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# A computational study of Rayleigh–Bénard convection. Part 2. Dimension considerations

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A study is made of the number of dimensions needed to specify chaotic Rayleigh–Bénard convection, over a range of Rayleigh numbers ($\gamma = Ra/Ra_c < 10^2$). This is based on the calculation of Lyapunov dimension over the range, as well as the notion of Karhunen–Loéve dimension. An argument suggesting a universal relation between these estimates and supporting numerical evidence is presented. Numerical evidence is also presented that the reciprocal of the largest Lyapunov exponent and the correlation time are of the same order of magnitude. Several other universal features are suggested. In particular it is suggested that the *intrinsic* attractor dimension is $O(Ra^{\frac{2}{3}})$, which is sharper than previous results.

## 1. Introduction

The general introduction to this paper may be found in our previous one (Deane & Sirovich 1991; hereinafter referred to as I). The main concern of this second part is the number of parameters or dimensions needed to describe turbulent thermal convection. Some indication of this number follows from the Karhunen–Loéve (K-L) procedure discussed in I, and this idea is given more precision in the discussion given in §2. A general and precise definition of dimension is given by the *capacity*, or Hausdorff dimension (see e.g. Bergé, Pomeau & Vidal 1986; Schuster 1984) and a means for estimating this dimension has been given by Kaplan & Yorke (1979), who use Lyapunov exponents for this purpose. Foiaş, Manley & Temam (1986) have shown, in general, that the Kaplan–Yorke formula gives an upper bound for the capacity. In order to verify our calculation of Lyapunov exponents, we supposed that the usual correlation time would furnish a baseline estimate for the largest Lyapunov exponent. Since this provoked some debate in certain circles, a numerical study of this point is made here, and to within the range of our calculations it indeed appears that the correlation time and the reciprocal of the largest Lyapunov exponent are of the same order of magnitude.

An attempt at relating the K-L dimension, $d_{KL}$, to the Lyapunov dimension, $d_L$, is made in §2. This is based on intuitive geometrical arguments, and is to be regarded as informal. On the other hand, as the Rayleigh number $Ra$ increases for a given cell, and the turbulence becomes more vigorous, we expect the flow to enter a scaling regime. When this occurs, one might suppose that the two different dimension definitions should bear a fixed relation to one another. This too has been investigated, and somewhat surprisingly, the fixed relation appears to hold rather well at our relatively low values of $Ra$.

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Several other results presented here also depend on scaling arguments. In particular, we give some numerical evidence that the Lyapunov exponents, on average, have a uniform density. This is used as a basis for the approximation of $d_L$ in cases when the actual calculations would require excessive computer time. In a related context, we give a dimensional argument that implies that the largest Lyapunov exponent is $O(Ra^{\frac{2}{3}})$, and from this that $d_L = O(Ra^{\frac{2}{3}})$. If our argument holds generally, it is sharper than previous estimates. For example a simple estimate of the degrees of freedom based on a calculation of Landau (1944) gives an estimate of $O(Ra)$ (Foiaş *et al.* 1986). Rigorous mathematical results are available in two-dimensions (Foiaş, Manley & Temam 1987; Ruelle 1982), in which case the estimate is $O(Ra)$, and by implication, it is $O(Ra^{\frac{3}{2}})$ in three dimensions. Possible reasons for our sharper bound are given in §6.

We have made extensive numerical calculations for the flows, and for their Lyapunov exponents over a range of Rayleigh numbers. However, this range is small, $Ra/Ra_c < 10^2$, and although we regard our arguments as having a sound basis, they have not been verified over a sufficiently large range of $Ra$. Thus, some of the remarks made above and in the text should be regarded as speculative and conjectural. In this regard we point out that the numerical data for the Nusselt number discussed in I fall on a $Nu = O(Ra^{\frac{1}{4}})$ curve, which as we pointed out in I is valid only at relatively low values of $Ra$. However, none of the arguments that we present here depend on this particular Nusselt-number relationship. In fact the scaling of dependent variables, which we do use in our deliberations, has proven to be reliable for $Ra/Ra_c = O(10^4)$ (Balachandar, Maxey & Sirovich 1989; Sirovich, Balachandar & Maxey 1989).

## 2. Dimension: Karhunen–Loéve

In I, chaotic Rayleigh–Bénard convective flow was analysed across a range of Rayleigh numbers in terms of the K-L procedure. One characterization of the K-L procedure is that, of all admissible orthonormal bases, it optimally captures on the average the most energy in any finite spanning basis. In other words if we wish to retain some preassigned fraction of the total energy of the flow, the fewest number of functions will be required if the K-L basis is chosen. This in turn infers that we can define an intrinsic dimension of the flow, based on the K-L description. It has been suggested (Sirovich 1989; Sirovich & Sirovich 1989) that for an informal definition of K-L dimension, $d_{KL}$, we proceed as follows. For a flow $\boldsymbol{v}$, and a K-L orthonormal basis $\{\phi_n\}$ define the projector

$$\boldsymbol{v}_N = P_N \boldsymbol{v} = \sum_{n=0}^{N} a_n \phi_n(\boldsymbol{x}). \tag{1}$$

