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Wave Propagation and Other Spectral Problems in Kinetic Theory

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A rigorous discussion of the spectrum of the linear Boltzmann equation and kinetic models is presented. Particular attention is given to plane-wave propagation for a general class of kinetic models. These models in general have a velocity-dependent collision frequency $\nu(\xi)$. The main results of this paper concern the relationship between the complex wavenumber and complex frequency for a plane wave. It is shown that the question of analyticity of this relation is reduced to considering ν in the neighborhood of infinity. Specifically, if

$$\lim_{\xi\to\infty} \nu/\xi = 0,$$

the relationship is not analytic. Otherwise, analyticity is obtained. (Although not specifically considered here, analyticity is closely connected to the convergence of the Chapman–Enskog procedure.) In a general discussion it is shown how the question of analyticity is closely connected with (i) the continuous spectrum of the underlying operator, (ii) the behavior of solutions at large distances from boundaries, and (iii) the nature of the cutoff in an intermolecular interaction.

1. INTRODUCTION

Aside from its own intrinsic interest and importance, the study of plane-wave propagation is a powerful tool for revealing the underlying structure of the equations used in kinetic theory. Of special interest is the functional relation between frequency σ and wavenumber s which is implied by a plane wave. For generality, we permit both these quantities to be complex.[1] The function relationship $\sigma(s)$ [or $s = s(\sigma)$] is usually found as a root of the dispersion relation of the underlying equations and, in general, a number of such relations are possible. A special role is played by those roots for which $\sigma(0) = 0$. These roots are referred to as the hydrodynamical roots since they are the kinetic theory form of the roots found from the hydrodynamical equations.[2]

The analytical form of $\sigma(s)$ [or $s(\sigma)$] in the neighborhood of the origin has a vital role in any discussion of the transition of molecular to continuum theory. More specifically, the convergence of the series expansion of $\sigma(s)$ at the origin is closely connected with the convergence of the Chapman–Enskog procedure.[3] Recently, two conflicting results have been reported in regard to this issue. In an earlier paper,[4] we constructively demonstrated (using a special kinetic model) that this series is, in fact, divergent. On the other hand, for the case of rigid sphere molecules, the series has been found to converge; that is, $\sigma(s)$ is analytic at the origin.[5,6] This conflict may be isolated, and in fact resolved, by considering the molecular collision frequency $\nu(\xi)$. [Section 2 contains a discussion of $\nu(\xi)$ as well as a number of results for and forms of the linear Boltzmann equation.]

The form which $\nu(\xi)$ takes depends greatly on the effective range of the intermolecular potential. For most infinite-range potentials the analytical definition of $\nu(\xi)$ leads to divergent integrals. For this and other reasons, some sort of interaction cutoff seems advisable. From the analytical standpoint the most effective part of the collision frequency is its behavior at large speeds, $\xi \gg 1$. For a rigid sphere gas one finds (Sec. 2):

$$\lim_{\xi\to\infty} \frac{\nu}{\xi} = \pi D^2 n,$$

where D is the molecular diameter and n the number density. The same result is obtained if an arbitrary potential is given a radial cutoff D. If the interaction cutoff is allowed to be velocity-dependent, a wide range of behaviors is obtained. We represent the behavior at infinity by

$$\nu = 0(\xi^\alpha).$$

For example, the recently discussed angular cutoff[7]

[1] For a disturbance in an unbounded media, i.e., a free wave, $s = ik$, with k real. On the other hand, for forced (steady-state) oscillations, $\sigma = i\omega$, ω real.

[2] L. Sirovich, Phys. Fluids **6**, 10 (1963).

[3] See for instance, S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1952).

[4] L. Sirovich and J. K. Thurber, J. Math. Phys. **8**, 888 (1967). Hereafter this will be referred to as I. The result quoted above was reported earlier in our lectures in *Statistical Mechanics and Spectral Theory*, J. Pincus, Ed. (Brookhaven National Laboratories, Brookhaven, N.Y., 1965).

[5] J. A. McLennen, Phys. Fluids **8**, 1580 (1965).

[6] A. A. Arseniev, Zh. Vitch. Math. Mat. Phys. (Moscow) **5**, 864 (1965).

