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# Non-linear effects in steady supersonic dissipative gasdynamics
## Part 1. Two-dimensional flow

By T. H. CHONG AND L. SIROVICH
Division of Applied Mathematics and Centre for Fluid Dynamics, Brown University

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Steady supersonic two-dimensional flows governed by the Navier–Stokes equations are considered. For flows past a thin body, the Oseen theory is shown to fail at large distances. An investigation of the equations bridging the linear and non-linear zones is made. From this, it follows that the resulting equations are a system of Burgers and diffusion equations. The Whitham theory is shown to result under the inviscid limit of our analysis. Various other limits are also obtained.

An explicit expression for flows past a thin airfoil is given, and the flow past a double wedge is exhibited in terms of known functions.

## 1. Introduction

It is our intention in the present investigation to consider simultaneously the effects of dissipation and non-linearity in two-dimensional supersonic flows. With the exception of the boundary layer, we will be interested in the total flow field past a body.

Steady dissipative linearized flows have been considered extensively (Sirovich 1968). For supersonic flow this investigation describes the flow field in terms of upper and lower Mach zones and a wake structured by entropy and vorticity behind the body. In a previous investigation (Chong & Sirovich 1970) we pointed out that, in general, the two-dimensional theory breaks down at sufficiently large distances from a body. In particular, if $h$ denotes the distance along a Mach line from the body, linear theory fails when $h \to \infty$. In § 3 a detailed demonstration of this is given.

Although linear theory does break down, one can anticipate that a simplified non-linear theory should govern the far flow field. Perturbations at large distances can be expected to be small, a fact which is supported by the linear theory. In § 4, we demonstrate that this is in fact the case and that the far field Mach zones are governed by equations which can be reduced to the Burgers equation (Burgers 1948; Cole 1951; Hopf 1950; Lighthill 1956). The breakdown of the linear theory at large distances has to some extent been anticipated by Ryzhov & Terent'ev (1967) in their treatment of the transonic problem. Also the validity of the Burgers equation in the far field region has been indicated by Lighthill (1956), who considered a propagating wave in one-dimensional unsteady gasdynamics (see also Hayes 1960). Allied treatments have also been given by Moran & Shen (1966), Su & Gardner (1969) and Parker (1969).

The need for the inclusion of non-linear effects in steady supersonic flow was pointed out by Whitham (1950, 1952, 1956), who considered inviscid theory. We show in § 5 that the Whitham theory follows from our analysis under the inviscid limit. This is interesting since our method of investigation is substantially different from his. Also in this connexion it is important to note that the inviscid limit is non-uniform. The breakdown of inviscid theory occurs in shock regions of course, but also in the distant flow field. The entire distant flow field is a region of competing viscous and non-linear terms. In a number of situations, the distant flow can be entirely viscous, and hence linear.

The basis for our method of investigation lies in searching for the equations bridging the linear and non-linear zones. This search leads to the above mentioned Burgers equations. An additional property is that the resulting equations are valid for $h \to 0$. That is, our analysis reduces to linear inviscid or viscous theory as the case requires – regions of abrupt change require viscous terms while slowly varying portions are influenced by the inviscid terms. Thus, we are able to give a representation which provides a description of the flow field, with the exception of the boundary layer.

As an illustration of our method, we solve for the flow past a thin airfoil in § 6. This solution is given in some detail. Also considered is the thin diamond-shaped airfoil for which a more explicit representation is given.

## 2. Governing equations

We consider steady two-dimensional supersonic flow past a body. The upstream velocity $\tilde{\mathbf{u}}_0$ is uniform in the $x$ direction. The equations of motion are

$$\left.\begin{aligned} \tilde{\nabla}\,.\,(\tilde{\rho}\tilde{\mathbf{u}}) &= 0,\\ \tilde{\nabla}\,.\,(\tilde{\rho}\tilde{\mathbf{u}}\tilde{\mathbf{u}}+\tilde{p}\mathbf{l}-\tilde{\mathbf{P}}) &= 0,\\ \tilde{\nabla}\,.\,(\tilde{\rho}\tilde{\mathbf{u}}(\tilde{e}+\tfrac{1}{2}\tilde{u}^2)+\tilde{p}\tilde{\mathbf{u}}-\tilde{\mathbf{P}}\,.\,\tilde{\mathbf{u}}-\kappa\tilde{\nabla}\tilde{T}) &= 0,\\ \tilde{P}_{ij} &= \mu(\tilde{u}_{i,j}+\tilde{u}_{j,i})+(\beta-\tfrac{2}{3}\mu)\,\tilde{\nabla}\,.\,\tilde{\mathbf{u}}\,\delta_{ij}. \end{aligned}\right\} \tag{1}$$

Introducing a normalization with respect to upstream density $\tilde{\rho}_0$, temperature $\tilde{T}_0$, isothermal speed of sound $\tilde{a}_0$, and a length scale $L$ to be specified later, we define the dimensionless quantities

$$\left.\begin{aligned} \mathbf{x} = \tilde{\mathbf{x}}/L, \quad \rho = \tilde{\rho}/\tilde{\rho}_0, \quad \mathbf{u} = \tilde{\mathbf{u}}/\tilde{a}_0, \quad \mathbf{P} = \tilde{\mathbf{P}}/\tilde{\rho}_0\tilde{a}_0^2.\\ \mathbf{U} = \tilde{\mathbf{u}}_0/\tilde{a}_0, \quad T = (c_v/\tilde{a}_0^2\tilde{T}_0)^{\frac{1}{2}}\tilde{T}, \quad c_v = (\partial\tilde{e}/\partial\tilde{T})_{\rho_0}, \quad \tilde{a}_0 = [(\partial\tilde{p}/\partial\tilde{\rho})_{T_0}]^{\frac{1}{2}}. \end{aligned}\right\} \tag{2}$$

This particular normalization was chosen to correspond to our linearized viscous theory (Sirovich 1968). Although the nature of the fluid can be arbitrary, for simplicity of calculations we consider a perfect gas which satisfies

$$p = R\rho T, \quad e = c_v T, \tag{3}$$

where $e$ is the internal energy and $c_v$ is a constant independent of temperature.

