2 B_x \sin^3\theta + U_x\gamma A^2 B_z \sin^2\theta\cos\theta\} \\ & \times F\, \frac{\Theta^{\frac{1}{2}} t \exp\,(\Theta t^2/4\Psi h)}{4\pi^{\frac{1}{2}}\Psi^{\frac{3}{2}}h^{\frac{3}{2}}}. \end{aligned} \tag{3.23}$$

The remaining lowest orders are given in Salathe (1965).)

We note that

$$\lim_{\Psi\to 0} \frac{\exp(-\Theta t^2/4\Psi h)}{(4\Theta\Psi\pi h)^{\frac{1}{2}}} = \frac{1}{\Theta}\delta(t). \tag{3.24}$$

Comparing this with the non-dissipative solution obtained by Salathe & Sirovich (1967), it is obvious that the two solutions are identical provided that

$$\Gamma d_\mu \cos^3\theta_\mu \prod_{\substack{l=1\\ l\neq\mu}}^{4} (d_\mu - d_l) = \Theta. \tag{3.25}$$

The function Θ is given explicitly in terms of $\sin\theta_\mu$, $\cos\theta_\mu$, but the left side of (3.25) involves all the roots, d_μ, of the fourth-order equation $D_{ND} = 0$, which cannot be solved conveniently. Equation (3.25) is proved in Salathe (1965, appendix C).

We note that the approach to the non-dissipative solution is non-uniform, in the same sense that we described for the wake solution.

4. Aligned fields case

For flows in the neighbourhood of the aligned fields case, $B_z \sim 0$, a special analysis is required. In this situation $k_1/k_3 = 0$ is a triple root of $D_{ND} = 0$, instead of a simple root. The two non-zero roots lead to structured waves and this is covered by the analysis of the last section. Hence we focus attention on the structure of the wakes. Applying the boundary-layer analysis to D we obtain

$$\begin{aligned} D \sim{} & ik_1^3k_3^2[-U_x^3\gamma - U_x^3A^2 + \gamma U_x A^2] + k_1^2k_3^4\left\{\frac{1}{Re}[-\gamma U_x^2 - A^2U_x^2]\right. \\ & \left. + \frac{1}{PrRe}[-\gamma U_x^2 - \gamma U_x^2A^2 + \gamma A^2] - \frac{1}{Rm}[\gamma U_x^2]\right\} + ik_1k_3^6\left\{\frac{1}{Re}\frac{1}{PrRe}\right. \\ & \left. \times[\gamma U_x A^2 + \gamma U_x] + \frac{1}{Re}\frac{1}{Rm}[\gamma U_x] + \frac{1}{Rm}\frac{1}{PrRe}[\gamma U_x]\right\} \\ & + k_3^8\frac{1}{Re}\frac{1}{PrRe}\frac{1}{Rm}[\gamma] = ik_3^2\Lambda(k_1 - i\alpha_1k_3^2)(k_1 - i\alpha_2k_3^2)(k_1 - i\alpha_3k_3^2), \end{aligned} \tag{4.1}$$

where $\Lambda = -U_x^3\gamma - U_x^3A^2 + \gamma U_x^3A^2$ and $i\alpha_1, i\alpha_2$, $i\alpha_3$ are the non-trivial roots of D.† The three factors occur because the wake now consists of the two waves which collapse on the x axis as $B_z \to 0$ as well as the entropy wake.

It is well known that if $\Lambda > 0$ ($U_x < 1/\{1/A^2 + 1/\gamma\}$) one of the waves collapses upstream, whereas if $\Lambda < 0$ they both collapse downstream (see, for example Salathe & Sirovich, 1967). Correspondingly, it can be shown that if $\Lambda < 0$ all the roots of (4.1) lie in the upper half of the complex k_1 plane, while if $\Lambda > 0$ one of the roots lies in the lower half plane.

† Fan (1964) states that the α_i are real. Although we have been able to show this in special cases, the general proof has eluded us. This assertion is plausible on physical grounds and we will henceforth assume that it is true.

