stability argument now parallels that of the general case.

6. ALGEBRAIC STABILITY

We now consider a situation omitted in the previous discussion of stability—perturbations independent of the direction along $\mathbf{B}_0$. In this situation $k = 0$ and the linearized equations are given by

$$\begin{aligned} &0 = (\partial B_\parallel/\partial t) + \nabla \cdot \mathbf{e}; \quad 0 = \partial \mathbf{B}_\perp/\partial t, \\ &0 = (\rho_0/B_0)(\partial \mathbf{e}/\partial t) + \nabla p, \\ &0 = \frac{\partial}{\partial t} g^\pm + (\mathbf{e} + v\mathbf{B}_\perp) \cdot \nabla g_0^\pm + g_0^\pm \frac{B^\pm}{B_0} \\ &\qquad \cdot \left\{ \alpha^\pm \left(\frac{\nabla P_\parallel^0}{\rho_0} \right)_\Delta + \left[\alpha^\pm \left(\frac{P_\perp - P_\parallel^0}{\rho_0} \right)_\Delta - \mu B_0 \right] \nabla B_0 \right\}. \end{aligned} \tag{15}$$

The time invariance of $\mathbf{B}_\perp$ implied by Eqs. (15) indicates an algebraic growth in $g^\pm$ due to gradients in the equilibrium configuration. The realization of this growth when k is a transform variable is at present undetermined.

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Plane Wave Propagation in Kinetic Theory*

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A number of results for plane wave propagation in kinetic theory are obtained. Among these are the following. It is shown that the Boltzmann equation for bounded collision operator has a cutoff frequency beyond which plane waves cease to exist. The analytic continuation of the dispersion relation is discussed and asymptotic results beyond the critical frequency are obtained. Using a kinetic model it is shown that the low-frequency formal power series for complex wavenumber in terms of frequency (and vice versa) are divergent. The last is contrary to the recent result obtained for the rigid sphere Boltzmann equation.

1. INTRODUCTION

ONE of the customary devices for examining the nature of a theory is through the study of plane wave propagation. In cases where a solution may be shown to be composed only of plane waves, such a study fully discloses the theory. In rarefied gas dynamics, a continuous spectrum appears in addition to the point spectrum and hence plane waves only partially reveal the theory. However, on physical grounds one expects the continuous spectrum not to play a role except at high frequencies.[1] Cercignani[2] and Weitzner[3] have shown the contrary to be true for the Krook model (this conclusion should hold for any model of the Gross–Jackson[4] type)—but this is a shortcoming of the model rather than a physical effect. In this paper our attention is focused on plane waves only, and more specifically on the low- and high-frequency regimes.

Following the general practice, we define a plane wave as a perturbation solution which has exponential space–time dependence $\sim e^{\sigma t + sx}$. For generality we regard both σ and s as complex; however, only two special cases are of physical interest. When $s = ik$, k real, the situation corresponds to a pure initial-value problem and we refer to this as a free wave. When $\sigma = i\omega$, ω real, the situation corresponds to an oscillating wall (boundary-value problem) and we refer to this as a forced wave. This terminology is

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[1] We use the term frequency generically to signify a modulus of oscillation in either space or time.

[2] C. Cercignani, Ann. Phys. (N.Y.) **30**, 154 (1964).

[3] H. Weitzner, in *Proceedings of Fourth International Conference on Rarefied Gasdynamics* (Academic Press Inc., New York, 1965).

[4] E. P. Gross and E. A. Jackson, Phys. Fluids **5**, 432 (1959).

the same used by us in our investigations of sound propagation.[5,6]

Formal power series methods can be employed in analyzing low-frequency plane waves, and these are by far the simplest to consider. This approach proved useful in a study which compared the Boltzmann equation, model equations, and various macroscopic approximations.[7] Such formal expansions, which can only be presumed valid for low frequencies, have in addition a variety of other uses, especially in solving problem. It is therefore natural to enquire as to the convergence of such expansions. In two recent studies of the rigid sphere Boltzmann equation,[8,9] it was shown that these expansions are convergent. In Sec. 6, using a kinetic model, we prove the contradictory result; i.e., the expansion is divergent.[10] The reason behind this conflict is that the collision frequency ν which is a constant for the model equations of Gross and Jackson,[4] behaves as $|\boldsymbol{\xi}|$, $|\boldsymbol{\xi}|$ large, for rigid spheres. For reasons given in Sec. 6, our conjecture is that the formal power series of $\sigma(k)$ and $s(\omega)$ diverge for $\nu(\xi) \to |\boldsymbol{\xi}|^\alpha$, $0 \leq \alpha < 1$, $|\boldsymbol{\xi}|$ large. And that they converge for $\nu(\xi) \to |\boldsymbol{\xi}|$, $|\boldsymbol{\xi}|$ large (exponents larger than unity are not of physical interest).