Thus, $\boldsymbol{v}_N$ represents the flow as seen in the space of the first $N+1$ eigenfunctions where these have been ordered so that

$$\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \dots. \tag{2}$$

Each eigenvalue, $\lambda_k$ represents the average *energy* of the flow along the corresponding direction $\phi_k$. Then $d_{KL}$ is chosen such that

$$d_{KL} = N | \min_N \langle(\boldsymbol{v}_N, \boldsymbol{v}_N)\rangle / \langle(\boldsymbol{v}, \boldsymbol{v})\rangle = \sum_{n=0}^{N} \lambda_n \bigg/ \sum_{n=0}^{\infty} \lambda_n = E_N/E > 0.9 \tag{3}$$

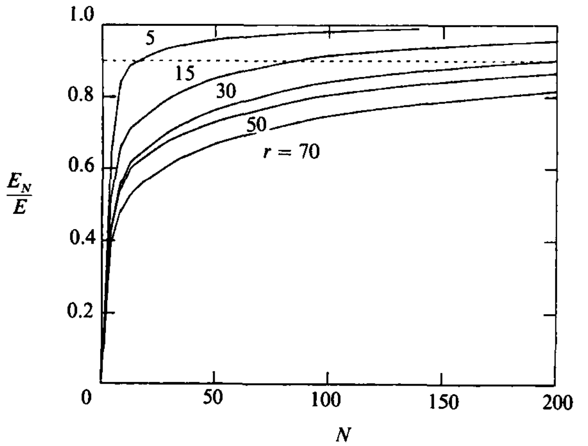


FIGURE 1. Cumulative energy contribution of K-L modes for various Rayleigh numbers. The dashed line at $E_N/E = 0.9$ indicates the K-L dimension $d_{KL}$.

| $r$ | 5 | 15 | 30 | 50 | 70 |
|---|---|---|---|---|---|
| $d_{KL}$ | 20 | 88 | 206 | 309 | 506 |

TABLE 1. The Karhunen–Loéve dimension for the indicated values of Rayleigh number

and
$$\lambda_N/\lambda_0 < 10^{-2}. \tag{4}$$

In (3) $E_N$ represents the average energy in the projected space of (1), and $E$ is the total average energy of the flow. Thus, condition (3) guarantees that 90% of the energy, on average, resides in the space spanned by (1), and condition (4) guarantees that, on average, the excluded amplitudes are less than 10% of the amplitude of the principal mode. We emphasize that this definition is informal and is only intended as a plausible criterion. With this cautionary remark made, we propose that $d_{KL}$ furnishes a practical approach for finding a low-dimensional description.

In figure 1 we show the energy captured, $E_N$, as a function of the number of spanning functions, $N$, for the five cases calculated in I. To repeat what was said in the previous paragraph, a description based on a system of functions other than the K-L basis would lead to curves lying below the corresponding curves in figure 1. The criterion (3) is shown as a horizontal dashed line in figure 1. (For each of the cases considered the second part of the criterion, (4), is met within the first part.) The corresponding K-L dimensions are given in table 1. Thus for $Ra = 70Ra_c$ the dimension is 506, but is still well below the representational phase space of $(12)^2 \times 32 \times 3 \approx 14000$. (A factor of 3 instead of 4 arises because the three dependent variables, $\boldsymbol{u}$, are linked by continuity.)

In the next section, we begin our discussion of the Hausdorff dimension of the attracting set for chaotic convection. To contrast this notion with the K-L construction, it is useful to make some preliminary remarks of a geometrical nature. If we consider a flow, say $v(\boldsymbol{x}, t)$, then provided it satisfies certain minimal mathematical properties, it has a representation in terms of some admissible and convenient basis set, perhaps suitable products of sinusoids. The K-L basis is generated, through a unitary transformation, from the original convenient basis. Amongst all unitary transformations, the K-L basis has the unique optimal properties described at the outset of this section. In particular the system point, $\boldsymbol{v}$, as represented in the function space, spends most of its time aligned along the first axis of the K-L system, the next longest time is spent along the second axis, and so forth. The variances of the motion along the K-L axes, are given by the corresponding eigenvalues $\{\lambda_n\}$. Thus in stating that there is an intrinsic dimension, such as $d_{\mathrm{KL}}$, we are saying that by passing a finite-dimensional hyperplane through the attractor, we capture the overwhelming majority of the energy – with little of it leaking out. Actually since the variances, $\{\lambda_n\}$, form a decreasing sequence, (2), the motion, more accurately, lies in an ellipsoid (in the hyperplane), becoming flatter with each additional dimension.

We expect the actual attractor to be highly irregular in shape. It can be highly convoluted, scarred, wrinkled, hairy, twisted and so forth. The K-L dimension should be regarded as giving a rough cut at the determination of the dimension of the attractor. The ellipsoid described in the previous paragraph makes an attempt at containing the highly irregular shape of the attractor. The Lyapunov dimension, which is discussed next, gives a finer estimate of the attractor dimension. However, it is not part of a constructive procedure, in the sense that it does not furnish us with a set of fitting functions, by which we might parameterize the flow on the attractor.