In terms of these normalized variables, our governing equations become

$$\left.\begin{aligned} &\nabla.(\rho\mathbf{u}) = 0,\\ &\rho(\mathbf{u}.\nabla)\mathbf{u}+\chi\nabla(\rho T)-\nabla.\mathbf{P} = 0,\\ &\rho(\mathbf{u}.\nabla)T+\chi^2\rho T\nabla.\mathbf{u}-\chi\mathbf{P}:\nabla\mathbf{u}+\nabla.(\xi\nabla T) = 0,\\ &P_{ij} = \zeta(u_{i,j}+u_{j,i}+(\nabla.\mathbf{u})\,\delta_{ij})+\hat{\eta}(\nabla.\mathbf{u})\,\delta_{ij},\\ \text{where}\quad &\xi = \kappa/c_v L\tilde{\rho}_0\tilde{a}_0,\quad \zeta = \mu/L\tilde{\rho}_0\tilde{a}_0,\quad \hat{\eta} = (\beta+\tfrac{1}{3}\mu)/L\tilde{\rho}_0\tilde{a}_0\end{aligned}\right\} \tag{4}$$

and $$\chi^2 = \gamma-1,\quad \gamma = c_p/c_v.$$

## 3. Linearized problems

The solution to the two-dimensional linearized Oseen problem was discussed earlier (Sirovich 1968). The solution of supersonic flow past a body can be represented in terms of the fundamental matrix solution $\mathbf{V}$ by

$$\mathbf{v} = \int_S \mathbf{V}(\mathbf{s}).\mathbf{F}(\mathbf{x}-\mathbf{s})\,dS(\mathbf{s}), \tag{5}$$

where $$\mathbf{F} = (0, (p\mathbf{l}-\mathbf{P}).\mathbf{n}, \{\mathbf{Q}-\mathbf{U}.(p\mathbf{l}-\mathbf{P})\}.\mathbf{n}).$$

The integration is over the body of surface $S$ and normal $\mathbf{n}$. From (5) we see that the solution is explicitly represented in terms of the heat flow, stress and flow variables at the surface. The fundamental matrix solution is given by [see Sirovich (1968) for the detailed derivation],

$$\mathbf{V}(\mathbf{x}) \sim \sum_{i=1}^{4} I^i(\mathbf{x})\,\boldsymbol{\omega}^i\boldsymbol{\omega}^i, \tag{6}$$

where

$$I^1 = \frac{H(x)\exp[-y^2\gamma U/4x\xi]}{(4\pi\xi\gamma Ux)^{\frac{1}{2}}},\quad \boldsymbol{\omega}^1 = [\chi, 0, 0, -1],$$

$$I^2 = \frac{H(x)\exp[-y^2U/4x\zeta]}{(4U\zeta\pi x)^{\frac{1}{2}}},\quad \boldsymbol{\omega}^2 = [0, 1, 0, 0],$$

$$I^{3,4} = \frac{H(x)\,M^2\exp[-\{y\mp x(M^2-1)^{\frac{1}{2}}\}^2/\alpha x]}{2\gamma^2U(M^2-1)\,(\pi\alpha x)^{\frac{1}{2}}},$$

$$\boldsymbol{\omega}^{3,4} = [\pm\gamma^{\frac{1}{2}}, \mp\gamma/M, \gamma(M^2-1)^{\frac{1}{2}}/M, \pm\chi\gamma^{\frac{1}{2}}],$$

$$\alpha = 2M^3\{\gamma(\eta+\zeta)+(\gamma-1)\,\xi\}/(M^2-1)^2\gamma^{\frac{3}{2}}.$$

The $\boldsymbol{\omega}^i$ are constant vectors, $I^1$ and $I^2$ govern the entropy and vorticity wakes respectively, and $I^3$, $I^4$ are the Mach regions above and below the body.

For a flow past a thin body given by

$$y^{\pm} = \pm\epsilon f^{\pm}(x)\quad (0 \leqslant x \leqslant l), \tag{7}$$

we can expand the source terms in powers of $\epsilon$. The leading term of (5) is then

$$\mathbf{v} \sim \int \mathbf{V}(s,y)\,\mathbf{M}^0(x-s)\,ds, \tag{8}$$

where $$\mathbf{M}^0 = (0, \mathbf{e}_y.[p\mathbf{l}-\mathbf{P}], \mathbf{e}_y[\mathbf{Q}+\mathbf{P}.\mathbf{U}]).$$

$\mathbf{e}_y$ is the unit vector in the $y$ direction, and the square brackets denote jumps across the thin body.

Since our aim now is to consider the limits of accuracy of the linear theory, we avoid the difficult problem of solving equation (8) for a particular problem. Instead, for purposes of illustration, we consider flow due to a distribution of sources. For simplicity, we take only a pressure jump. That is, corresponding to a thin body of length $l$, we take

$$[p] = \epsilon\{H(x)-H(x-l)\}, \quad [\mathbf{P}] = 0, \quad [\mathbf{T}] = 0, \tag{9}$$

where $H(x)$ is the Heaviside function. (This is related to flow past a finite wedge of angle $\epsilon$.) Also for this section, we take the normalization length in (2) to be that of the body, so that $l = 1$.

Substituting (9) into (8) we find

$$u \sim \frac{\epsilon H(x)}{2U(M^2-1)^{\frac{1}{2}}} \int_{x-1}^{x} \exp\left\{-\frac{R[y-s/(M^2-1)^{\frac{1}{2}}]^2}{s}\right\} \left(\frac{R}{\pi s}\right)^{\frac{1}{2}} ds. \tag{10}$$

$R$ $(= 1/\alpha)$ is effectively the Reynolds number when $L$ is taken to be the body length. Certain simplifications result when $R \gg 1$ is imposed on (10), however we do not pursue this.

The representative assumption underlying the linearization of the Navier–Stokes equations is that

$$u\frac{\partial u}{\partial x} = o\left(\frac{1}{R}\frac{\partial^2 u}{\partial x^2}\right). \tag{11}$$

If we consider the very distant flow field, $x \gg R$ and $y - x/(M^2-1)^{\frac{1}{2}} = O(x/R)^{\frac{1}{2}}$ we easily find from (10) that

$$u = O(\epsilon(R/x)^{\frac{1}{2}}), \quad u_x = O(\epsilon R/x), \quad u_{xx} = O(\epsilon(R/x)^{\frac{3}{2}}), \tag{12}$$

so that taking a derivative increases the order by the factor $(R/x)^{\frac{1}{2}}$.

Comparing condition (11) with distant flow field (12) we see that the linear theory is not self-consistent if

$$\epsilon R \geqslant O(1). \tag{13}$$

Since (13) seems to typify the condition met in practice, we regard the linear theory as breaking down on approaching the far field and try to determine the location of this breakdown.

We consider (10) for $R \gg 1$, $R \gg x$ and $y - x/(M^2-1)^{\frac{1}{2}} = O((x/R)^{\frac{1}{2}})$. Then (10) may be evaluated by Laplace's formula and we find

$$u = O(\epsilon);$$

also

$$u_x = O(\epsilon(R/x)^{\frac{1}{2}}), \quad u_{xx} = O(\epsilon R/x).$$

Again a derivative introduces the factor $(R/x)^{\frac{1}{2}}$. Imposing condition (11) we see that breakdown occurs at

$$Rx = O(1/\epsilon^2) \tag{14}$$

and the flow is self-consistently linear for $Rx \ll 1/\epsilon^2$.

Studying the above analysis, we see that breakdown occurs in the neighbourhood of the shock wave, that is, the linearized location of the shock wave. Also condition (14) does not really place this breakdown very far from the body since the product $Rx$ is effectively the ratio of distance to mean free path. The cause of breakdown arises when non-linear steepening becomes competitive with the diffusive broadening. Away from the region of the shocks, the flow is still linear (and really inviscid).