In addition to the leading terms of the determinant, we need the leading terms of M^*. These are:

$$M^*_{11} = k_1^2k_3^2[-A^2U_x^2-(\gamma-1)\,U_x^2+(\gamma-1)\,A^2]+ik_1k_3^4 \times\left\{\frac{\gamma}{PrRe}[A^2U_x]+\frac{1}{Re}[(\gamma-1)\,U_x+A^2U_x]+\frac{1}{Rm}[(\gamma-1)\,U_x]\right\} + k_3^6\left\{\frac{1}{Re}\frac{1}{Rm}(\gamma-1)+\frac{1}{Re}\frac{\gamma}{PrRe}A^2\right\}, \tag{4.2}$$

$$M^*_{12} = M^*_{21} = k_1^2k_3^2[U_xA^2]+ik_1k_3^4\left\{\frac{\gamma}{PrRe}[-A^2]\right\},$$

$$M^*_{14} = M^*_{41} = k_1^2k_3^2[U_x^2-A^2]+ik_1k_3^4\left\{-U_x\left(\frac{1}{Re}+\frac{1}{Rm}\right)\right\}-k_3^6\frac{1}{Re}\frac{1}{Rm},$$

$$M^*_{22} = k_1^2k_3^2[-(\gamma+A^2)\,U_x^2] + ik_1k_3^4\left\{\frac{\gamma}{PrRe}[U_x+U_xA^2]+\frac{1}{Rm}[\gamma U_x]\right\}+k_3^6\left\{\frac{1}{Rm}\frac{\gamma}{PrRe}\right\},$$

$$M^*_{42} = (\gamma-1)\,M^*_{24} = k_1^2k_3^2(\gamma-1)\,U_xA^2,$$

$$M^*_{44} = k_1^2k_3^2[A^2-U_x^2-U_x^2A^2]+ik_1k_3^4\left\{[U_x+U_xA^2]\frac{1}{Re}+U_x\frac{1}{Rm}\right\}+k_3^6\frac{1}{Re}\frac{1}{Rm},$$

$$M^*_{15} = -k_3^2A^2M^*_{51} = ik_1^2k_3^3A^2U_x^2+k_1k_3^5A^2U_x\left(\frac{\gamma}{PrRe}+\frac{1}{Re}\right)-ik_3^7A^2\frac{\gamma}{PrRe}\frac{1}{Re},$$

$$M^*_{25} = -k_3^2A^2M^*_{52} = -ik_1^2k_3^3\gamma U_xA^2-k_1k_3^5\frac{\gamma}{PrRe}A^2,$$

$$M^*_{45} = -k_3^2A^2(\gamma-1)\,M^*_{54} = \left(k_1^2k_3^2U_x^2-ik_1k_3^4U_x\frac{1}{Re}\right)ik_3(\gamma-1)\,A^2,$$

$$M^*_{55} = \gamma k_3^2\left(ik_1U_x+\frac{1}{Re}k_3^2\right)\left(ik_1U_x+\frac{1}{PrRe}k_3^2\right).$$

Neglected entries are of higher order.

These elements can be written in the form

$$\left.\begin{aligned} M^*_{\mu\nu} &= \bar{M}_{\mu\nu}k_3^2(k_1-i\beta_1^{\mu\nu}k_3^2)\,(k_1-i\beta_2^{\mu\nu}k_3^2) \quad \text{for} \quad \mu,\quad \nu \leqslant 4;\quad \mu=\nu=5,\\ M^*_{5\nu} &= (\bar{M}_{5\nu}/ik_3)\,k_3^2(k_1-i\beta_1^{5\nu}k_3^2)\,(k_1-i\beta_2^{5\nu}k_3^2) \quad \text{for} \quad \nu = 1,4,\\ M^*_{\mu5} &= ik_3\bar{M}_{\mu5}k_3^2(k_1-i\beta_1^{\mu5}k_3^2)\,(k_1-i\beta_2^{\mu5}k_3^2) \quad \text{for} \quad \mu = 1,4, \end{aligned}\right\} \tag{4.3}$$

where the $\bar{M}_{\mu\nu}$ are constants.