Detailed results for $\sigma(k)$, $s(\omega)$ for all values of the arguments do not seem attainable without numerical aid.[5,6] For large values of frequency, however, a variety of results may be obtained. In Sec. 3 we consider the finite moments method of Wang Chang and Uhlenbeck.[11] (This method, which is macroscopic in approach, is the same used in the exhaustive numerical investigations of Pekeris *et al.*[12]) We show in this case that for large ω, Im $s(\omega) \sim \omega$ and Re $s(\omega)$ is bounded. The latter effect accounts for the poor agreement this theory has with experiment.[5,6]

In considering the initial value problem for model equations, we have shown that plane wave propagation ceases past some critical frequency.[13] More recently Cercignani,[2] using the Krook model, has shown the same result for shear waves. In Sec. 4 we consider Boltzmann equations with bounded collision operators (this includes the kinetic models of Gross and Jackson[4]) and show these always have a critical frequency beyond which plane waves do not exist.

In Sec. 5 we consider the formulation of a specific problem and from it we introduce the analytic continuation of the dispersion relation. This results in relations for $\sigma(k)$ and $s(\omega)$ past the critical frequency. This notion has proven valuable in the numerical studies of Refs. 5 and 6, and in fact leads to plots of speed and attenuation rate which are in close agreement with experiment. Using the analytically continued dispersion relation we obtain asymptotic forms for $\sigma(k)$ and $s(\omega)$. In particular, for forced sound waves we obtain

$$s \sim \frac{-|\omega|}{[\ln \omega^2]^{\frac{1}{2}}}\left\{1 - \frac{3\pi i}{2 \ln \omega^2}\right\}.$$

2. FORMULATION

We follow the notation of an earlier paper.[5] In brief, the linearized Boltzmann equation, in one dimension, for the perturbed distribution function g is written as

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x}\right) g = \mathfrak{L}(g) = \int f_*^0 [g'_* + g' - g_* - g] \times B(\theta, |\boldsymbol{\xi}_* - \boldsymbol{\xi}|)\, d\theta\, d\epsilon\, d\boldsymbol{\xi}_*. \quad (2.1)$$

This equation is made dimensionless with respect to an as yet unspecified frequency $\nu = 1/\tau$, as follows:

$$t' = \nu t, \quad x' = x\nu/(RT_0)^{\frac{1}{2}},$$
$$\boldsymbol{\xi}' = \boldsymbol{\xi}/(RT_0)^{1/2} B' = \rho_0 \nu B/m. \quad (2.2)$$

Substituting and then removing the primes we obtain the dimensionless equation

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x}\right) g = L(g) = \int \Omega_* [g'_* + g' - g_* - g] B\, d\theta\, d\epsilon\, d\boldsymbol{\xi}_* \quad (2.3)$$

with

$$\Omega = e^{-\xi^2/2}/(2\pi)^{\frac{3}{2}}. \quad (2.4)$$

Introducing the inner product (asterisk denotes complex conjugate)

$$(p, q) = \int \omega p^*(\boldsymbol{\xi}) q(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \quad (2.5)$$

one can show that for

$$\Psi_i = S_{l(i)+\frac{1}{2}}^{r(i)}(\xi^2/2)\xi^{l(i)} \times P_{l(i)}(\xi_1/\xi) \Big/ \left[\frac{2^{l(i)+1}\Gamma(r(i) + l(i) + \frac{3}{2})}{\pi^{\frac{1}{2}} r(i)!\,(2l(i) + 1)}\right]^{\frac{1}{2}}, \quad (2.6)$$

$$(\Psi_i, \Psi_j) = \delta_{ij}. \quad (2.7)$$

[5] L. Sirovich and J. K. Thurber, J. Acoust, Soc. Am. 37, 329 (1965).
[6] L. Sirovich and J. K. Thurber, in *Proceedings of Fourth International Conference on Rarefied Gasdynamics* (Academic Press Inc., New York, 1965).
[7] L. Sirovich, Phys. Fluids 6, 10 (1963).
[8] J. A. McLennan, Phys. Fluids 8, 1580 (1965).
[9] A. A. Arseniev, Zh. Vitch. Mat. i Mat. Phys. (Moscow) 5, 864 (1965).
[10] See also the papers by L. Sirovich and J. Thurber, in "Notes of a in Summer Conference Statistical Mechanics," Brookhaven National Laboratories (1965).
[11] C. S. Wang Chang and G. Uhlenbeck, "On the Propagation of Sound in Monoatomic Gases," Univ. Mich. Eng. M999 (1952).
[12] C. Pekeris, Z. Alterman, L. Finkelstein, and K. Frankowski, Phys. Fluids 5, 1608 (1962).
[13] L. Sirovich and J. K. Thurber, "Sound Propagation According to Kinetic Models," NYU Inst. Math. Sci. Rept. AFOSR-1380 MF-17 (1961) [see also *Rarefied Gas Dynamics* (Academic Press Inc., New York, 1963)].