[7] H. Grad in *Third International Rarefied Gasdynamics Symposium* (Academic Press Inc., New York, 1963), Vol. 1, p. 26.

leads to $0 \leq \alpha < 1$ (negative α can be obtained by considering interactions which are softer than the Maxwell potential). The kinetic models considered in I correspond to an exponent $\alpha = 0$. Returning to the question of the analyticity of $\sigma(s)$, it was conjectured in I that analyticity is obtained if $\alpha \geq 1$ and that nonanalyticity occurs otherwise. This is proven in Secs. 5 and 6 for a very general class of kinetic models. Section 5 contains the proof for models of the type considered in I, $\alpha = 0$. Section 6 contains the proof for velocity-dependent collision-frequency models.

The velocity-dependent collision-frequency models of Sec. 6 are developed from the Boltzmann equation in Sec. 3. As a starting point we use the model recently introduced by Cercignani.[8] The development then follows the methods[9,10] used in extending the Krook model.[11] Other velocity-dependent collision-frequency models have also been introduced in neutron-diffusion theory.[12] These do not satisfy all the conservation laws and we do not specifically consider them (most of our results still apply, however).

In the past, great use has been made of the BGK model[11] and its extensions.[9,10] In recent years, however, it has become increasingly clear that for many purposes these are too crude an approximation to the Boltzmann equation. In order to produce a more faithful model, it should more accurately mimic the spectrum of the Boltzmann equation. For this reason a general discussion of the spectrum for initial and boundary problems is given in Sec. 4. In the course of preparing this section, it was found that a mathematically rigorous treatment could be given even for the Boltzmann equation. (This too is included in Sec. 4). Although all previous discussions of the spectrum of the Boltzmann equation[13–15] are based only on plausibility arguments, no significantly different results are found here.

Although our main results (Secs. 5 and 6) apply to plane waves, and hence to the discrete modes or spectra, there are a number of interesting and informative connections with other diverse problems and effects. These we discuss now in the introduction. To begin, we consider the continuous spectrum.

To discuss this we write the linear Boltzmann equation and the model equations in the form

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial}{\partial \mathbf{x}} + \nu(\xi)\right) g = Kg,$$

where K represents an integral operator. Provided the collision frequency $\nu(\xi)$ exists, this can always be done. Two canonical situations are considered: the initial-value problem in an unbounded domain with sinusoidal initial data of wavenumber k. This leads to a discussion of the spectrum of the operator

$$L_i = K - \nu(\xi) - ik\xi_1 = K - T_i.$$

Second, the case of steady-state oscillations of frequency ω, in a half-space. This leads to a discussion of the spectrum of the operator

$$L_b = \frac{1}{\xi_1} K - \frac{\nu(\xi) + i\omega}{\xi_1} = \frac{1}{\xi_1} K - T_b.$$

Ford[14] has already considered the spectrum of L_i for a rigid sphere gas and for a certain kinetic model (see Sec. 3). Also, Grad[15] considers the spectrum of both L_i and L_b in a number of situations. Both of these studies, however, are based only on plausibility arguments, and in Sec. 4 we give a mathematically rigorous discussion of the spectrum of L_i and L_b for both kinetic models and the Boltzmann equation.

In Sec. 4 we demonstrate that if K is compact, then the continuous spectra of L_i and $-T_i$ are the same. Using the additional information that in all cases K is isotropic, it also follows, then, that L_b and $-T_b$ have the same continuous spectra. Since both $-T_i$ and $-T_b$ are multiplicative operators, their spectra are given by their respective ranges. That is, for the continuous spectra of $-T_i$ (and hence L_i), we hold k (real) fixed and vary $\boldsymbol{\xi}$, and for the continuous spectra of $-T_b$ (and hence L_b), we hold ω (real) fixed and vary $\boldsymbol{\xi}$. Sketches of the continuous spectrum have been given by Ford[14] and Grad.[15] For completeness we repeat some of these.