Let $\mathbf{w} = \{\rho, u_x, u_z, T, b_x)$. Then we can obtain the wake solution in the form

$$w_\mu = \frac{1}{(2\pi)^2 i\Lambda}\left\{\sum_{\nu=1}^{4}\bar{M}_{\mu\nu}S_\nu+\bar{M}_{\mu5}S_5\frac{\partial}{\partial z}\right\} \times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{(k_1-i\beta_1^{\mu\nu}k_3^2)\,(k_1-i\beta_2^{\mu\nu}k_3^2)\exp{(ik_1x+ik_3z)}\,dk_1dk_3}{(k_1-i\alpha_1k_3^2)\,(k_1-i\alpha_2k_3^2)\,(k_1-i\alpha_3k_3^2)}$$

$$= \left\{\sum_{\nu=1}^{4}\bar{M}_{\mu\nu}S_\nu+\bar{M}_{\mu5}S_5\frac{\partial}{\partial z}\right\}\sum_{i=1}^{3}\operatorname{sgn}\alpha_i\,H(x\operatorname{sgn}\alpha_i)\frac{\prod_{m=1}^{2}(\alpha_i-\beta_m^{\mu\nu})}{\prod_{\substack{j=1\\ j\neq 1}}^{3}(\alpha_i-\alpha_j)} \times(4\pi\alpha_ix)^{-\frac{1}{2}}\exp\left(\frac{-z^2}{4\alpha_ix}\right). \tag{4.4}$$

The solution exhibits three structured layers, all downstream for $\Lambda < 0$, two downstream and one upstream for $\Lambda > 0$. Before discussing this solution we shall write down the gasdynamic limit, obtained by setting $A = 0$. The denominator becomes a quadratic, factoring exactly with one factor containing only $1/Re$, the other $1/PrRe$. We find

$$\left.\begin{aligned} \rho &= \frac{H(x)}{\gamma U_x}[(\gamma-1)M-E]\left(\frac{U_x PrRe}{4\pi x}\right)^{\frac{1}{2}}\exp\left(-\frac{z^2 PrRe U_x}{4x}\right), \\ T &= -\rho, \\ u_x &= \frac{M_x}{U_x}H(x)\left(\frac{U_x Re}{4\pi x}\right)^{\frac{1}{2}}\exp\left(-\frac{z^2 U_x Re}{4x}\right), \end{aligned}\right\} \quad (4.5)$$

which is the same as found by Sirovich (1967*b*). We see from this that the gasdynamic wake consists of two distinct layers, one carrying only vorticity and structured only by viscosity, the other an entropy wake structured by thermal conductivity.

The magnetohydrodynamic wake does not, in general, exhibit this splitting of effects. Except under special conditions all three wakes are structured by all three dissipative parameters, and all three carry vorticity, current and entropy disturbances.

In the limit $\gamma \to 1$ the energy equation decouples so that temperature can be solved for separately (although the other variables still depend on the energy equation). This is manifested in the wake solution in a number of ways. The denominator now contains the factor $\{k_1 - i(1/U_x PrRe)k_3^2\}$ times a quadratic containing only Re and Rm. Hence one wake is structured only by thermal conductivity while the other two are structured by viscosity and electrical conductivity. Furthermore, inspection of the cofactor matrix elements shows that this wake is excited only by the energy source, $S_4 = H$, although it carries all the disturbances (i.e. ρ, u_x, T, b_x). We also note that temperature depends only on H and is carried only in this heat conduction wake:

$$T = \frac{H}{U_x}H(x)\left(\frac{U_x PrRe}{4\pi x}\right)^{\frac{1}{2}}\exp\left(-\frac{z^2 U_x PrRe}{4x}\right). \quad (4.6)$$

Non-dissipative limit

If the equations governing non-dissipative aligned fields magnetohydrodynamic flow are considered, we are led to the evaluation of the following integral:

$$\Phi = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\exp(ik_1 x + ik_3 z)\,dk_1 dk_3}{k_1^3(k_1^2 + d^2 k_3^2)}. \quad (4.7)$$

The third-order pole at $k_1 = 0$ corresponds to the wake, while the poles at $k_1/k_3^2 = \pm id$ correspond to the remaining (hyperbolic or elliptic) flow field.