From Whitham's investigation of inviscid theory (Whitham 1950, 1952, 1956) we know that linearized theory breaks down due to secularities – more precisely the accumulated effect of local changes in the characteristic direction. In anticipation of the results of subsequent sections, we remark that the non-linear rectification of the shock zone will also include corrections of the Whitham type. (Actually the scaling and perturbation procedure is altered in the inviscid case, but we do not go into this here.)

## 4. Method of multiple scales

We are now interested in finding a flow description which bridges the regions $Rx = O(1/\epsilon)$ with $Rx = O(1/\epsilon^2)$. That is, we wish to obtain the connexion between the linear zone and the non-linear zone. For this purpose, we redefine the scale $L$ so that the dissipative dimensionless ratios in (14) are $O(1)$. Instead of introducing tiresome variable changes, we now consider $\mathbf{x}$ as the spatial variable in the new normalization and seek a description which bridges the ranges $x = O(1/\epsilon)$ and $x = O(1/\epsilon^2)$.

Starting with the small parameter $\epsilon$ (e.g. in flow past a thin airfoil, we would take the thickness ratio of the airfoil to be $\epsilon$), we introduce the multiple scales

$$\mathbf{x}_i = \epsilon^i \mathbf{x} \quad \text{for} \quad i = 1, 2, 3, \ldots \tag{15}$$

and the corresponding perturbation series

$$\left.\begin{aligned} \rho &= 1 + \epsilon\rho_1 + \epsilon^2\rho_2 + \ldots, \\ \mathbf{u} &= \mathbf{U} + \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \ldots, \\ T &= (1/\chi) + \epsilon T_1 + \epsilon^2 T_2 + \ldots, \\ \nabla &= \epsilon\nabla_1 + \epsilon^2\nabla_2 + \ldots. \end{aligned}\right\} \tag{16}$$

All variables are assumed to be functions of $\mathbf{x}_i$, $i = 1, 2, \ldots$.

The temperature dependence of the dissipative parameters will be immaterial up to the required perturbation order; i.e. setting

$$\left.\begin{aligned} \beta &= \beta_0 + \beta_1(T - T_0) + \ldots, \\ \mu &= \mu_0 + \mu_1(T - T_0) + \ldots, \\ \kappa &= \kappa_0 + \kappa_1(T - T_0) + \ldots, \end{aligned}\right\} \tag{17}$$

we will only need $\beta_0$, $\mu_0$ and $\kappa_0$. Hence, we avoid unnecessary details and take

$$\xi = \kappa_0/c_v L\tilde{\rho}_0\tilde{a}_0, \quad \zeta = \mu_0/L\tilde{\rho}_0\tilde{a}_0, \quad \eta = (\beta_0 + \tfrac{1}{3}\mu_0)/L\tilde{\rho}_0\tilde{a}_0. \tag{18}$$

Substituting expansions (16) into the governing equations (4) and formally carrying out the perturbation procedure, we get

$$\epsilon^2\{\nabla_1.\mathbf{u}_1 + \mathbf{U}.\nabla_1\rho_1\} + \epsilon^3\{\nabla_2.\mathbf{u}_1 + \nabla_1.\mathbf{u}_2 + \mathbf{U}.\nabla_2\rho_1 + \mathbf{U}.\nabla_1\rho_2 + \nabla_1.(\rho_1\mathbf{u}_1)\} + O(\epsilon^4) = 0, \tag{19}$$

$$\begin{aligned} &\epsilon^2\{\nabla_1\rho_1 + (\mathbf{U}.\nabla_1)\mathbf{u}_1 + \chi\nabla_1 T_1\} + \epsilon^3\{\nabla_1\rho_2 + (\mathbf{U}.\nabla_1)\mathbf{u}_2 + \chi\nabla_1 T_2 + \nabla_2\rho_1 \\ &\quad + \mathbf{U}.\nabla_2 u_1 + \chi\nabla_2 T_1 + \rho_1(\mathbf{U}.\nabla_1)\mathbf{u}_1 + \chi\rho_1\nabla_1 T_1 + (\mathbf{u}_1.\nabla_1)\mathbf{u}_1 + T_1\nabla_1\rho_1 \\ &\quad - \zeta\nabla_1^2\mathbf{u}_1 - \eta\nabla_1(\nabla_1.\mathbf{u}_1)\} + O(\epsilon^4) = 0, \end{aligned} \tag{20}$$

$$\begin{aligned} &\epsilon^2\{\mathbf{U}.\nabla_1 T_1 + \chi\nabla_1.\mathbf{u}_1\} + \epsilon^3\{\mathbf{U}.\nabla_1 T_2 + \chi\nabla_1.\mathbf{u}_2 + \mathbf{U}.\nabla_2 T_1 + \mathbf{u}_1.\nabla_1 T_1 \\ &\quad + \rho_1\mathbf{U}.\nabla_1 T_1 + \chi\nabla_2.\mathbf{u}_1 + \chi^2 T_1\nabla_1.\mathbf{u}_1 + \chi\rho_1\nabla_1.\mathbf{u}_1 - \xi\nabla_1^2 T_1\} + O(\epsilon^4) = 0. \end{aligned} \tag{21}$$

*Lowest-order problem*

For conciseness, we write $\mathbf{v} = [\rho, u, v, T]$ and hence

$$\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1 + \epsilon^2 \mathbf{v}_2 + \ldots, \tag{22}$$

where $\mathbf{v}_0 = [1, U, 0, 1/\chi]$. Then the $O(\epsilon^2)$ equations become just the linearized inviscid equations

$$\mathbf{L}\mathbf{v}_1 = \left(\mathbf{A}\frac{\partial}{\partial x_1} + \mathbf{B}\frac{\partial}{\partial y_1}\right)\mathbf{v}_1 = 0, \tag{23}$$

where

$$\mathbf{A} = \begin{bmatrix} U & 1 & 0 & 0 \\ 1 & U & 0 & \chi \\ 0 & 0 & U & 0 \\ 0 & \chi & 0 & U \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \chi \\ 0 & 0 & \chi & 0 \end{bmatrix}.$$

Although the solution of (23) is straightforward, we solve it using a method which is useful in the higher orders. We simultaneously diagonalize $\mathbf{A}$ and $\mathbf{B}$. Thus, let $\mathbf{l}^i$ and $\boldsymbol{\alpha}^i$ be vectors such that

$$\left.\begin{aligned} \mathbf{l}^i\mathbf{A} &= \boldsymbol{\alpha}^i \\ \mathbf{l}^i\mathbf{B} &= \lambda^i\boldsymbol{\alpha}^i \end{aligned}\right\} \quad (i = 1,2,3,4). \tag{24}$$