Figure 1 shows a sketch of the initial-value problem continuous spectrum in the rigid sphere (and radial cutoff) case. As is easily seen, the portion closest to the imaginary axis is due to the slow-moving molecules. The comparable boundary-value case is sketched in Fig. 2. We note that in this case two branches are obtained and that the portion closest to the imaginary axis is due to the fast-moving molecules, $|\xi_1| \to \infty$. As is shown in Figs. 3 and 4, a constant collision frequency leads to a one-dimensional continuous spectrum. In Fig. 4 the contribution in the neighborhood of origin comes from $|\xi_1| \to \infty$. The sketches in Figs. 5 and 6 are comparable to those discussed by Grad.[15] As in Fig. 4 the neighborhood of the origin in Fig. 6 is due to $|\xi_1| \to \infty$.

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[8] C. Cercignani, Ann. Phys. (N.Y.) **40**, 469 (1966).
[9] E. P. Gross and E. A. Jackson, Phys. Fluids **2**, 432 (1959)
[10] L. Sirovich, Phys. Fluids **5**, 908 (1962).
[11] P. R. Bhatnager, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).
[12] N. Corngold, P. Michael, and W. Wollman, Nucl. Sci. Eng. **15**, 13 (1963).
[13] N. Corngold, Nucl. Sci. Eng. **19**, 80 (1964).
[14] G. W. Ford, "Dispersion of Sound in Monatomic Gases," in Proceedings of Midwest Conference on Theoretical Physics, 1963 (unpublished).
[15] H. Grad, SIAM J. Appl. Math. **14**, 932 (1966).

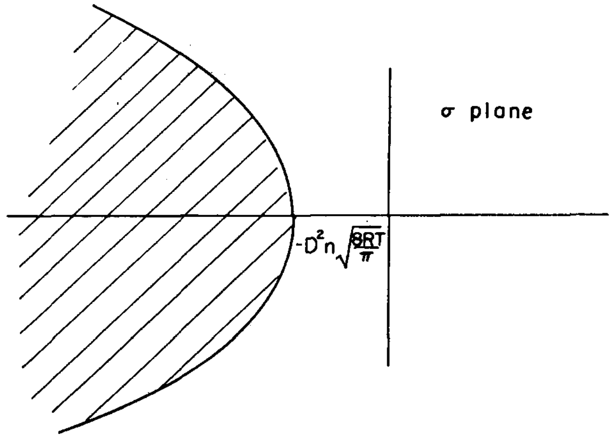

FIG. 1. Continuous spectra for initial-value problem; $\nu \sim |\xi|$.

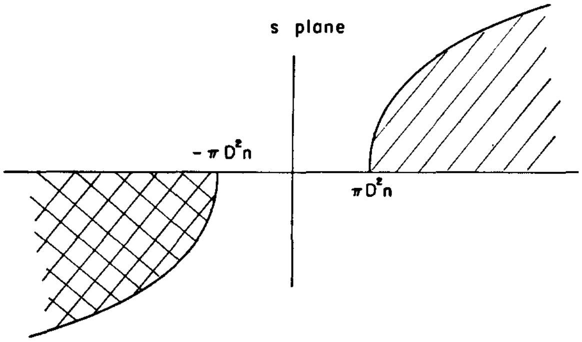

FIG. 2. Continuous spectra for steady-state oscillations; $\nu \sim |\xi|, \omega > 0$.

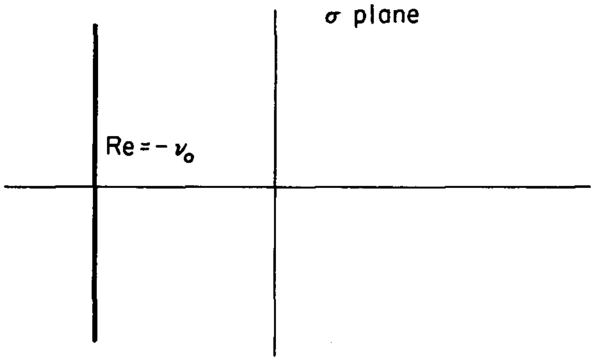

FIG. 3. Continuous spectra for initial-value problem; ν a constant.