S_q^p and P_l are the Laguerre and Legendre polynomials, respectively. The notation $r(i)$, $l(i)$ signifies that the double subscript rl has been reduced to a single subscript i.[14] For $B = B(\theta)$ Wang Chang and Uhlenbeck[11] have shown ψ_i to be eigenfunctions of L, i.e., for $B = B(\theta)$

$$L(\Psi_i) = \lambda_i \Psi_i, \tag{2.8}$$

where

$$\lambda_i = (\Psi_i, L(\Psi_i)). \tag{2.9}$$

More generally we define

$$\Lambda_{ij} = (\Psi_i, L(\Psi_j)). \tag{2.10}$$

We find it convenient to introduce the projection operator

$$P_N = \sum_{i \le N} \Psi_i(\Psi_i). \tag{2.11}$$

Hence

$$g = P_\infty g = \sum_{i=1}^{\infty} a_i \Psi_i \tag{2.12}$$

with

$$a_i = (\Psi_i, g). \tag{2.13}$$

Plane wave solutions to the Boltzmann equation are of the form

$$g = e^{\sigma t + sx} g(\xi), \tag{2.14}$$

where we allow both σ and s to be complex. In order for such solutions to exist we must find σ and s such that the homogeneous equation

$$(\sigma + s\xi_1 - L)g = 0 \tag{2.15}$$

is satisfied. Various aspects of this problem have been considered in the literature. With the exception of low-frequency phenomena, these results are generally obtainable only by numerical methods. In this paper we analytically show certain general features of both high- and low-frequency wave propagation.

In keeping with common practice we refer to the relation in σ and s which results from

$$(\sigma + s\xi_1 - L) = 0 \tag{2.16}$$

as the dispersion relation. As mentioned in the Introduction we refer to the case when $\text{Re}\,(s) = 0$ as the free-wave problem, and to case when $\text{Re}\,(\sigma) = 0$ as the forced-wave problem.

3. TRUNCATED BOLTZMANN EQUATION

In order to solve (2.16) Wang Chang and Uhlenbeck[11] consider the truncated Boltzmann equation

$$P_N(\sigma + s\xi_1 - L)P_N g = 0. \tag{3.1}$$

In considering the forced wave problem they show for Maxwell molecules, that the coefficients in the formal power series of $s = s(i\omega)$ can be exactly determined by successive truncations. Pekeris and co-workers[12] in a numerical investigation consider the exact dispersion relation which results from (3.1), for relatively large N. In this section we give an analytical description of the truncated dispersion relation (3.1) for high-frequency phenomena.

Equation (3.1) leads to the equivalent $N \times N$ coefficient system

$$(\sigma \mathbf{1} + s\mathbf{A} - \mathbf{\Lambda})\mathbf{a} = 0, \tag{3.2}$$

where

$$A_{ij} = (\Psi_i, P_N \xi_1 \Psi_j), \quad i, j \le N \tag{3.2a}$$

and $\mathbf{\Lambda}$ is defined by (2.10), and $\mathbf{a}$ by (2.13). It is convenient to define the finite inner product equivalent to (2.5)

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{N} a_i^* b_i. \tag{3.3}$$

We now observe that $\mathbf{\Lambda}$ is negative semidefinite,

$$\langle \mathbf{a}, \mathbf{\Lambda a} \rangle = (P_N g, L(P_N g)) \le 0 \tag{3.4}$$

the latter relation being a well-known property of the Boltzmann equation. Also as is well known, equality in (3.4) holds if and only if $N = 3$.[15]

From this it follows that there exists an $(N - 3)$-order square matrix $\tilde{\mathbf{S}}$ such that

$$\mathbf{S} = \begin{pmatrix} 1_3 & 0 \\ 0 & \tilde{\mathbf{S}} \end{pmatrix}$$

diagonalizes Λ under similarity (1_n denotes the $n \times n$ unit matrix),

$$\mathbf{S\Lambda S}^{-1} = \mathbf{D}.$$

$\mathbf{D}$ is a nonpositive diagonal matrix in which the first three diagonal elements are zero.

Because of its natural importance we focus attention on the forced wave problem. We therefore wish to determine $s = s(\omega)$ such that

$$\begin{aligned} 0 &= \det\,(i\omega\mathbf{1} + s\mathbf{A} - \mathbf{\Lambda}) \\ &= \det\,(\mathbf{1}i\omega + s\mathbf{SAS}^{-1} - \mathbf{D}). \end{aligned} \tag{3.5}$$

Setting $s/i\omega = z$, we seek $z(\omega)$ for $\omega \to \infty$ in

$$\det\,[\mathbf{1} + z\mathbf{H} - (i/\omega)\mathbf{D}] = 0, \tag{3.6}$$

where $\mathbf{H}$ is the real symmetric matrix

$$\mathbf{H} = \mathbf{S}^{-1}\mathbf{AS}.$$

Since the degree of (3.6) remains unchanged as

[14] The degree of Ψ_{rl} is $2r + l$. A variety of reductions can be based on the degree of the polynomials.

[15] Since the first three Ψ_i are linear combinations of the collision invariants.

$\omega \to \infty$, z possesses a power series in $1/\omega$.[16] Since **H** is real symmetric, and from the form of (3.6), we have

$$z(\infty) \neq 0,$$
$$\operatorname{Im} z(\infty) = 0.$$

Further it is easily shown that (3.6) is even in z. Finally, if we denote the normalized eigenvectors of the matrix of (3.5) by **u**, we easily have

$$\operatorname{Im}(z) = -\langle \mathbf{u}, \mathbf{\Lambda u}\rangle/\omega\langle \mathbf{u}, \mathbf{Au}\rangle.$$

Since

$$\langle \mathbf{u}, \mathbf{\Lambda u}\rangle \leq 0$$

and

$$\langle \mathbf{u}, \mathbf{Au}\rangle \to -1/z(\infty),$$

we have

$$\operatorname{sgn}(\operatorname{Im} z) = -\operatorname{sgn}\big(z(\infty)\omega\big).$$

Therefore we have, for $\omega \to \infty$,

$$z = \frac{s}{i\omega} \sim \pm\left(\frac{1}{c_0} - \frac{i\alpha}{\omega}\right) \tag{3.7}$$

with c_0 and α nonnegative, or

$$s \sim \pm(i\omega/c_0 + \alpha). \tag{3.8}$$

This has the basic property of forced waves, i.e., that they decay in propagating to either $\pm\infty$. A noteworthy feature of the truncated Boltzmann equation is that it leads to $\operatorname{Re} s$ bounded as (3.8) indicates.