## 3. Lyapunov spectrum

In a turbulent flow small perturbations to a particular realization depart rapidly as the system evolves and after a while the flow bears little detailed resemblance to the unperturbed flow. In a dynamical systems context, this is termed, a *sensitive dependence on initial conditions*. Of course there is a subtle distinction between initial conditions for a flow that has not yet fully developed and initial conditions in the form of perturbations to an evolved flow. However, the spirit of the definition, and its consequences on quantities such as Lyapunov exponents, which are based on long-term evolution, are the same.

If we consider an $n$-dimensional ellipsoid of infinitesimally close solutions, then their long-term evolution determines the Lyapunov characteristic exponents,

$$\mu_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{p_i(t)}{p_i(0)}, \tag{5}$$

where $p_i$ are the principal axes of the $n$-ellipsoid. A positive value indicates that two solutions diverge, on average, exponentially at a rate $\mu_i$, and a negative value indicates that they converge exponentially fast. Note that $\mu_i$ has units of inverse time. The exponents are presumed to be ordered

$$\mu_1 > \mu_2 > \ldots > \mu_n. \tag{6}$$

Kaplan & Yorke (1979) conjectured that the Hausdorff dimension (capacity) of the strange attractor found in a dynamical system may be obtained by the formula

$$d_{\mathrm{L}} = j + \frac{\sum_{i=1}^{j} \mu_i}{|\mu_{j+1}|}, \tag{7}$$

where $j$ is the largest integer for which the sum is positive. The formula is plausible in that an infinitesimal $j$-volume on average grows, but a $(j+1)$-dimensional volume shrinks as the attractor is sampled. While for some systems (7) has been shown to indeed give the Hausdorff dimension (Russel, Hanson & Ott 1982), for other systems it may be incorrect (see Kaplan, Mallet-Paret & Yorke 1984). Foiaş *et al.* (1986) have shown under general conditions that $d_L$ is an upper bound for the Hausdorff dimensional of the attractor. Sidestepping this issue we shall determine the Lyapunov dimension, $d_L$, based on the Kaplan–Yorke formula given by (7).

In principle one need only monitor the evolution of such $n$-volumes to determine the desired exponents. We implement this procedure by what is termed the *standard method* (Shimada & Nagashima 1979; Benettin *et al.* 1980; Wolf *et al.* 1985). First a solution, $\boldsymbol{v}$, to the nonlinear equation, termed the *fiducial solution* is obtained; then the *linearized* equations of Boussinesq convection in $(\boldsymbol{u}', T')$,

$$\nabla \cdot \boldsymbol{u}' = 0, \tag{8}$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -\boldsymbol{u}' \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u}' - \nabla p' + Ra\, Pr\, \boldsymbol{e}_z T' + Pr\, \nabla^2 \boldsymbol{u}', \tag{9}$$

$$\frac{\partial T'}{\partial t} = -(\boldsymbol{u}' \cdot \nabla)\, T - (\boldsymbol{u} \cdot \nabla)\, T' + w' + \nabla^2 T', \tag{10}$$

plus the appropriate boundary conditions are solved for each of the $n$ desired exponents. Here the primed quantities refer to linearized variables and the unprimed to the fiducial solution. Since we solve the equations on a computational grid each field variable is a vector of length $4 \times 12 \times 12 \times 32$. Under the action of the differential operator in (8)–(10), initially orthogonal vectors collapse, in time, towards the direction of greatest dilation (locally). Thus, magnitudes of the initially unit vectors diverge for directions that correspond to the positive exponents. To avoid exceedingly large values of the magnitudes, and the collapse of the directions, the vectors are regularly replaced by an orthonormal set obtained by a Gram–Schmidt procedure. At times of normalization, the growth of a vector can be compared to its initial value of unity. It is this growth, when averaged over a long period, that leads to the limiting value of (5).

Since our numerical procedure does not distinguish between linear and nonlinear equations, the same scheme is used for solving the linear and nonlinear equations. The calculation is started by choosing the $n$ mutually orthogonal vectors to be the harmonics in the vertical and horizontal such that

$$\boldsymbol{v}_j(0) = \sin(2\pi x)\sin(2\pi y)\cos(j\pi z), \quad j = 1, 2, \ldots n. \tag{11}$$

The initial conditions for the fiducial solution is the solution that resulted at the end of the eigenfunction decomposition (see I). All calculations are done in spectral space (except for the product terms), including the Gram–Schmidt procedure, since inner products are unchanged between physical and spectral spaces. The code minimizes the amount of storage required by utilizing the same working arrays for the linear and fiducial solutions, and allows the restarting of calculations if more exponents are needed.

We have monitored the collapse of the directions towards that corresponding to the largest exponent and the divergence of the magnitudes of vectors. It was found that a Gram–Schmidt procedure could be safely applied about every 50 time steps, where the directions remain to within 65° and the magnitudes have diverged negligibly. No unreasonable errors are introduced by this choice.