Then

$$\mathbf{l}^i(\lambda^i\mathbf{A} - \mathbf{B}) = 0, \tag{25}$$

which requires $\lambda^i$ to be the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$. Straightforward calculations show

$$\lambda^{1,2,3,4} = 0, 0, \pm(M^2-1)^{-\frac{1}{2}}. \tag{26}$$

Strict supersonic flow implies $M^2 - 1 > 0$. Thus we have four real roots corresponding to two wakes (vorticity and entropy) and two Mach waves. Substituting (26) into (25), we find

$$\left.\begin{aligned} \mathbf{l}^1 &= [0,1,0,0], \\ \mathbf{l}^2 &= [\chi, 0, 0, -1], \\ \mathbf{l}^3 &= [1, -\gamma/U, \gamma(M^2-1)^{\frac{1}{2}}/U, \chi], \\ \mathbf{l}^4 &= [1, -\gamma/U, -\gamma(M^2-1)^{\frac{1}{2}}/U, \chi]. \end{aligned}\right\} \tag{27}$$

From (27) and (24), we get

$$\left.\begin{aligned} \boldsymbol{\alpha}^1 &= [1, U, 0, \chi], \\ \boldsymbol{\alpha}^2 &= [\chi U, 0, 0, -U], \\ \boldsymbol{\alpha}^3 &= [\gamma(M^2-1)/U, 0, \gamma(M^2-1)^{\frac{1}{2}}, \chi\gamma(M^2-1)/U], \\ \boldsymbol{\alpha}^4 &= [\gamma(M^2-1)/U, 0, -\gamma(M^2-1)^{\frac{1}{2}}, \chi\gamma(M^2-1)/U]. \end{aligned}\right\} \tag{28}$$

Setting $\omega^i = \boldsymbol{\alpha}^i.\mathbf{v}_1$, we have from (23)

$$\left.\begin{aligned} 0 &= \mathbf{l}^i.\left(\mathbf{A}\frac{\partial}{\partial x_1} + \mathbf{B}\frac{\partial}{\partial y_1}\right)\mathbf{v}_1 \\ &= \left(\frac{\partial}{\partial x_1} + \lambda^i\frac{\partial}{\partial y_1}\right)\boldsymbol{\alpha}^i.\mathbf{v}_1 \\ &= \left(\frac{\partial}{\partial x_1} + \lambda^i\frac{\partial}{\partial y_1}\right)\omega^i \end{aligned}\right\} \quad (i = 1,2,3,4). \tag{29}$$

Thus

$$\left.\begin{aligned} \omega^1 &= g_1(y_1, x_2, y_2, \ldots), \\ \omega^2 &= g_2(y_1, x_2, y_2, \ldots), \\ \omega^3 &= g_3(x_1 - (M^2-1)^{\frac{1}{2}} y_1, x_2, y_2, \ldots), \\ \omega^4 &= g_4(x_1 + (M^2-1)^{\frac{1}{2}} y_1, x_2, y_2, \ldots). \end{aligned}\right\} \quad (30)$$

From the definition of $\omega^i$, we also have

$$\left.\begin{aligned} \omega^1 &= \rho_1 + Uu_1 + \chi T_1, \\ \omega^2 &= \chi U \rho_1 - UT_1, \\ \omega^3 &= \gamma(M^2-1)\rho_1/U + \gamma(M^2-1)^{\frac{1}{2}} v_1 + \gamma\chi(M^2-1)\, T_1/U, \\ \omega^4 &= \gamma(M^2-1)\rho_1/U - \gamma(M^2-1)^{\frac{1}{2}} v_1 + \gamma\chi(M^2-1)\, T_1/U. \end{aligned}\right\} \quad (31)$$

We define a matrix $\mathbf{S}$ whose rows are composed of four row vectors $\boldsymbol{\alpha}^i$, and let $\boldsymbol{\omega} = [\omega^1, \omega^2, \omega^3, \omega^4]$. Then, by definition, we have $\boldsymbol{\omega} = \mathbf{S}\mathbf{v}_1$, and hence

$$\mathbf{v}_1 = \mathbf{S}^{-1}\boldsymbol{\omega}. \quad (32)$$

Carrying out the calculations, we find

$$\mathbf{v}_1 = \left[\frac{\chi}{U\gamma} g_2, \frac{g_1}{U}, 0, -\frac{g_2}{(\gamma U)}\right] + \frac{g_3}{2\gamma(M^2-1)}\left[\frac{U}{\gamma}, -1, (M^2-1)^{\frac{1}{2}}, \frac{\chi U}{\gamma}\right]$$
$$+ \frac{g_4}{2\gamma(M^2-1)}\left[\frac{U}{\gamma}, -1, -(M^2-1)^{\frac{1}{2}}, \frac{\chi U}{\gamma}\right] = \mathbf{v}_1^0 + \mathbf{v}_1^+ + \mathbf{v}_1^-, \quad (33)$$

where we have decomposed the solution into modes: $\mathbf{v}_1^{\pm}$ describe the Mach zones in the upper and lower half planes, while $\mathbf{v}_1^0$ describes the 'wake', which is really the superposition of two wakes.

A point of importance with regard to $\mathbf{v}_1$ is that when applied to a thin body the modes disengage; that is, products of variables of different modes are effectively zero. For example $u^+ \partial u^-/\partial x$ can be taken as zero.

### *The non-linear field equation*

Equations for $O(\epsilon^3)$ in (19), (20) and (21) can be grouped into two parts with one involving operations on $\mathbf{v}_1$ and the other on $\mathbf{v}_2$. That is, we can write formally

$$\mathbf{L}\mathbf{v}_2 = \mathbf{M}(\mathbf{v}_1) = \mathbf{M}(\mathbf{v}_1^+) + \mathbf{M}(\mathbf{v}_1^-) + \mathbf{M}(\mathbf{v}_1^0), \quad (34)$$

where $\mathbf{L}$ is the same linear operator as in (23) and $\mathbf{M}(\mathbf{v}_1)$, which is non-linear and rather lengthy, is implicitly given by (19), (20) and (21). The decoupling in (34) follows from the discussion in the previous paragraph. Multiplying (34) from the left by $\mathbf{l}^i$, we have

$$\left(\frac{\partial}{\partial x_1} + \lambda^i \frac{\partial}{\partial y_1}\right) \boldsymbol{\alpha}^i . \mathbf{v}_2 = \mathbf{l}^i . \mathbf{M}(\mathbf{v}_1). \quad (35)$$

The solution $\mathbf{v}_2$ of (35) will involve a complementary solution and a particular solution. The complementary solution will be of a similar form to $\mathbf{v}_1$, but the particular solution will involve terms which grow faster than $\mathbf{v}_1$ in the far field. To demonstrate this let us consider, e.g. the wave region in the upper half plane, i.e. $i = 3$. Then recalling that $\mathbf{v}_1^+$ corresponds to $i = 3$, we have (since $\mathbf{v}_1^- = \mathbf{v}_1^0 = 0$ in the upper half plane)

$$\left(\frac{\partial}{\partial x_1} + \lambda^3 \frac{\partial}{\partial y_1}\right) \boldsymbol{\alpha}^3 . \mathbf{v}_2 = \mathbf{l}^3 . \mathbf{M}(\mathbf{v}_1^+). \quad (36)$$