Denoting the viscosity of a gas by μ and the pressure by p, we specify the ν, (2.2), by

$$p/\mu.$$

Therefore, if $\bar{\omega}$ denotes the dimensional frequency we have

$$\omega = \bar{\omega}\mu/p = 1/r. \tag{3.9}$$

The quantity r was introduced by Greenspan[17] and is the parameter used in plots of sound results. We now note that as $\omega \to \infty$ the sound speed and attenuation rate are given by

$$\operatorname{Im} z/(\tfrac{5}{3})^{\frac{1}{2}} = a/a_0 \sim \text{const}, \tag{3.10}$$
$$\operatorname{Re} z \sim \tau$$

(a_0 denotes the adiabatic sound speed).

We note in passing that the free wave results follow if one inverts (3.7) and (3.8).

4. BOLTZMANN EQUATION WITH BOUNDED COLLISION OPERATOR

A useful class of equations in kinetic theory are the kinetic models.[4] To obtain these we first introduce

$$l = \operatorname{lub} -\langle \mathbf{x}, \mathbf{\Lambda x}\rangle/\langle \mathbf{x}, \mathbf{x}\rangle,$$

where

$$\Lambda_{ij} = (\Psi_i, \mathfrak{L}\Psi_j), \quad i, j \leq N + 1$$

[$\mathfrak{L}$ is the dimensional collision operator introduced in (2.1)]. Then taking

$$\nu = l$$

in (2.2), the kinetic models are defined to be

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\xi}\cdot\frac{\partial}{\partial \mathbf{x}} + 1\right)g = P_N(L+1)P_N g = Kg. \tag{4.1}$$

It is clear by construction that the norm $\|K\|$ of **A** satisfies

$$0 < \|K\| \leq 1. \tag{4.2}$$

The discussion which now follows also applies to Boltzmann equations for which K satisfies (4.2).

We focus attention on the forced wave problem

$$(i\omega + s\xi_1 + 1)g = Kg, \tag{4.3}$$

and for convenience we take

$$(g, g) = 1.$$

Taking the inner product of (4.3) with g, the real and imaginary parts may be written as

$$s_r = [-1 + (g, Kg)]/(g, \xi_1 g), \tag{4.4}$$
$$(g, \xi_1 g) = -\omega/s_i, \tag{4.5}$$

where we have written $s = s_s + is_i$. Equation (4.5) states that $(g, \xi_1 g) \geq 0$ for waves moving to the right and $(g, \xi_1 g) \leq 0$ for waves moving to the left. With (4.4) we have $s_r < 0$ for waves moving to the right and $s_r > 0$ for waves moving to the left. This is a basic feature of forced waves.

We can in full generality focus attention only on waves moving to the right, and also take $\omega > 0$. Defining

$$z = x + iy = (1 + i\omega)/s,$$

substituting for s in (4.4) and (4.5), and eliminating x, we find

$$y = -\omega(g, \xi_1 g)(g, Kg)/[\omega^2 + (g, Lg)^2]. \tag{4.6}$$

From this it is clear that $y \geq 0$.

We now show that $(g, \xi_1 g)$ is bounded away from zero. For if $(g, \xi_1 g) = 0$ then

$$i\omega = (g, Kg) - 1$$

implies

$$\omega = 0$$

and

$$g = a_1\Psi_1 + a_2\Psi_2 + a_3\Psi_3;$$

i.e., g is composed of only the collisional invariants, and therefore $(1 - K)g = 0$. The Boltzmann equation now becomes

$$s\xi_1 g = 0,$$

[16] If the roots are not distinct a Puiseux series may result.
[17] M. Greenspan, J. Acoust. Soc. Am. **28**, 644 (1956).

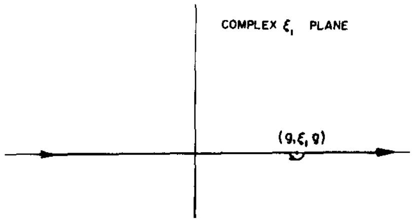

FIG. 1. Path of integration in the complex ξ_1 plane.

which implies a_i, $i = 1, 2, 3$ are constant. These may be eliminated by linearization about an appropriate Maxwellian in the definition of g.