This expression is, in fact, the inviscid counterpart of the integral which occurs in (4.4). In order to evaluate (4.7) it is necessary to write, instead of k_1^3 in the denominator, the expression $(k_1 - i\epsilon_1)(k_1 - i\epsilon_2)(k_1 - i\epsilon_3)$ where the ϵ_i are real. (They can, in fact, be complex, but (4.4) implies that they be real.) If a flow has only downstream wakes the ϵ_i are all positive and the value of Φ is independent of the way the ϵ_i vanish. On the other hand, if an upstream wake occurs one of the ϵ_i must be negative (the other two must be positive) and a simple calculation shows that an infinite variety of forms for Φ is possible as the ϵ_i vanish at different rates. It is clear from (4.4) that the correct non-dissipative solution follows only by considering the ratios of the dissipative parameters as they vanish. Therefore the correct non-dissipative solution follows only from a consideration of the underlying dissipative problem, for flows where an upstream wake occurs.

In a paper on the non-dissipative theory (Salathe & Sirovich 1967) we obtained an aligned fields wake by letting the angle between the applied magnetic field and the free stream velocity approach zero. The effect of non-alignment is to split the triple pole into three simple poles along the real axis, one being at the origin. The integration can then be performed because it is again clear physically whether the path should go under or over the poles. The aligned fields solution obtained in this way agrees with the above-mentioned solutions when all wakes are downstream ($\Lambda < 0$), but when one wave collapses upstream ($\Lambda > 0$) the solution obtained cannot be realized as a non-dissipative limit of our dissipative aligned fields wake. Therefore this method of obtaining the solution is invalid, as are the other methods which have been proposed (Sears & Resler 1959; Stewartson 1960; Leibovich & Ludford 1966).

We shall now consider various non-dissipative limits of the dissipative solution. The determinant D can be abbreviated in the form

$$D \sim k_3^8(ik^3\Lambda + k^2\mathscr{B} + ik\mathscr{C} + \mathscr{D}), \tag{4.8}$$

where $k = k_1/k_3^2$ and the definitions of $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ are obvious from (4.1). If one of the dissipative parameters ($1/Re$, $1/Rm$, $1/RePr$) approaches zero, $\mathscr{D} \to 0$, one layer collapses to a non-dissipative wake, and D becomes a quadratic. It is easy to see that the roots of the remaining quadratic are both in the upper half plane if $\Lambda < 0$, and one in each half plane if $\Lambda > 0$. Hence it is always one of the downstream wakes which collapses. Letting a second parameter approach zero, we obtain two-non-dissipative wakes, and the third still structured. The root of D is now $k = i\mathscr{B}/\Lambda$.

It is easy to see that $\Lambda < 0$ implies $\mathscr{B} < 0$ so that the root is in the upper half plane. However, if $\Lambda > 0$ we can have either $\mathscr{B} < 0$ or $\mathscr{B} > 0$, the latter being the case only when $U_x < 1/\{1+1/A^2\}^{\frac{1}{2}} = P$ and the $1/PrRe$ term in $\mathscr{B}$ dominates over the $1/Re$ and $1/Rm$ terms. Except for this case, we have that the last remaining diffusing wake is upstream. When heat conduction dominates and $0 < U < P$ the flow pattern consists of one upstream and one downstream non-dissipative wake and a downstream diffusing wake.

The point $\Lambda = 0$ corresponds to the cusp along the B-axis on the second Friedrichs diagram, and is given in terms of dimensional variables by

$$U^2 = 1/\{1/a^2 + 1/\mathscr{A}^2\}$$

where $\mathscr{A}$ is the Alfvén speed and a the isothermal sound speed. The point P is obtained by replacing a by c, the adiabatic sound speed.

A flow field containing some diffusing, some non-diffusing wakes is not contradictory, since the non-diffusing wakes always carry only those disturbances which would not disperse under the action of the remaining dissipation. For example, if heat conduction is the only dissipation present, the two non-dissipative wakes contain vorticity and current, but no temperature disturbances.

We shall now give the solutions in some limiting cases. Suppose $RePr \to \infty$ (i.e. $\kappa \to 0$). One of the roots of D, say α_3, approaches zero and the wake becomes a delta function. However, as has been pointed out before, α_i appears multiplied by x in the solution, so no matter how small α_i is we may choose x such that $\alpha_i x$ is not small. As $\alpha_3 \to 0$, therefore, we do not obtain the delta function, but a structured wake very much thinner than the other two. We shall therefore retain the lowest order in α_3 and display the structure of this thin sublayer.