The left-hand side is a derivative in the direction $(1, \lambda^3)$. But according to (29), $\mathbf{l}^3.\mathbf{M}(\mathbf{v}_1^+)$, since it is composed of $\mathbf{v}_1^+$ or alternately $\boldsymbol{\alpha}^3.\mathbf{v}_1$, is a constant with respect to differentiation in this direction. Therefore the integral of (36) has a particular solution which grows linearly in the $(1, \lambda^3)$ direction. Thus to preserve the ordering of the perturbation scheme, we impose the secularity conditions

$$\mathbf{l}^i.\mathbf{M}(\mathbf{v}_1) = 0 \quad \text{for} \quad i = 1, 2, 3, 4. \tag{37}$$

The decoupling shown in (34) further reduces (37).

*Wake region*

In the wake, $\mathbf{v}_1 \sim \mathbf{v}_1^0$. Thus the secularity conditions are

$$\mathbf{l}^1.\mathbf{M}(\mathbf{v}_1^0) = 0 \quad \text{and} \quad \mathbf{l}^2.\mathbf{M}(\mathbf{v}_1^0) = 0. \tag{38}$$

Straightforward calculation shows

$$\left(\frac{\partial}{\partial x_2} - \frac{\zeta}{U}\frac{\partial^2}{\partial y_1^2}\right) g_1(y_1, x_2) = 0 \tag{39}$$

and

$$\left(\frac{\partial}{\partial x_2} - \frac{\xi}{U\gamma}\frac{\partial^2}{\partial y_1^2}\right) g_2(y_1, x_2) = 0. \tag{40}$$

These equations can be solved explicitly to yield

$$g_1(y_1, x_2) = \int_{-\infty}^{\infty} g_1^0(s) \frac{\exp\left[-U(y_1 - s)^2/4\zeta x_2\right]}{2[\pi(\zeta/U)\,x_2]^{\frac{1}{2}}}\, ds \tag{41}$$

and

$$g_2(y_1, x_2) = \int_{-\infty}^{\infty} g_2^0(s) \frac{\exp\left[-\gamma U(y_1 - s)^2/4\xi x_2\right]}{2[\pi(\xi/\gamma U)\,x_2]^{\frac{1}{2}}}\, ds, \tag{42}$$

where $g_1^0(y_1)$ and $g_2^0(y_1)$ are 'initial' conditions for the wake for $x_2 \to 0$. We defer a discussion of the initial conditions until later.

*Mach zones*

In the Mach zones, we have $\mathbf{v}_1 \sim \mathbf{v}^+, \mathbf{v}^-$. Thus the secularity condition is

$$\mathbf{l}^{3,4}.\mathbf{M}(\mathbf{v}^{\pm}) = 0. \tag{43}$$

After simplification this becomes

$$\frac{\partial}{\partial x_2} g_{3,4} \pm \frac{1}{(M^2-1)^{\frac{1}{2}}}\frac{\partial}{\partial y_2} g_{3,4} - \frac{(\gamma+1)M^3}{4\gamma^{\frac{3}{2}}(M^2-1)^2} g_{3,4}\frac{\partial}{\partial \tau_1} g_{3,4} = \frac{\{\gamma(\zeta+\eta)+\chi^2\xi\}\,M^3}{2(M^2-1)\,\gamma^{\frac{3}{2}}}\,\frac{\partial^2}{\partial \tau_1^2} g_{3,4}, \tag{44}$$

where

$$\tau_1 = x_1 \mp (M^2-1)^{\frac{1}{2}} y_1.$$

By defining new independent variables, we can transform (44) into the form of a Burgers equation. Let

$$\tau_2 = x_2 \mp (M^2-1)^{\frac{1}{2}} y_2, \quad Y^2 = \pm y_2, \tag{45}$$

then

$$\left.\begin{aligned}\frac{\partial}{\partial Y_2} g_{3,4} - c_1 g_{3,4} \frac{\partial}{\partial \tau_1} g_{3,4} &= \nu \frac{\partial^2}{\partial \tau_1^2} g_{3,4},\\ c_1 &= \frac{(\gamma+1) M^3}{4\gamma^{\frac{1}{2}}(M^2-1)^{\frac{3}{2}}},\\ \nu &= \frac{[\gamma(\eta+\zeta)+\chi^2\xi] M^3}{2(M^2-1)^{\frac{1}{2}}\gamma^{\frac{3}{2}}}.\end{aligned}\right\} \qquad (46)$$

where

Equation (46) has been solved by Cole (1951) and Hopf (1950). We represent the solution in the form

$$g_{3,4}(\tau_1, Y_2) = \frac{\displaystyle\int_{-\infty}^{\infty} g_{3,4}^0(s) \exp[(1/2\nu) F(\tau_1, Y_2, s)]\, ds}{\displaystyle\int_{-\infty}^{\infty} \exp[(1/2\nu) F(\tau_1, Y_2, s)]\, ds}, \qquad (47)$$

where
$$F(\tau_1, Y_2, s) = c_1 \int_{s_0}^{s} g_{3,4}^0(t)\, dt - (\tau_1 - s)^2/2Y_2$$

and the 'initial' data $g_{3,4}^0(\tau_1)$ is to be determined by allowing $Y_2 \to 0$ in $g_{3,4}$. In defining $F$, the lower limit of integration $s_0$ is actually arbitrary. Changing $s_0$ amounts to multiplying both the numerator and the denominator of (47) by the same constant. However, for definiteness we will take $s_0$ to be the minimum point in the support of $g_{3,4}^0$. It should be recalled that through (33), we can represent all flow variables in the upper and lower waves in terms of $g_3$ and $g_4$ respectively.

## 5. Structure of the general solution

To begin with, we will find it useful to return to the normalization based on body length. This is effected by taking $L$ in (4) to be the body length. It is also convenient to eliminate the small parameter $\epsilon$ by simply setting it to unity. Finally, rather than considering $g_3$ or $g_4$, we, for illustrative reasons, consider the velocity perturbation in the $x$ direction which we write as $u^{\pm}$ (suppressing the subscript 1). Then corresponding to (46), we have

$$\frac{\partial}{\partial y} u^{\pm} + c_2 u^{\pm} \frac{\partial}{\partial \tau} u^{\pm} = \frac{1}{2R} \frac{\partial^2}{\partial \tau^2} u^{\pm}, \qquad (48)$$

where
$$c_2 = \frac{(\gamma+1) M^3}{(4\gamma(M^2-1))^{\frac{1}{2}}}, \quad \tau = x \mp (M^2-1)^{\frac{1}{2}} y,$$

and $R = 1/(2\nu)$, see (46), is essentially the Reynolds number based on body length. The solution to (48) is given by

$$u^{\pm}(x, y) = \frac{\displaystyle\int_{-\infty}^{\infty} h^{\pm}(s) \exp\{R\mathscr{F}(\tau, y, s)\}\, ds}{\displaystyle\int_{-\infty}^{\infty} \exp\{R\mathscr{F}(\tau, y, s)\}\, ds}, \qquad (49)$$

where
$$\mathscr{F}(\tau, y, s) = -c_2 \int_{s_0}^{s} h^{\pm}(t)\, dt - \frac{(\tau-s)^2}{2y}. \tag{50}$$

The function $h^{\pm}(t)$ is determined by the data.