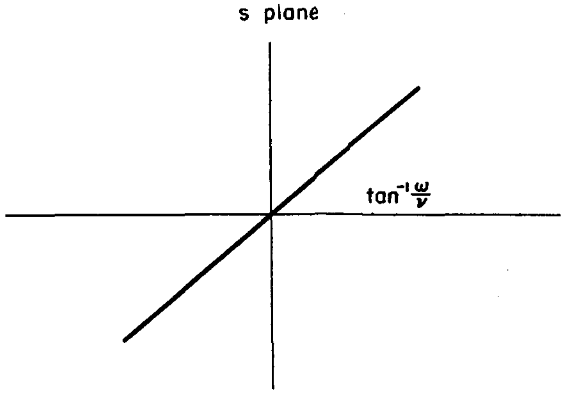

FIG. 4. Continuous spectra for steady-state oscillations; ν a constant, $\omega > 0$.

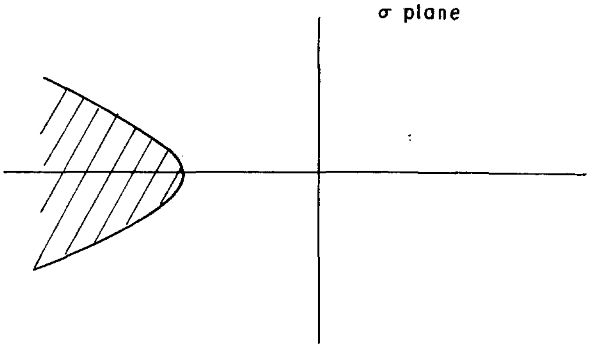

FIG. 5. Continuous spectra for initial-value problem; $\nu \sim |\xi|^\alpha, 0 < \alpha < 1$.

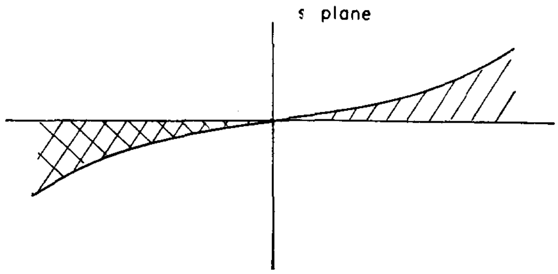

FIG. 6. Continuous pectra for steady-state oscillations; $0 < \alpha < 1, \omega > 0$.

Based on the above discussion, the question of the analyticity of $\sigma(s)$ at $s = 0$ [or $s(\sigma)$ at $\sigma = 0$] can be connected to the continuous spectrum for the steady-state oscillations problem. If the region of the continuous spectrum does not reach the origin, $\sigma(s)$ is analytic; otherwise it is not analytic. (Actually this effect plays a role in the proofs given in Secs. 5 and 6.)

Next we discuss the canonical sound propagation problem, i.e., the problem of an infinite plane oscillating normal to itself in anotherwise unbounded space. A solution is then soughtin one of the half-spaces, say $x > 0$. In addition to the continuous spectra, there is the point spectra which in Figs. 2, 4, 6 lies in either the first or third quadrant. (As was shown in I at sufficiently high frequncies, $\omega \gg 1$, the discrete modes disappear.) For $x > 0$, only the discrete points in the third quadrant enter (which for $\omega > 0$ can be shown to lie of the imaginary axis), and also the continuous spectra in that quadrant. Therefore if we consider steady stte oscillations in a rigid sphere gas, Fig. 2 shows tht the solution falls off at least exponentially for $x \gg 1$. The same statement cannot be made for the constant collision-frequency case shown in Fig. 4. In fact, from the work on this problem using the Krook model,[16–18] or from the related Rayleigh problem considered by Cercignani,[19] we can expect that, for $x \gg 1$, the solution falls off as $\exp(-kx^{\frac{2}{3}})$ (k a constant). Although no calculations have been performed for the cases depicted in Fig. 6, we can anticipate that the far-field solutions will fall off as $\exp(-kx^{\beta})$, $\frac{2}{3} \leq \beta < 1$ (k a constant). Hydrodynamics leads only to discrete modes, and hence to exponential decay. Therefore, in those cases for which

$$\lim_{\xi \to \infty} \left(\frac{\nu}{\xi}\right) = 0,$$

the far field is not hydrodynamical, even when $0 < \omega \ll 1$. Only for the rigid sphere gas or, more generally, for the finite-interaction case is the far field hydrodynamical in nature for small ω. Although there is no reason to regard hydrodynamics as sacrosanct, there does not seem to be any experimental evidence that it does not give rise to the dominant effect for smooth phenomena. Certainly, examining the far field for $0 < \omega \ll 1$ falls into this category. A careful experiment under these conditions would go far in resolving the question of interaction cutoff for the Boltzmann equation.