Before leaving this section we point out that the Lyapunov spectrum must contain three zero values. As is well known, an autonomous system immediately has a zero Lyapunov exponent. This follows from the fact that the time derivative of the fiducial solution satisfies the linearized equations. In particular

$$(\boldsymbol{u}', T') = \left(\frac{\partial \boldsymbol{u}}{\partial t}, \frac{\partial T}{\partial t}\right) \tag{12}$$

satisfies the linearized system, (8)–(10), as well as all of the boundary conditions. This shows that nearby solutions on the fiducial trajectory remain nearby. Two other solutions of (8)–(10) are given by

$$(\boldsymbol{u}', T') = \left(\frac{\partial \boldsymbol{u}}{\partial x}, \frac{\partial T}{\partial x}\right) \tag{13}$$

and

$$(\boldsymbol{u}', T') = \left(\frac{\partial \boldsymbol{u}}{\partial y}, \frac{\partial T}{\partial y}\right). \tag{14}$$

It is clear that each of these satisfies all of the boundary conditions of the problem. Again, each expresses the fact that certain nearby solutions remain nearby and hence correspond to a zero Lyapunov exponent.

## 4. Dimension: Lyapunov

The definition of the Lyapunov exponent involves the limiting process $t \to \infty$, (10). A local value of $\mu_i(t)$ is obtained for finite time, with $\mu_i(t) \to \mu_i$ when $t \to \infty$. Goldhirsh, Sulem & Orszag (1986) have shown that for typical systems, $\mu_i(t)$ evolves as

$$\mu_i(t) = \mu_i + \frac{b_i + \chi_i(t)}{t}. \tag{15}$$

Here $\chi_i$ represents a noise term and $\mu_i$ the true value. Our data appear to be in accordance with (15), and we use them to extrapolate the Lyapunov values at $t \to \infty$. In figure 2 we show typical $\mu_i(t)$ *vs.* $1/t$ for the three values $r = 5$, 15, 70. Superposed on one of the data points is the least-squares, straight-line fit, to indicate the $1/t$ behaviour. The intercepts at $1/t = 0$ give the values $\mu_i$. All of the Lyapunov exponents that we report have been calculated in this way.

For attractors of large dimension the Kaplan–Yorke formula becomes impractical as a tool for computation in view of the number of exponents which are required. In figure 3 we plot the calculated Lyapunov spectra for $r = 5$, 15, 30, 50, 70. For all but $r = 5$ and 15 only partial spectra are plotted. On the basis of the five plots in figure 3 we conjecture: (*a*) that on average the Lyapunov exponents fill in uniformly; and (*b*) that the density of the Lyapunov spectrum becomes independent of the Rayleigh number as it becomes large. From (*a*) it follows that $\mu_n$ *vs.* $n$ can be thought of as lying on a straight line, and from (*b*) that this straight line has the same slope as $Ra \to \infty$. In figure 3 we also show the least-squares fit to the data by a straight line and table 2 shows the resulting slopes. Thus to within about 15% of all slopes are $-1/14$.

Keefe & Moin (1987), in the case of channel flow, have also found that the Lyapunov spectrum is reasonably linear. In a general context it has also been found

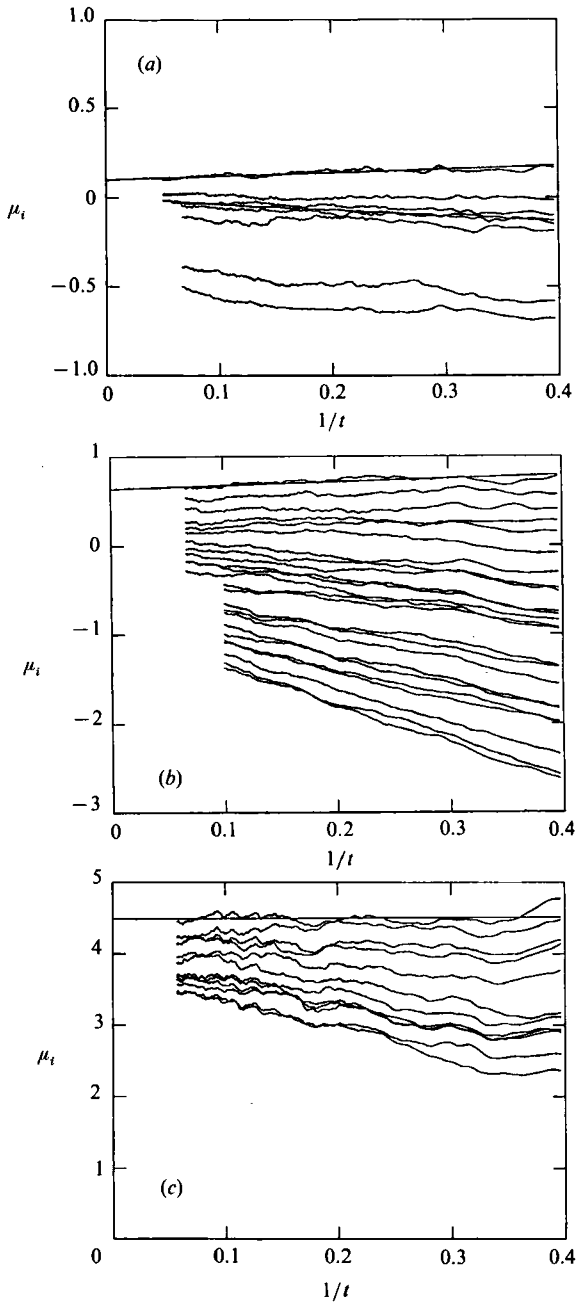


FIGURE 2. The Lyapunov exponents as a function of $1/t$ for the indicated values of Rayleigh number. The least-squares straight line superposed on the trace of the largest exponent indicates how the asymptotic value of the exponent has been obtained. (*a*) $r = 5$, (*b*) 15, (*c*) 70.