*Inner limit of the solution*

The inner limit $y \to 0$ is facilitated by multiplying numerator and denominator of (49) by $(R/y)^{\frac{1}{2}}$ and noting that
$$\lim_{y\to 0} (R/2\pi y)^{\frac{1}{2}} \exp\{-(\tau-s)^2 R/2y\} = \delta(\tau - s).$$

We then obtain from (49) that
$$u^{\pm}(x,y) \sim h^{\pm}(x \mp (M^2-1)^{\frac{1}{2}} y). \tag{51}$$

These clearly are solutions of the wave equations – or in other words solutions of the linearized inviscid theory. This observation is important in the determination of $h^{\pm}$, i.e. the 'boundary condition'. Before doing this, however, we discuss the inviscid limit.

*The inviscid limit*

Inviscid limits for Burgers equation were first considered by Burgers (1948). Later, rigorous mathematical treatments were given by Hopf (1950), Lax (1957), Olejnik (1956) and Ladyzhenskaya (1956). Our discussion here is based on that given by Lighthill (1956).

Under the limit $R \to \infty$ on (47), the main contribution comes from the neighbourhood of the maximum points of $\mathscr{F}$. Stationary points of $\mathscr{F}$ are given by
$$\frac{\partial}{\partial s}\mathscr{F}(\tau, y, s) = -c_2 h^{\pm}(s) + \frac{(\tau - s)}{y} = 0.$$

If $\mathscr{F}$ has only a single global maximum $s_1$, then it is determined implicitly by
$$s_1(\tau, y) = \tau - c_2 y h^{\pm}(s_1).$$

From (49), the leading term becomes
$$u^{\pm}(x,y) \sim h^{\pm}(s_1(\tau, y)). \tag{52}$$

It is clear by direct substitution that (52) satisfies the inviscid Burgers equation
$$\frac{\partial}{\partial y} h^{\pm} + c_2 h^{\pm} \frac{\partial}{\partial \tau} h^{\pm} = 0. \tag{53}$$

The same equation results if we substitute (47) into (46) and let $R = 1/2\nu \to \infty$.

If the maximum of $\mathscr{F}$ is achieved at two or more points, shocks occur. That is, suppose that $\mathscr{F}$ has two equal global maxima at $s_1$ and $s_2$, then they are defined implicitly by
$$c_2 h^{\pm}(s_{1,2}) = (\tau - s_{1,2})/y.$$

Hence $\mathscr{F}(\tau, y, s_1) = \mathscr{F}(\tau, y, s_2)$ gives
$$c_2 \int_{s_1}^{s_2} h^{\pm}(t)\, dt = \frac{1}{2y}[(\tau - s_1)^2 - (\tau - s_2)^2]$$
$$= \tfrac{1}{2} c_2 (s_2 - s_1)[h^{\pm}(s_2) + h^{\pm}(s_1)] \tag{54}$$

which determines the shock trajectory. The slope of the shock is given by differentiating (54) implicitly with respect to $x$. Thus

$$dy/dx = \{\pm (M^2-1)^{\frac{1}{2}} + \tfrac{1}{2}c_2[h^{\pm}(s_1)+h^{\pm}(s_2)]\}^{-1}. \tag{55}$$

Equation (55) can also be derived from inviscid theory. If we consider the integral form of the inviscid Burgers equation (53), it allows discontinuous solutions subject to the condition

$$-m[h^{\pm}] + \tfrac{1}{2}c_2[(h^{\pm})^2] = 0, \tag{56}$$

where [ ] denotes the jump across the shock and $m$ is $d\tau/dy$ along the shock trajectory. Equation (55) can then be recovered from (56).

A number of the results under the inviscid limit above can also be found in the paper by Whitham (1952). Whitham obtains his result by using a non-linear transformation on the linear inviscid solution. He represents his solution in the form

$$u = -\frac{F(\xi)}{(M^2-1)^{\frac{1}{2}}}, \tag{57}$$

where $\xi$ is the characteristic curve defined by

$$x = (M^2-1)^{\frac{1}{2}}y - kF(\xi)\,y + kF(\xi)\,R(\xi) + \xi \tag{58}$$

with $k = \frac{1}{2}(\gamma+1)\,M^4/(M^2-1)$ a constant, and $x = R(y)$ is the body. Differentiating (58) along the characteristics $\xi$ = constant yields

$$\begin{aligned} dx &= (M^2-1)^{\frac{1}{2}}\left[1 - \frac{kF(\xi)}{(M^2-1)^{\frac{1}{2}}}\right]dy \\ &= (M^2-1)^{\frac{1}{2}}[1+ku]\,dy. \end{aligned} \tag{59}$$

Hence, $u$ is the solution to the partial differential equation

$$\frac{\partial}{\partial x}u + \frac{1}{(M^2-1)^{\frac{1}{2}}}\frac{\partial}{\partial y}u + ku\frac{\partial}{\partial x}u = 0. \tag{60}$$

Upon appropriate transformations, (60) can be shown to be just the inviscid form of (48).

This accounts for the fact that Whitham's inviscid result is identical to ours. Whitham's theory may also be developed directly from inviscid theory. A straightforward perturbation analysis on the inviscid equations for a slender body reveals a secularity at the second order. This signals the use of multiple scales and Whitham's results follow.

We remark here that the inviscid limit is not uniform in $y$. Since $1/R$ and $y$ appear in combination in (49), it is clear that the inviscid limit we have derived will not necessarily be correct as $y \to \infty$.

Another non-uniformity appears in the calculation of the shock wave. We recall that the shock trajectory was obtained by locating two equal global maxima of $\mathscr{F}$. Therefore, one should also consider the solution when two relative maxima (one of which is global) are close to one another in value. This analysis yields the shock structure. (For the unsteady version, see Cole (1951), Hopf (1950) and Lighthill (1956).) We do not repeat this calculation here.

*Boundary conditions*

The key to the proper choice of boundary conditions lies in relation (51) which tells us that in the limit $y \to 0$, the solution approaches a solution to the linearized inviscid equation. On this basis, it is tempting to take the inviscid linearized solution and from it obtain the functions $h^{\pm}$. This is especially appealing since, as one may verify, the relations between $\rho$, $u$, $v$ and $T$ as given by (33) are the same as those obtained from linearized inviscid theory.