In the limit $RePr \to \infty$ we obtain immediately from (4.1) that $\alpha_3 = 1/(U_x PrRe)$. The leading terms for this portion of the wake are

$$\left.\begin{aligned} \rho &= \frac{1}{\gamma U_x}[(\gamma-1)M-E]\,H(x)\left(\frac{U_x PrRe}{4\pi x}\right)^{\frac{1}{2}}\exp\left(-\frac{U_y PrRez^2}{4x}\right),\\ T &= -\rho. \end{aligned}\right\} \tag{4.9}$$

(The factor in the brackets approaches $\delta(z)$ as $x/PrRe \to 0$.) This is the entropy wake that appeared in the non-aligned fields case, and is also one of the two wakes that appear in the gasdynamic solution.

Next consider the limiting case $1/Re \ll 1$, i.e. the limit of vanishing viscosity. We obtain (for the collapsing wave only) to lowest order

$$u_x = M_x H(x)\left(\frac{U_x Re}{4\pi x}\right)^{\frac{1}{2}}\exp\left(-\frac{U_x Rez^2}{4x}\right). \tag{4.10}$$

This will be recognized as one of the two wakes that occur in gasdynamics (i.e. the vorticity wake). The collapsing wake for $1/Rm \ll 1$ is given by

$$\left.\begin{aligned} \rho &= \frac{A^2}{U_x(A^2+1)}MH(x)\left[\frac{U_x(1+A^2)\,Rm}{4\pi x}\right]^{\frac{1}{2}}\exp\left(-\frac{U_x(1+A^2)\,Rmz^2}{4x}\right),\\ b_x &= -\frac{1}{A^2}\rho. \end{aligned}\right\} \tag{4.11}$$

For the two cases $1/Rm \gg 1/Re \gg 1/PrRe$ and $1/Rm \gg 1/PrRe \gg 1/Re$ we have two downstream wakes, one much thinner than the other, and both much thinner than the upstream wake. One of these waves is given by (4.9) and the other by (4.10). The upstream wake can easily be computed, but since it does not assume any particularly special form we shall not write it out. We note, however, that it does carry vorticity and entropy, which are diffused by Rm. This is a result of the coupling that occurs in magnetohydrodynamics.

Consider now the two cases: (1) $1/PrRe \gg 1/Rm \gg 1/Re$, and (2) $1/PrRe \gg 1/Re \gg 1/Rm$. We again have three wakes, each a different order of thickness. The thickest wake, however (the one structured by $PrRe$) may now be either downstream or upstream, as pointed out previously. The thinnest wake is always downstream and the remaining one is in the opposite direction to the $PrRe$ wake.

In summary, for $\Lambda > 0$ the non-dissipative wake solution ($x/Re \to 0$, etc.) consists of a downstream and upstream delta function singularity. However, the strength of these singularities, what disturbances they carry, and what sources they depend on are related to how the dissipation was taken to approach zero. This is important, since in actual fact the dissipation is never zero; some dissipative effects may dominate over others and in each case the non-dissipative limit has a different meaning.

We saw that one solution would be obtained if electrical diffusion dominated, another if vorticity diffusion dominated. Two different solutions are obtained if thermal diffusion is dominant, depending on the relative value of U_x. Other limits could be obtained, by, for example, letting all dissipation approach zero while remaining of the same order.

5. The decoupled mode

In this section equations (2.12) for the perturbation of velocity and magnetic field in the y direction are solved. The transformed equations are

$$\begin{bmatrix} ik_1 U_x & -ik_1 B_x - ik_3 B_z \\ -ik_1 A^2 B_x - ik_3 A^2 B_z & ik_1 U_x \end{bmatrix} \begin{bmatrix} b_y \\ U_y \end{bmatrix} + \begin{bmatrix} (1/Rm)(k_1^2+k_3^2) & 0 \\ 0 & (1/Re)(k_1^2+k_3^2) \end{bmatrix} \begin{bmatrix} b_y \\ U_y \end{bmatrix} = \begin{bmatrix} ik_1 F_z - ik_3 F_x \\ M_y \end{bmatrix}. \quad (5.1)$$

Following the procedure of §3, we have for the leading term of the determinant

$$\begin{aligned} D \sim mn\{&2\cos\theta\sin\theta(U_x^2 - A^2B_x^2) + (\cos^2\theta - \sin^2\theta)\, 2A^2B_xB_z \\ &+ 2\sin\theta\cos\theta\, A^2B_z^2\} - im^3U_x\sin\theta\left(\frac{1}{Re}+\frac{1}{Rm}\right) \\ = mnP &- im^3Q \end{aligned} \quad (5.2)$$

valid for A and B_z bounded away from zero.