that the density of the Lyapunov spectrum is constant under parameter changes (Ruelle 1982; Foiaş *et al.* 1983; Nicolaenko 1986). In particular this has been shown for the Kuramoto–Sivanshinsky equation (Manneville 1985).

Under the assumption that the spectrum is approximately linear, we see that the

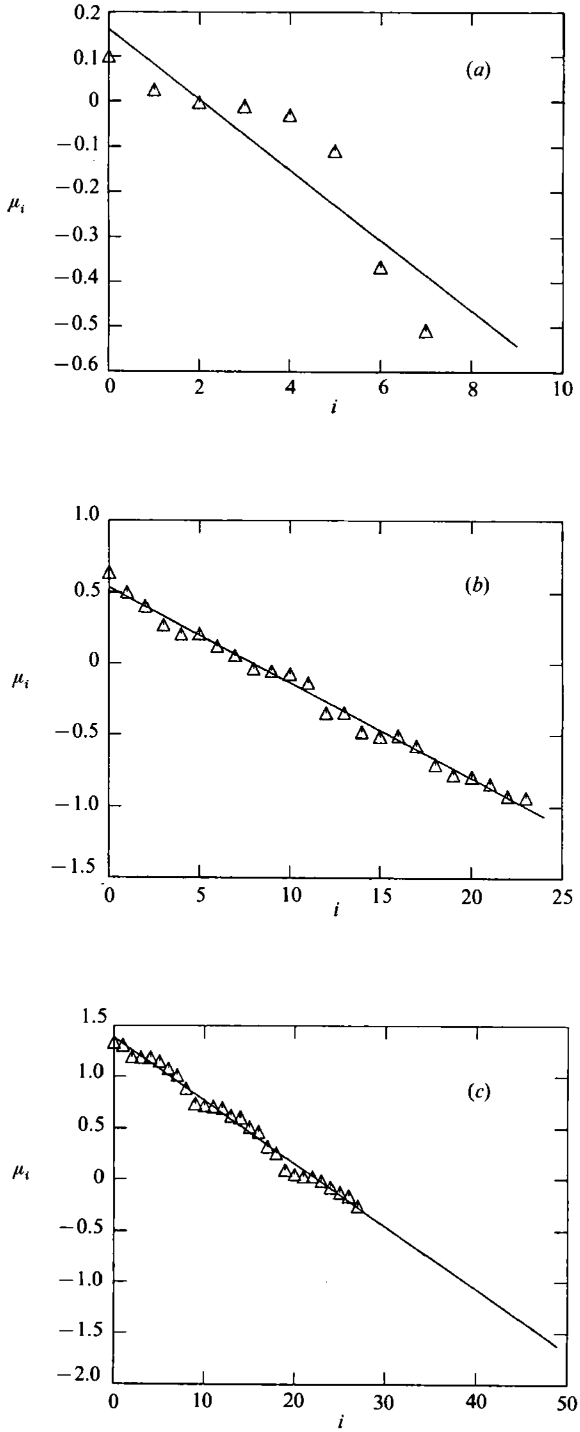


FIGURE 3 (*a–c*). For caption see facing page.

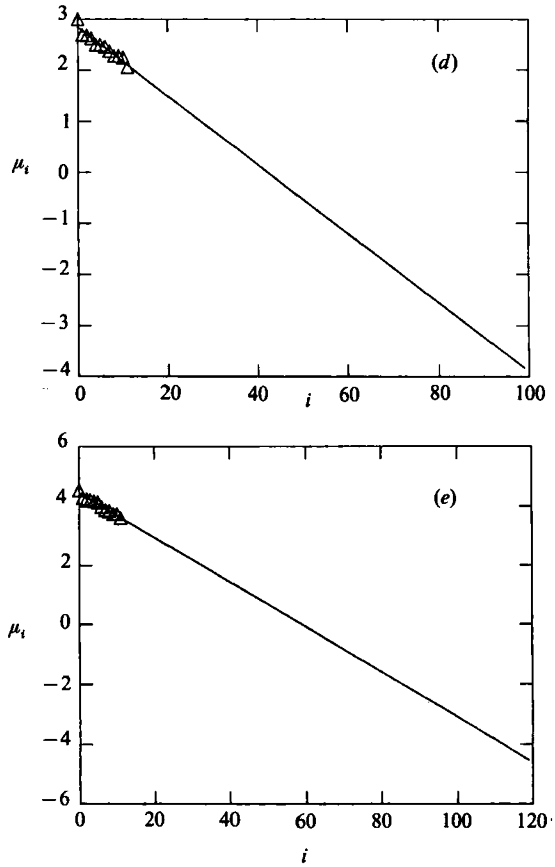


FIGURE 3. The Lyapunov exponent spectrum at various Rayleigh numbers: (*a*) $r = 5$, (*b*) 15, (*c*) 30, (*d*) 50, (*e*) 70. The line is a least squares fit to the spectrum for (*a*) and (*b*). See text §4 for details of (*c*), (*d*) and (*e*).