In a real flow, once out of the boundary layer, the flow field may be considered as inviscid. The shock wave regions below and above a body, however, incorporate effects of the boundary layer, e.g. viscous drag. Therefore a more nearly correct way of evaluating $h^{\pm}$ would be to obtain the exact 'get away' solution which comes off a body, and then use this to construct our $h^{\pm}$ functions. Such a program would be too difficult to carry out in practice. A possible approximate method would be to compute the linearized inviscid flow past the shape given by the body plus the boundary layer. This can then be used to obtain the linearized inviscid 'get away' solution for the construction of $h^{\pm}$.

Our main purpose in the next section is to illustrate our solutions. We therefore avoid boundary-layer effects and obtain solutions based on the linearized inviscid 'get away' solution. Therefore, the above remarks about the dissipative effects from the boundary layer should be regarded as cautionary.

## 6. Flow past a thin airfoil

We consider an airfoil whose upper and lower surfaces are given by

$$y = \pm \epsilon f_{\pm}(x), \tag{61}$$

where $\epsilon \ll 1$ is the thickness ratio, and $f(x) = 0$ for $x$ outside the interval $[-\frac{1}{2}, \frac{1}{2}]$. (The slope of $f_{\pm}$ is assumed to be $O(1)$.) Let $\overline{u}$ denote the linearized inviscid velocity perturbation in the $x$ direction for flow past the airfoil (61), then as is well known

$$\overline{u}^{\pm} = -\frac{\epsilon U}{(M^2-1)^{\frac{1}{2}}} f'_{\pm}(\tau). \tag{62}$$

The other flow variables can then be determined from the relations implicit in (33).

Keeping in mind the discussion at the end of the last section, we use (62) as the 'get away' flow in (49). In fact from (62) we have

$$h^{\pm}(s) = -\frac{\epsilon U}{(M^2-1)^{\frac{1}{2}}} f'_{\pm}(s), \tag{63}$$

so that

$$u^{\pm} = \frac{-\epsilon U}{(M^2-1)^{\frac{1}{2}}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f'(s) \exp\{R\mathscr{F}^{\pm}(\tau, y, s)\}\, ds \Big/ \int_{-\infty}^{\infty} \exp\{R\mathscr{F}^{\pm}(\tau, y, s)\}\, ds, \tag{64}$$

where
$$\mathscr{F}^{\pm}(\tau, y, s) = \frac{\epsilon U c_2}{(M^2-1)^{\frac{1}{2}}} f_{\pm}(s) - \frac{(\tau - s)^2}{2y}. \tag{65}$$

Using integration by parts, (64) may be written as

$$u = \int_{-\infty}^{\infty} \frac{(\tau - s)}{y} \exp\{R\mathscr{F}(\tau, y, s)\}\, ds \Big/ c_2 \int_{-\infty}^{\infty} \exp\{R\mathscr{F}(\tau, y, s)\}\, ds. \tag{66}$$

The repetitious superscripts $\pm$ have been dropped in (66).

We analyze the solution (66) by considering a sequence of zones in the flow field.

*Zone* 1. $y \ll 1/\epsilon$

In this zone, we know from the inner limit of § 5 that linearized inviscid theory is valid. Or in other words, the solution is simply given by (62). This limit, however, is not uniform in the neighbourhood of $\tau \approx \pm\frac{1}{2}$. In the neighbourhood of these lines, viscous effects come into play and the solution is given by an integral such as (5) where the source term is based on (62). A finer analysis then reveals that the linear description of this region breaks down when $y = O(1/\epsilon^2 R)$. This agrees with the result obtained in § 3, equation (14).

*Zone* 2. $y = O(1/\epsilon)$

The inviscid limit of § 5 gives the proper description of this zone. We recall that the inviscid non-linear description fails in the interior of the shock as described in § 5.

*Zone* 3. $y \gg O(1/\epsilon)$

To evaluate (66) in the limit $R \to \infty$ in this region, we first write (66) in the form

$$u = N/(c_2 D),$$

where an obvious notation is used. For this discussion it is convenient to write

$$\begin{aligned} N &= \int_{-\infty}^{\infty} \frac{(\tau - s)}{y} \exp\left\{-R\frac{(\tau-s)^2}{2y}\right\} ds - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(\tau - s)}{y} \exp\left\{-R\frac{(\tau-s)^2}{2y}\right\} ds \\ &\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(\tau - s)}{y} \exp\{R\mathscr{F}(\tau, y, s)\}\, ds \\ &= \frac{1}{R}\left\{\exp\left[-\frac{R(\tau+\frac{1}{2})^2}{2y}\right] - \exp\left[-\frac{R(\tau-\frac{1}{2})^2}{2y}\right]\right\} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(\tau - s)}{y} \exp\{R\mathscr{F}(\tau, y, s)\}\, ds \\ &= Q_1 + I_1, \end{aligned} \tag{67}$$

where again an obvious notation is used. And similarly

$$\begin{aligned} D &= \left(\frac{\pi y}{2R}\right)^{\frac{1}{2}} \left\{2 + \operatorname{erf}\left[(\tau - \tfrac{1}{2})\left(\frac{R}{2y}\right)^{\frac{1}{2}}\right] - \operatorname{erf}\left[(\tau + \tfrac{1}{2})\left(\frac{R}{2y}\right)^{\frac{1}{2}}\right]\right\} \\ &\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp\{R\mathscr{F}(\tau, y, s)\}\, ds \\ &= Q_2 + I_2. \end{aligned} \tag{68}$$

For $R \to \infty$, the major contribution of $I_1$ and $I_2$ come from the maximum of $\mathscr{F}$ for $s$ in $[-\frac{1}{2}, \frac{1}{2}]$. Simple estimates show that

$$u = O(\tau (yR)^{-\frac{1}{2}} \exp\{-R\tau^2/2y\}), \tag{69}$$

for $\tau > O(\epsilon y)$. For $\tau < O(\epsilon y)$, $\mathscr{F}$ has a maximum at $s = s_0 \in (-\frac{1}{2}, \frac{1}{2})$. Thus we use Laplace's formula to evaluate $I_1$ and $I_2$ to yield

$$\begin{aligned} u = &\left\{\frac{1}{c_2 R}\left[\exp\left(-R\frac{(\tau+\frac{1}{2})^2}{2y}\right) - \exp\left(-R\frac{(\tau+\frac{1}{2})^2}{2y}\right)\right]\right. \\ &\left. + \frac{\tau - s_0}{c_2 y}\left(\frac{2\pi}{-\epsilon R k f''(s_0)}\right)^{\frac{1}{2}} \exp\left[R\left(\epsilon k f(s_0) - \frac{(\tau-s_0)^2}{2y}\right)\right]\right\} \\ &\times\left\{\left(\frac{\pi y}{2R}\right)^{\frac{1}{2}}\left[2 + \operatorname{erf}\left((\tau-\tfrac{1}{2})\left(\frac{R}{2y}\right)^{\frac{1}{2}}\right) - \operatorname{erf}\left((\tau+\tfrac{1}{2})\left(\frac{R}{2y}\right)^{\frac{1}{2}}\right)\right]\right. \\ &\left. + \left(\frac{2\pi}{-\epsilon R k f''(s_0)}\right)^{\frac{1}{2}} \exp\left[R\left(\epsilon k f(s_0) - \frac{(\tau-s_0)^2}{2y}\right)\right]\right\}^{-1}, \end{aligned} \tag{70}$$

where

$$k = \frac{U c_2}{(M^2-1)^{\frac{1}{2}}} = \frac{(\gamma+1) M^4}{2(M^2-1)}. \tag{71}$$