We next show that $(1 + i\omega + s\xi)$, for $\omega \pm 0$, does not vanish for plane waves. For on substituting from (4.4) and (4.5)

$$1 + i\omega + s\xi_1 = 1 + \frac{-1 + (g, Kg)}{(g, \xi_1 g)}\xi_1 + i\left[\omega - \frac{\omega}{(g, \xi_1 g)}\xi_1\right]. \quad (4.7)$$

And hence this expression can be zero only for g, such that $(g, Kg) = 0$. But once K is nonnegative this implies that g belongs to the null space of K and for such a g,

$$(1 + i\omega + s\xi_1)g = 0.$$

This in turn implies that $g = 0$ except at the point $\xi_1 = (g, \xi_1 g)$ and hence $(g, g) = 1$ is violated. This proves our assertion. We may therefore form

$$g = [1/(1 + i\omega + s\xi_1)]Kg$$

and from this

$$1 = (g, [1 + i\omega + s\xi_1]^{-1}Kg) = \int \frac{\Omega g^* Kg\, d\xi}{1 + i\omega + s\xi_1}. \quad (4.8)$$

The argument leading to (4.6) then states that ω and s for plane waves stay to one side of the cut of the integral in (4.8).

Finally, we demonstrate that, in order for (4.7) to be satisfied, ω must be bounded. [Then (4.4) and (4.5) imply that s is also bounded.] Using (4.7) in (4.8), it is easily seen that as $\omega \to \infty$

$$\int \frac{\Omega g^* Kg\, d\xi}{1 + i\omega + s\xi_1} \sim \frac{1}{i\omega} \int \frac{\Omega g^* Kg\, d\xi}{1 - \xi_1/(g, \xi_1 g)},$$

where the path of integration is shown in Fig. 1. Therefore, using the Plemelj formula[18] we see that if $(g, \xi_1 g)$ is bounded as $\omega \to \infty$, the integral is $O(1/\omega)$ and if $(g, \xi_1 g)$ is unbounded the integral also vanishes since $(g, g) = 1$. Hence in either case (4.8) is violated as $\omega \to \infty$ and therefore ω and hence s must remain bounded for plane waves.

[18] In using the Plemelj formula we are assuming that g and Kg are, for example, Holder continuous.

A similar discussion can be achieved by even more elementary means for the free-wave problem.

The appearance of a cutoff in kinetic models for free-wave propagation was demonstrated in an earlier paper[13] and more recently was demonstrated by Cercignani[2] for the Krook equation for shear waves.

5. ANALYTIC CONTINUATION OF THE DISPERSION RELATION AND HIGH-FREQUENCY EXPANSIONS

The solution to both the initial- and boundary-value problems may be regarded as resulting from a normal mode analysis. Briefly, in this method one sums over the discrete modes and integrates over the continuum modes. For Boltzmann equations of the type considered in the previous section the discrete modes vanish at high frequencies and only the continuum modes remain. A more straightforward method of dealing with such problems is by means of transforms. The two approaches are, of course, equivalent.[19] For certain purposes the latter method seems superior. To anticipate the results of this section we first show that the discrete modes have an analytic continuation past the critical frequency. This is important since the cutoff frequency of the equations of Sec. 4 are mathematical rather than physical.[20] For example, the cutoff value of frequency for the Gross–Jackson type models increases with the index N in (4.1). The analytically continued mode furnishes an approximation of propagation speed and attenuation rate when a physical wave appears even though the theory excludes waves at such frequencies. It has been found for sound propagation that the analytic continuation is in excellent agreement with experiment.[5]

Since our only intention is to introduce the analytic continuation, we do so by means of the simplest problem. Consider the pure initial-value problem for

$$\left(\frac{\partial}{\partial t} + ik\xi_1 + 1\right) g = P_N(L + 1)P_N g = \sum_{m,n \le N} a_n \beta_{mn} \Psi_n,$$
$$g(t = 0) = g_0(k). \quad (5.1)$$

This is just (4.1) for "monochromatic" initial data. Introducing the Laplace transform, we obtain

$$(\sigma + ik\xi_1 + 1)g = \sum_{m,n \le N} a_m \beta_{mn} \Psi_n + g_0(k), \quad (5.2)$$

where the same letter has been used for the transformed dependent variable, and σ for the transform variable. Taking σ with a sufficiently large real part, we may

[19] K. M. Case, Ann. Phys. (N.Y.) 7, 349 (1959).

[20] Plane wave cutoff very possibly may occur and a "physically correct" Boltzmann should predict it. The equations of Sec. 4, however, predict the onset of "cutoff" at too low a frequency.

divide by $(\sigma + ik\xi_1 + 1)$ in (5.2) and form the moments

$$a_i = (\Psi_i, [\sigma + ik\xi_1 + 1]^{-1}[P_N(L+1)P_N g + g_0]). \quad (5.3)$$

Then introducing the following definitions,

$$u_i = (\Psi_i, \{\sigma + ik\xi_1 + 1\}^{-1} g_0),$$
$$C_{ij} = (\Psi_i, \{\sigma + ik\xi_1 + 1\}^{-1} \sum_{n \le N} \beta_{jn}\Psi_n, \quad i, j \le N \quad (5.4)$$

the solution for a_i $(i = 1, N)$ is given by

$$\mathbf{a} = \frac{1}{2\pi i}\int_{\uparrow} (\mathbf{1} - \mathbf{C})\mathbf{u}e^{\sigma t}\, d\sigma. \quad (5.5)$$

The arrow denotes the Bromwich path in the right half of the σ plane.[21]