| $r$ | 5 | 15 | 30 | 50 | 70 |
|---|---|---|---|---|---|
| Slope ($\times 10^2$) | −7.81 | −6.68 | −6.14 | −6.74 | −7.56 |

TABLE 2. Slope of the linear approximation to the Lyapunov exponent spectrum, $\mu_n$, *vs.* $n$, for the indicated values of the Rayleigh number

Lyapunov dimension is given by twice the index of the zero crossing. In figure 4 we plot the Lyapunov dimension, $d_L$, determined by us for the five cases. The values for $r = 5$ and 15 were determined by the Kaplan–Yorke formula, (7), and the remaining values by the assumption that the spectrum is fitted by a straight line. For comparison the K-L dimension, $d_{KL}$, has also been indicated. As expected it is larger than $d_L$, but it nevertheless is of the same order of magnitude. Although the $d_L$ *vs.* *Ra* curve appears near-linear we present an argument in §5 which suggests that this might be less than linear.

As the Rayleigh number increases, and turbulence intensifies, the flow can be expected to enter a scaling regime. The best example of such a scaling regime in turbulence is the inertial range of Kolmogorov (see e.g. Tennekes & Lumley 1972) in which the famous $O(k^{-\frac{5}{3}})$ law holds. For this case, as the Reynolds number increases,

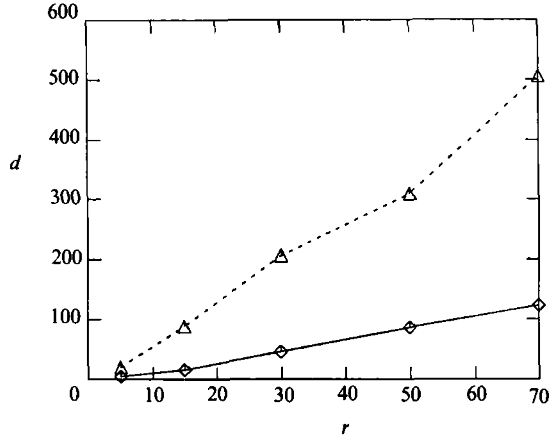


FIGURE 4. Attractor dimension is a function of Rayleigh number. The solid line is the Lyapunov dimension $d_L$, and the dashed line is the K-L dimension $d_{KL}$.

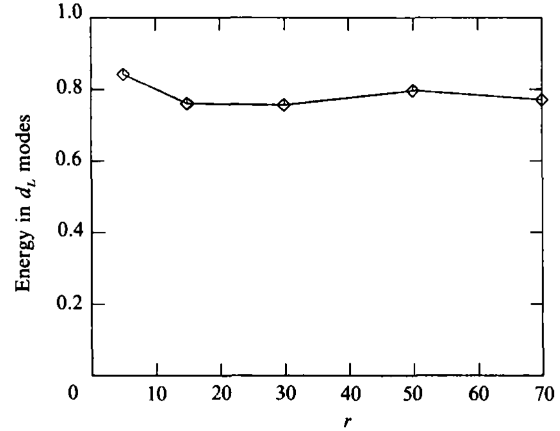


FIGURE 5. Amount of energy captured in $d_L$ K-L modes as a function of Rayleigh number.

the inertial range simply widens, i.e. its dimension increases. We now argue that the Lyapunov dimension gives us some estimate of the point beyond which the flow becomes purely dissipative (Stokesian). It is therefore of interest to consider the percentage of energy, in the optimal K-L coordinates, captured by the dimension $d_L$. Once in the scaling region this should become a constant for a fixed geometry. According to this argument it should become constant as $Ra \to \infty$. Figure 5 contains this plot and the constancy ($\approx 76\%$) of the curve is remarkable in view of the relatively low values of $Ra$ that we consider. Calculations at much higher values of $Ra$ will be necessary to be fully confident of the result.

## 5. Autocorrelation

From the point of view of fluid mechanics the timescales associated with the Lyapunov spectrum do not appear to be immediately relevant. A more natural timescale, theoretically and experimentally, is the correlation time, $\tau_c$. The autocorrelation

$$C_{aa}(t) = \frac{\langle a(t)\, a(t+\tau)\rangle}{\langle a^2(t)\rangle}, \tag{16}$$