Integrating first with respect to m, we find that the pole is in the upper or lower half plane depending on whether QP is positive or negative, respectively. Since the contour of integration is closed on the half plane containing the pole, we therefore conclude that $h > 0$ if $QP > 0$, $h < 0$ if $QP < 0$.

If the transposed cofactor matrix is also transformed and only the leading term retained, the dissipative solution is found to be

$$\left.\begin{aligned} b_y &= H[h\,\mathrm{sgn}\,(QP)]\,\mathrm{sgn}\,(QP)\left\{(B_x\sin\theta - B_z\cos\theta)\frac{M_y}{(4\pi PQh)^{\frac{1}{2}}}\right. \\ &\quad \left. + (\sin\theta\, F_z + \cos\theta F_x)(U_x\sin\theta)\frac{t}{4\pi^{\frac{1}{2}}}\frac{1}{Qh}\left(\frac{P}{Qh}\right)^{\frac{1}{2}}\right\}\exp\left(-\frac{Pt^2}{4Qh}\right), \\ u_y &= H[h\,\mathrm{sgn}\,(QP)]\,\mathrm{sgn}\,(QP)\left\{U_x\sin\theta\frac{M_y}{(4\pi PQh)^{\frac{1}{2}}}\right. \\ &\quad \left. + A^2(B_x\sin\theta - B_z\cos\theta)(\sin\theta\, F_z + \cos\theta F_x)\frac{t}{4\pi^{\frac{1}{2}}}\frac{1}{Qh}\left(\frac{P}{Qh}\right)^{\frac{1}{2}}\right\}\exp\left(-\frac{Pt^2}{4Qh}\right). \end{aligned}\right\} \quad (5.3)$$

In the limit $R \to \infty$, the non-dissipative solution obtained by Salathe & Sirovich (1967) is recovered as a non-uniform limit.

Aligned fields

When $B_z = 0$, we have

$$D = (A^2 - U_x^2)\,k_1^2 + ik_1U_x\left(\frac{1}{Re}+\frac{1}{Rm}\right)k_3^2 + \frac{1}{Re}\frac{1}{Rm}k_3^4$$
$$= (A^2 - U_x^2)\,(k_1 - i\alpha_1 k_3^2)\,(k_1 - i\alpha_2 k_3^2).$$

In this case we can have α_1, α_2 positive ($U_x^2 > A^2$) or one positive, one negative ($U_x^2 < A^2$). The solution is

$$u_y = \sum_{i=1}^{2} \operatorname{sgn}(\alpha_i)\,H[x(\operatorname{sgn}\alpha_i)]\,\frac{(-)^l}{U_x^2 - A^2}\left\{U_x M_y \frac{\alpha_i - (1/U_x Rm)}{\alpha_2 - \alpha_1} + A^2\left(F_z\frac{\partial}{\partial x} - F_x\frac{\partial}{\partial z}\right)\frac{\alpha_i}{\alpha_2 - \alpha_1}\right\}\frac{\exp(-z^2/4\alpha_i x)}{(4\pi\alpha_i x)^{\frac{1}{2}}},$$

$$b_y = \sum_{i=1}^{2} \operatorname{sgn}(\alpha_i)\,H[x(\operatorname{sgn}\alpha_i)]\,\frac{(-)^l}{U_x^2 - A^2}\left\{M_y\frac{\alpha_i}{\alpha_2 - \alpha_1} + U_x\left(F_z\frac{\partial}{\partial x} - F_x\frac{\partial}{\partial z}\right)\frac{\alpha_i - (1/U_x Re)}{\alpha_2 - \alpha_1}\right\}\frac{\exp(-z^2/4\alpha_i x)}{(4\pi\alpha_i x)^{\frac{1}{2}}}.$$

As before, the wake consists of two structured layers, either both downstream or one upstream and one downstream. Each wake is structured by both electrical conductivity and viscosity.

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