Each of the entries in the matrix $\mathbf{C}$ is of the form

$$(1+\sigma)\int \frac{p(\boldsymbol{\xi})\Omega\, d\boldsymbol{\xi}}{\sigma + 1 + s\xi_1} = \lambda \int \frac{p(\boldsymbol{\xi})\Omega\, d\boldsymbol{\xi}}{\lambda - i\xi_1}, \quad \lambda = (1+\sigma)/is, \quad (5.6)$$

where $p(\boldsymbol{\xi})$ is a polynomial in $\boldsymbol{\xi}$. For generality we have introduced the complex wavenumber s. By straightforward manipulation, (5.6) may be reduced to a polynomial in λ plus a polynomial in λ multiplied by

$$M = \lambda \int \Omega\, d\boldsymbol{\xi}/(\lambda - i\xi_1). \quad (5.7)$$

The explicit form of each C_{ij} can be given,[6] but this is of no interest here. The integral in (5.7) actually defines two functions M^+ and M^- for $\mathrm{Re}\,\lambda > 0$, $\mathrm{Re}\,\lambda < 0$, respectively. (This lies at the root of the cutoff shown in the last section.) On the other hand, in distorting the Bromwich path of (5.5), one makes use of the Cauchy integral theorem. This requires the analytic continuation of the integrand of (5.5) and hence of M.[22] For example, in the above initial-value problem if $k > 0$, M^- must be used in (5.5). In distorting the integration path to the left of $\mathrm{Re}\,\sigma = -1$ $(\mathrm{Re}\,\lambda < 0)$, M^- must still be used; i.e., the analytic continuation of the function M^- defined by (5.7). This yields the analytic continuation of the discrete spectra past the cutoff (i.e., in this case as k passes the critical wavenumber k_c). It should be noted, however, that the last section proves that no plane wave now actually exists in the theory—but for reasons mentioned earlier a plane wave may still physically exist.

In practice there is a somewhat simpler method of solving the above problem. If we form the system

$$(\Psi_i, \{\sigma + ik\xi_1 + 1\}g) = (\Psi_i, P_N(L+1)P_N g + g_0), \quad i = 1, N-2, \quad (5.8)$$

this system of $(N-2)$ equations by a proper arrangement of the Ψ_i, will only contain a_i for $i = 1, N$. Then, augmenting (5.8) by (5.3) for $i = N-1, N$ we have a determined system. The advantage to this mixed formulation is that the M function now occurs only in only two rows of the coefficient matrix of $\mathbf{a}$.

Using the analytic continuation we now obtain the high-frequency expansions for the roots of the dispersion relation. This is now carried out specifically for the Krook model $(N = 3)$.

$$\left(\frac{\partial}{\partial t} + \xi_1\frac{\partial}{\partial x} + 1\right)g = \Psi_1(\Psi_1, g) + \Psi_2(\Psi_2, g) + \Psi_3(\Psi_3, g). \quad (5.9)$$

Forming the matrix system according to the procedure already given we are then led to the consideration of the zeros of

$$\det(\mathbf{1} - \mathbf{C}) = 0.$$

For (5.9) this can be shown to be[13]

$$0 = 1 + \frac{i}{s}\left[\frac{\lambda^3(M-1)}{6} + \frac{11M}{6\lambda} - \frac{2\lambda M}{3} + \frac{5\lambda}{6}\right] - \frac{1}{s^2}\left[\frac{2M^2}{3\lambda^2} - \frac{2M(M-1)}{3} + \frac{\lambda^2(M-1)}{3} - \frac{2M}{3} + 1\right] + \frac{i}{s^3}\left[-\frac{2M(M-1)}{3\lambda} + \frac{\lambda(M-1)}{6} + \frac{M}{6}\right]. \quad (5.10)$$

The function M can be shown to be given by[6]

$$M^{\pm} = \pm(\tfrac{1}{2}\pi)^{\frac{1}{2}}\lambda e^{\lambda^2/2}[1 \mp \phi(\lambda/\sqrt{2})], \quad \mathrm{Re}\,\lambda \gtrless 0,$$

where

$$\phi(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{x\sqrt{2}} e^{-t^2/2}\, dt \quad (5.11)$$

is the error function.

The asymptotic expansions are easily found to be

$$M^{\pm}(\lambda) = 1 - \frac{1}{\lambda^2} + \frac{1\cdot 3}{\lambda^4} \mp \cdots, \quad |\arg\lambda| < \tfrac{1}{4}\pi,$$

$$M^{\pm}(\lambda) = (-)^{\frac{1}{2}\pm\frac{1}{2}}(2\pi)^{\frac{1}{2}}\lambda e^{\lambda^2/2} + 1 - \frac{1}{\lambda^2} + \frac{1\cdot 3}{\lambda^4} \mp \cdots, \quad \tfrac{1}{4}\pi < |\arg\lambda| < \tfrac{3}{4}\pi. \quad (5.12)$$

[21] The use of transforms for equations of the type (4.1) has been given a mathematically rigorous basis in the following: L. Sirovich and J. K. Thurber, Quart. Appl. Math. (to be published).

[22] Since this also requires the analytic continuation of u, this step cannot be accomplished unless g_0 is analytic. It is felt that this difficulty is mathematical rather than physical. One should be able to overcome this by approximating the data by analytic data (polynomials, for example). In Ref. 21 it is shown that the initial value is well posed for kinetic models, and the above remarks therefore follow.