When $y$ is large such that $\tau \ll O(\epsilon y)$, the maximum of $\mathscr{F}$ for $s$ in $[-\frac{1}{2}, \frac{1}{2}]$ occurs near the point where $f$ is maximum. Thus (70) can be further simplified by taking $s_0$ to be the maximum point of $f$. The region where (70) is valid has been sketched in figure 1.

We note that (70) is in the form of an $N$-wave for a wide range of $y$ (Lighthill 1956). But in the limit $y \to \infty$, the solution becomes entirely viscous, i.e. a linear description of the flow in the interior is no longer valid.

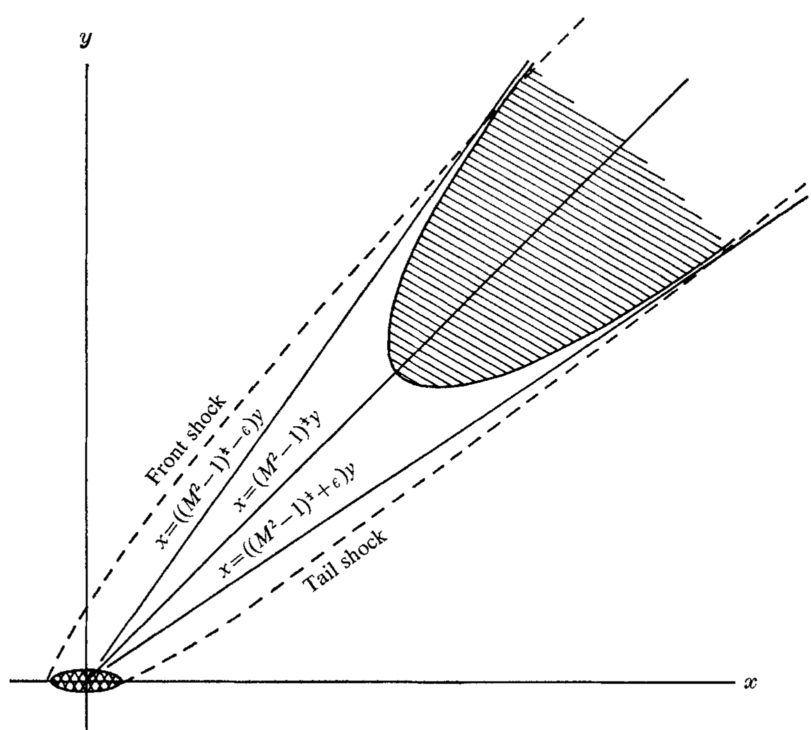


FIGURE 1. Region of validity of far field solution.

*Flow past a thin diamond-shaped wing*

For flow past a diamond-shaped wing of thickness ratio $\epsilon$, the function $f$ is given by

$$f(x) = \begin{Bmatrix} \tfrac{1}{2}+x & -\tfrac{1}{2} \leqslant x < 0, \\ \tfrac{1}{2}-x & 0 \leqslant x < \tfrac{1}{2}, \\ 0 & \text{otherwise.} \end{Bmatrix} \tag{72}$$

In this case, (64) can be exactly evaluated and yields

$$\begin{aligned} u = & -\epsilon U(M^2-1)^{\frac{1}{2}}\{\exp(\epsilon Rk\tfrac{1}{2}(1+\epsilon ky+2\tau))\,[\operatorname{erf}((\tfrac{1}{2}+\epsilon ky+\tau)\,(R/2y)^{\frac{1}{2}}) \\ & -\operatorname{erf}((\tau+\epsilon ky)\,R/2y)] - \exp(\epsilon Rk\tfrac{1}{2}(1+\epsilon ky-2\tau))\,[\operatorname{erf}(\tau-\epsilon ky)\,(R/2y)^{\frac{1}{2}} \\ & +\operatorname{erf}((\tfrac{1}{2}+\epsilon ky-\tau)\,(R/2y)^{\frac{1}{2}})]\}/\{2-\operatorname{erf}((\tfrac{1}{2}-\tau)\,(R/2y)^{\frac{1}{2}})-\operatorname{erf}((\tfrac{1}{2}+\tau)\,(R/2y)^{\frac{1}{2}}) \\ & +\exp(\epsilon Rk\tfrac{1}{2}(1+\epsilon ky+2\tau))\,[\operatorname{erf}(\tfrac{1}{2}+\epsilon ky+\tau)\,(R/2y)^{\frac{1}{2}}) \\ & -\operatorname{erf}((\tau+\epsilon ky)\,(R/2y)^{\frac{1}{2}})] + \exp(\epsilon Rk\tfrac{1}{2}(1+\epsilon ky-2\tau)) \\ & \times[\operatorname{erf}((\tfrac{1}{2}+\epsilon ky-\tau)\,(R/2y)^{\frac{1}{2}})+\operatorname{erf}((\tau-\epsilon ky)\,(R/2y)^{\frac{1}{2}})]\}. \end{aligned} \tag{73}$$

The far field representation corresponding to (70) can be evaluated using a method which does not require the differentiability of $f$ and yields,

$$\begin{aligned} u \sim & \left\{\frac{1}{c_2 R}\left[\exp\left(-R\frac{(\tau+\frac{1}{2})^2}{2y}\right)-\exp\left(-R\frac{(\tau-\frac{1}{2})^2}{2y}\right)\right]\right. \\ & \left.+\frac{2\tau}{\epsilon c_2 kRy}\exp\left(R\frac{(\epsilon ky-\tau^2)}{2y}\right)\right\}\Bigg/\left\{\left(\frac{\pi y}{2R}\right)^{\frac{1}{2}}\left[2+\operatorname{erf}\left((\tau-\tfrac{1}{2})\left(\frac{R}{2y}\right)^{\frac{1}{2}}\right)\right.\right. \\ & \left.\left.-\operatorname{erf}\left((\tau+\tfrac{1}{2})\left(\frac{R}{2y}\right)^{\frac{1}{2}}\right)\right]+\frac{2}{\epsilon Rk}\exp\left(\frac{R}{2y}(\epsilon ky-\tau^2)\right)\right\}. \end{aligned} \tag{74}$$

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