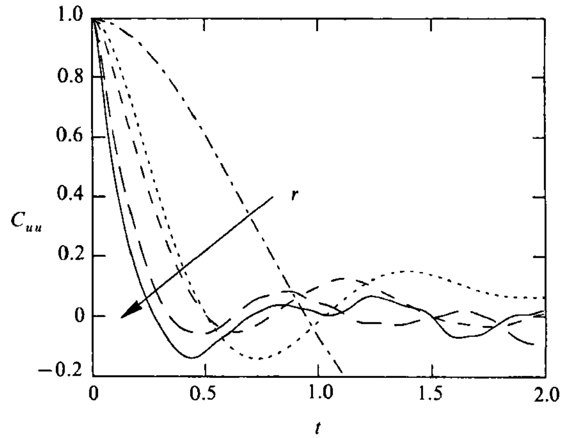


FIGURE 6. The autocorrelation of the horizontal velocity in the midplane for different Rayleigh numbers. The direction of the arrow indicates increasing values of $r$.

where $a$ is some fluid variable, has been a traditional measure of the loss of information. For turbulent flow, such correlations rapidly fall to zero as the delay time, $\tau$, is increased. Beyond a certain correlation time $\tau_c$ the signal is effectively uncorrelated with itself. As a working hypothesis we adopted the assumption that,

$$\tau_c = O(1/\mu_1), \tag{17}$$

to obtain some baseline verification of our calculations of Lyapunov exponents. It has also been suggested instead that the correlation time is related to the metric entropy which is another measure of the loss of information (see e.g. Shaw 1981; Farmer *et al.* 1980 and particularly Badii *et al.* 1988). In this section we explore the connections between these notions on the basis of our calculations.

Figure 6 shows the autocorrelation of the horizontal velocity $C_{uu}$ as sampled in the midplane, $z = \frac{1}{2}$, for several values of $r$. The $C_{uu}$ were constructed by calculating an autocorrelation for each of 16 points in the mid-plane and the averaging these together. The time signal corresponded to $2^{14}$ data points in time and $C_{uu}$ obtained by the Fourier transform method.

The results indicate that, as expected, there is a loss of correlation with increasing values of $\tau$. For all the cases there is a small residual periodicity present, so that $C_{uu}$ does not go monotonically to zero for large $\tau$, but oscillates with decreasing amplitude about zero. While some part of this is doubtless due to the insufficiency in the number of cycles captured to calculate $C_{uu}$, the oscillation reflects the weakly turbulent nature of the flows we have explored, which are dominated by a pair of rolls which randomly line up with the $x$- or $y$-axis. Away from the midplane of the layer, for the cases we have considered, the loss in correlation is less pronounced. The layer feels the presence of the walls, and correlations in the boundary layer are stronger, owing to the reduced level of turbulence.

For autocorrelations of the type shown in figure 6, there is no finite time such that $C_{uu}$ can sensibly be taken to be zero. We thus choose as a criterion for identification of a correlation time $\tau_c$ the appearance of the first minimum in the autocorrelation, which is identifiable at, and away from, the midplane. For some systems it has been shown that $\tau_c$ equals the metric entropy, $\sigma = \sum_{i=1}^{m} \mu_i$ where the sum is over all $\mu_i > 0$. In figure 7 we show $\tau_c$ as calculated here as a function of Rayleigh number and

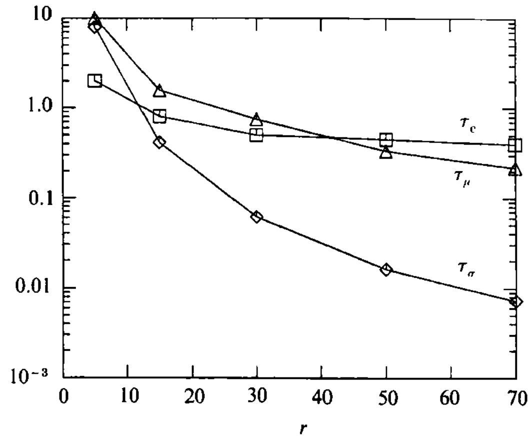


FIGURE 7. Comparison of timescales based on different measures of the loss of information as functions of Rayleigh number. $\tau_c$ is the correlation time, $\tau_\mu$ is the Lyapunov time and $\tau_\sigma$ is the time based on the metric entropy.

also show $\tau_\mu = 1/\mu_1$ and $\tau_\sigma = 1/\sigma = 1/(\mu_1+\mu_2+\ldots\mu_m)$. For large $r$ there are many exponents and $\tau_\mu$ can be much larger than $\tau_\sigma$ but, for small $r$, $\sigma \approx \mu_1$ and so $\tau_\sigma \approx \tau_\mu$. Note that all the values of $\tau_\mu$ have been obtained explicitly since $\mu_1$ has been obtained for all the Rayleigh numbers. To within an order of magnitude $\tau_c$ is approximated by $\tau_\mu$ but not by $\tau_\sigma$.

The issue being discussed may have its origin in semantics. Examination of the metric entropy shows that it measures the rate of information loss as a result of a volumetric increase in the phase space. The reciprocal correlation time, on the other hand, estimates the rate of information loss based on the examination of just one dependent variable. A more telling way to see that metric entropy does not give a measure of the correlation time is to observe that $\sigma$ increases as the aspect ratio increases ($\sigma$ is extensive) whereas $\tau_c$, being intensive, does not change in any significant way under such a change.