It should be noted that the rays

$$\arg \lambda = \pm \tfrac{3}{4}\pi$$

are Stokes lines for M.

From (5.11) and (5.10) we see that λ bounded implies that s and σ are bounded. To find high-frequency roots we therefore restrict attention to large λ. Due to the nature of the asymptotics this must be carried out in three regions,

I. $|\arg \lambda| < \frac{3}{4}\pi$: Substituting (5.12) into (5.10) and solving the cubic we find

$$s \sim -\frac{i}{\lambda} + \frac{i}{\lambda^3},$$
$$s \sim -\frac{i}{\lambda} \pm \frac{1}{\lambda^2}\left(\frac{5}{3}\right)^{\frac{1}{2}} + \frac{i8}{3\lambda^3}. \qquad (5.13)$$

Solving for the free wave roots

$$\sigma \sim -k^2,$$
$$\sigma \sim \pm(\tfrac{5}{3})^{\frac{1}{2}} ik - k^2, \qquad (5.14)$$

and for the forced wave roots

$$s \sim \pm(i\omega)^{\frac{1}{2}}(1 + i\omega),$$
$$s \sim \pm(\tfrac{3}{5})^{\frac{1}{2}} i\omega(1 + 2i\omega). \qquad (5.15)$$

By inspection we see that these are the low-frequency or Chapman–Enskog expansions[7] and we do not pursue these further.

II. $|\arg \lambda| \sim \frac{3}{4}\pi$. The previous discussion showed that no high-frequency roots lie along rays such that $|\arg \lambda| < \frac{3}{4}\pi$. There are, however, roots in the neighborhood of $|\arg \lambda| = \frac{3}{4}\pi$ as $\lambda \to \infty$. The analysis in this case is tedious and we consider instead the following model,[23]

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + 1\right) g = \Psi_1(\Psi_1, g). \qquad (5.16)$$

The dispersion relation for this model is easily found to be

$$s = -iM/\lambda. \qquad (5.17)$$

It is sufficient to consider the ray $\arg \lambda = \frac{3}{4}\pi$, and the function M^+. Then along this ray

$$\lambda = \rho e^{i(3\pi/4)},$$
$$\rho = |\lambda|.$$

Substituting into (5.17) and using the appropriate asymptotic form of M^+, we have

$$is/(2\pi)^{\frac{1}{2}} = e^{-i(\rho^2/2)} + O(1/\rho), \qquad (5.18)$$

[23] An extensive study of this model is to be found in Ref. 13. Since this model preserves (only) the continuity equation, it is referred to as the isosteric model.

and therefore

$$|is/(2\pi)^{\frac{1}{2}}| = 1 + O(1/\rho).$$

The free-wave roots are

$$k \sim (2\pi)^{\frac{1}{2}},$$
$$-\mathrm{Re}\, \sigma = \mathrm{Im}\, \sigma \sim 2\pi n^{\frac{1}{2}}, \qquad (5.19)$$

with integers $n \to \infty$. This is relatively low-frequency propagation. For the forced-wave roots, on the other hand, we find

$$s \sim -i(2\pi)^{\frac{1}{2}} e^{-i\omega^2/4\pi}, \qquad (5.20)$$

with $\omega \to \infty$.

The Krook model (5.9) has in addition to the above another set of qualitatively similar roots.[13]

III. $\pi > |\arg \lambda| > \frac{3}{4}\pi$. For this region the exponential is the dominant term in M. The solutions to the cubic in s [(5.10)] are (using M^+),

$$s = -i(2\pi)^{\frac{1}{2}}(\tfrac{1}{6}\lambda^4)(1 - 4/\lambda^2)e^{\lambda^2/2} + O(e^{\lambda^2/2}), \quad (5.21)$$

$$s = \frac{4i(2\pi)^{\frac{1}{2}}}{\lambda^2}\left(1 + \frac{3}{\lambda^2}\right)e^{\lambda^2/2} + O\left(\frac{e^{\lambda^2/2}}{\lambda^6}\right), \qquad (5.22)$$

$$s = (i/\lambda) + O(1/\lambda^2). \qquad (5.23)$$

The last root [(5.23)] is seen to be of low frequency and therefore is of no interest in this discussion. To solve (5.21) and (5.22) we take logarithms, for example, for (5.21) to obtain

$$2 \ln is = 2 \ln [\tfrac{1}{6}(2\pi)^{\frac{1}{2}}] + 8 \ln \lambda + \lambda^2 + 4\pi i n + O(1/\lambda^2). \quad (5.24)$$

The branches of the logarithm appear through the term $4\pi i n$ (n takes on integer values). The resulting equations may then be solved iteratively without difficulty. For free waves we find to lowest order in the real and imaginary part,

$$\sigma_n \sim -|k| (\ln k^2)^{\frac{1}{2}} - \pi i n\, |k|/(\ln k^2)^{\frac{1}{2}}. \quad (5.25)$$

It is clear that σ_0 is the continuation of the hydrodynamical diffusion root (since this must be real). Numerical estimates may be made which show that $\sigma_{\pm 1}$ are the continuation of the hydrodynamical sound propagation roots.

For the forced-wave problem we find

$$s_n^{\pm} \sim -\frac{|\omega|}{(\ln \omega^2)^{\frac{1}{2}}} - i\frac{(4n \pm 1)}{2}\pi \frac{|\omega|}{(\ln \omega^2)^{\frac{3}{2}}}. \quad (5.26)$$

Here we consider only the waves moving to the right. The complex wavenumber s_0^+ corresponds to the hydrodynamical diffusion and s_1^- to the hydrodynamical sound propagation. We see therefore that

$$\text{speed of sound} \sim [\ln \omega^2]^{\frac{1}{2}}, \qquad (5.27)$$

$$\text{attenuation rate} \sim |\omega|/[\ln \omega^2]^{\frac{1}{2}}. \qquad (5.28)$$

This is a marked contrast to the asymptotic behavior found by means of the Wang Chang–Uhlenbeck method, see Eqs. (3.9), (3.10).

Although (5.26) results in particular from the Krook model, it holds with slight modification quite generally for models of the type (4.1). To see this recall the discussion in connection with (5.8). It was pointed out there that the dispersion relation is of degree 2 in the function M. Because of this one finds, as with (5.10), that there are only two high-frequency roots for $\pi \geq |\arg \lambda| > \frac{3}{4}\pi$. With the exception of "cross section" constants these take the same form as (5.21), (5.22) and lead to results similar to (5.25), (5.26).

The results contained in (5.25) and (5.26) can be used in a rough way to obtain a plot of speed and attenuation rate versus frequency. To do this, one plots the roots of the Navier–Stokes dispersion relation and the appropriate branches of (5.25) or (5.26) for all values of frequency. A patch of these two plots turns out to be in good agreement with experiment.

6. LOW-FREQUENCY EXPANSIONS

We now consider the nature of the formal power series expansions of $\sigma(s)$ [or $s(\sigma)$]. Our analysis is constructive and therefore, in order to avoid heavy analysis, we consider the following simplified model[23]:

$$[(\partial/\partial t) + \xi_1(\partial/\partial x) + 1]g = a_1\Psi_1. \qquad (6.1)$$

Taking the inner product of (6.1) with $\Psi_1 = 1$ we obtain

$$a_1 = a_1 \int \frac{\Omega\, d\boldsymbol{\xi}}{\sigma + s\xi_1 + 1}. \qquad (6.2)$$

Carrying out the integration of ξ_2 and ξ_3 we are left with the dispersion relation [see (5.17)]

$$\sigma = \lambda \int_{-\infty}^{\infty} \frac{e^{-\zeta^2/2}\, d\zeta}{\lambda - i\zeta} - 1 = M(\lambda) - 1, \qquad (6.3)$$

where as in the last section

$$\lambda = (1 + \sigma)/is. \qquad (6.4)$$

As can be easily verified the function M satisfies

$$dM/d\lambda = (\lambda + 1/\lambda)M - \lambda. \qquad (6.5)$$

Setting

$$-\kappa = s^2 \qquad (6.6)$$

and using (6.3) and (6.4) we find after some manipulation

$$2\frac{d\sigma}{d\kappa} = \frac{\sigma}{\kappa} + \frac{1}{\kappa} + \frac{1}{\sigma}. \qquad (6.7)$$

Further setting

$$\phi = \sigma/\kappa \qquad (6.8)$$

we find

$$2\kappa^2\phi(d\phi/d\kappa) + \kappa\phi^2 = \phi + 1. \qquad (6.9)$$

Formally writing

$$\phi = \sum_{r=0}^{\infty} a_r \kappa^r \qquad (6.10)$$

and substituting we find

$$a_{r+1} = (r + 1)\sum_{j=0}^{r} a_{r-j}a_j. \qquad (6.11)$$

For the isosteric model, only one discrete root exists,[13] and in fact $\phi = 0 = \kappa$ satisfies (6.3). From (6.9) we see that $\phi = -1$ for $\kappa = 0$, so that $a_0 = -1$. An induction argument immediately yields that $a_{2r} > 0$ and $a_{2r+1} < 0$. Induction also shows that $|a_r| > r!$, for $r > 1$. We therefore conclude that $\sigma(s)$ [or $s(\sigma)$] is not analytic at the origin.

This last result provides an interesting contrast to the recent work of MacLennan[8] and Arseniev[9] (an account of this is given in the review article of Guiraud[24]). They show for the rigid sphere Boltzmann equation that $\sigma(s)$ [or $s(\sigma)$] are analytic at the origin. Preliminary calculations indicate that analyticity hinges on the relative growths of $|\sigma + s\xi_1|$ and the collision frequency $\nu(\xi)$ for large $|\xi|$. The conjecture is that $\sigma(s)$ is not analytic for

$$\lim \nu(\xi) \to |\xi|^\alpha, \quad \alpha < 1,$$

and that analyticity is obtained for $\alpha = 1$. The latter corresponds to the rigid sphere case.

Note added in proof: Since submitting our paper for publication we have obtained a proof of the above conjecture. As a sidelight of this, one may show that, under requirements on the data, the Chapman–Enskog expansion is divergent for $\alpha < 1$ and convergent for $\alpha = 1$. These results will appear in a forthcoming paper.

[24] J. P. Guiraud, "Kinetic Theory and Rarefied Gas Dynamics," ONERA Rept. (1966); also to appear in *Fifth Symposium on Rarefied Gas Dynamics* (Academic Press Inc., New York, to be published).