## 6. Discussion

A simple argument for estimating the degrees of freedom active in a turbulent convective flow can be based on the argument given by Landau (1944) for general turbulent flows. The smallest relevant scale in the convective problem is $\delta$, the thickness of the thermal layer at a wall (it is tacitly being assumed that $Pr = O(1)$). Thus, the number of degrees of freedom in the vertical direction is

$$\frac{H}{\delta} = 2Nu, \tag{18}$$

which, as indicated, is twice the Nusselt number. Neglecting factors of order 1, one can argue that the number of degrees of freedom $d$ is

$$d = O(Nu^3). \tag{19}$$

If it is further assumed that the Nusselt number takes on the *free-fall* relation, $Nu = O(Ra^{\frac{1}{3}})$, then we obtain

$$d = O(Ra) \tag{20}$$

(Foiaş *et al.* 1986). A mathematically rigorous estimate of the degrees of freedom has been given by Foiaş *et al.* (1987) and Ruelle (1982) in two space dimensions and shown to be $O(Ra)$. This, however, implies that instead of (20), one obtains $O(Ra^{\frac{3}{2}})$ in three dimensions.

Before proceeding further, we observe that if the width and depth of the convective cell are denoted by $W$ and $D$, respectively, then

$$d = O(V/\delta^3), \tag{21}$$

where $V = HWD$ is the volume. This relation underlines the basic extensive quality of dimension estimates. It also underlines a somewhat impractical feature of such deliberations. For, in the limit of large aspect ratios, $d$ becomes unbounded. This is due to the inclusion of long-range correlations, which in any practical sense are down in the noise of any real problem. As has been pointed out (Sirovich 1989; Sirovich & Sirovich 1989), for turbulent flow, the spatial correlation length, or integral scale, should be taken as the relevant lengthscale in the horizontal directions, in order to obtain a dimension estimate of practical significance.

To apply (19), it was necessary to know $Nu$ as a function of $Ra$. As we discussed in I, a variety of power laws arise from experiment and theory, and our use of the free-fall relation should be regarded as nominal. In the recent experiments on Rayleigh–Bénard convection by the Chicago group (Castaing *et al.* 1989), $Nu = O(Ra^{\frac{1}{3}})$ was found to hold in the *soft* turbulence region, implying (20). But in the *hard* turbulence regime $Nu = O(Ra^{\frac{2}{7}})$ was found, which if inserted into (19) yields

$$d = O(Ra^{\frac{6}{7}}). \tag{22}$$

We now approach the problem of determining dimension from a different perspective, one which appears to lead to a sharper result.

If we consider a boundary layer, thickness $\delta$, the velocity of an eddy emerging from it is

$$w = (2g\alpha\Delta T\delta)^{\frac{1}{2}}. \tag{23}$$

This in turn defines a timescale

$$\tau' = \frac{\delta}{w}, \tag{24}$$

which when normalized in the standard way (with respect to $H^2/\kappa$) yields

$$\tau = \left(\frac{1}{Nu\,Pr\,Ra}\right)^{\frac{1}{2}}. \tag{25}$$

This we assert can be identified with the correlation time. Thus if we adopt $Nu = O(Ra^{\frac{1}{3}})$ we are led to

$$\tau = O(Ra^{-\frac{2}{3}}). \tag{26}$$

But by virtue of the remarks in §5 this is proportional to $1/\mu_1$, the leading Lyapunov exponent. Thus on the basis of the assumptions (*a*) and (*b*) in §4 we conjecture that

$$d_{\mathrm{L}} = O(Ra^{\frac{2}{3}}). \tag{27}$$

Equation (27) is an unverified conjecture and a cautionary discussion already appears in the Introduction. We contrast (27) with (20), by noting a somewhat subtle but essential difference in the premises that go into the derivation of the two formulas. Equation (20) was an estimate of the degrees of freedom while (27) hopes to estimate the attractor dimension, the point being that if certain scales find a relationship or correlation amongst themselves, then fewer fitting functions will be necessary. This lies at the heart of the difference in these estimates, and perhaps sheds light on why (27) is sharper. One additional point should be mentioned, that (27) is an intrinsic measure. It does not include the extrinsic affect mentioned in connection with (21). In practical terms this would permit unnatural correlations at scales larger than the integral integral scale. This is deemed unimportant in any real situation.

Finally, we remark on a concern which the reader may have. Are the numerical estimates of dimension robust under increasing spatial resolution of the flow? To answer this we first point out that the working definition of $d_{KL}$ through (3) and (4) lead to $d_{KL} = 512$ and 320 for grids of $(12)^2 \times 32$ and $(32)^3$ respectively. Clearly, even neglecting energetically unimportant modes raises $d_{KL}$ by almost a factor of 2. However, we note that the energy curve for $r = 70$ (figure 1) is very flat for large values of energy cutoff. Thus this error in $d_{KL}$ is of the same order as taking the cutoff in energy not at 90% but, say, at 88% and thus not of real significance. Second, we re-emphasize that for the larger values of $r$ that we have considered, we have based our estimates on a small number of exponents and this is likely to be a greater source of error. For the smaller values of $r$ the flow is completely resolved and here we have calculated all the exponents and the dimension estimate is precise.

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