The effects just described, and their sensitivity to the collision frequency, may be viewed in another way which to some extent explains, physically, why they occur. Nonhydrodynamical behavior occurs in those cases for which the continuous spectra reaches the origin (Figs. 4 and 6). The continuous spectra in the neighborhood of the origin is, in turn, due to the molecules for which $|\xi_1| \gg 1$, and hence $\xi \gg 1$. Now the free path of a molecule moving with a speed ξ is

$$l(\xi) = \xi/\nu.$$

Therefore, if $\alpha < 1$, then (ξ) becomes unbounded for ξ large. Physically this staes that there are molecules which can travel any distance, no matter how large, and on the average not encounter another molecule. It is clear that such molecules carry signals which are not hydrodynamical in naure. On the other hand, for a finite cutoff D, say, we have

$$l_0 = l(\infty) = 1/\pi D^2 n.$$

In this case, no molecule, no matter how large its speed, can travel on the average a distance larger than l_0 without colliding with another molecule. Therefore, the signals carried by the molecules in this case are collision dominated and, at least for $0 < \omega \ll 1$, are hydrodynamical.

In this introduction we have discussed the question of analyticity of $\sigma(s)$ [or $s(\sigma)$], the topology of the continuous spectra, the range of intermolecular forces, the far-field solution and the free path of fast molecules. It is a remarkable and interesting fact that all of these seemingly diverse effects are so intimately interwoven. So much so that an accurate knowledge of any one aspect resolves for us the nature of the other effects.

2. THE LINEAR BOLTZMANN EQUATION AND THE COLLISION FREQUENCY

Using the notation of I, the linear Boltzmann equation has the form

$$Dg = \left(\frac{\partial}{\partial t} + \boldsymbol{\xi} \cdot \nabla\right) g = \int \Omega_*(g'_* + g' - g_* - g) \times B(\theta, |\boldsymbol{\xi}_* - \boldsymbol{\xi}|)\, d\theta\, d\epsilon\, d\boldsymbol{\xi}_* = Lg. \quad (2.1)$$

g is the dimensionless perturbed distribution function and Ω is the normalized Gaussian:

$$\Omega = [\exp(-\xi^2/2)](2\pi)^{-\frac{3}{2}}. \quad (2.2)$$

Using two common descriptions, the collision frequency is defined by

$$\nu(\xi) = \int \Omega_* B(\theta, |\boldsymbol{\xi}_* - \boldsymbol{\xi}|)\, d\epsilon\, d\theta\, d\boldsymbol{\xi}_* = \int \Omega_* |\boldsymbol{\xi} - \boldsymbol{\xi}_*|\, b\, db\, d\epsilon\, d\boldsymbol{\xi}_*. \quad (2.3)$$

Using operator notation,

$$Lg = kg - \nu g,$$

where

$$kg = \int \Omega_* k(\boldsymbol{\xi}, \boldsymbol{\xi}_*) g_*\, d\boldsymbol{\xi}_*. \quad (2.4)$$

The symmetry of $k(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ is easily demonstrated and explicit forms for it have been known for some time.[3] In connection with (2.4), it is natural to consider the inner product

$$(f, g) = \int \Omega \bar{f} g\, d\boldsymbol{\xi} \quad (2.5)$$

($\bar{f}$ denotes the complex conjugate of f) and the resulting Hilbert space which we denote by h. For some purposes it is more convenient to remove the weight

[16] H. Weitzner in *Rarefied Gasonamics*, J. H. de Leeuw, Ed. (Academic Press Inc., New York, 1965), Vol. 1, p. 1.
[17] R. Mason in *Rarefied Gasdyamics*, J. H. de Leeuw, Ed. (Academic Press Inc., New York, 1165), Vol. 1, p. 44.
[18] H. S. Ostrowski and D. J. Kleman, Nuovo Cimento **64b**, 49 (1966).
[19] C. Cercignani, "Elementary Sutions of Linearized Kinetic Models and Boundary Value Problems in the Kinetic Theory of Gases," Brown University Report, 195 (unpublished).

function Ω. Setting

$$g = \Omega^{-\frac{1}{2}}G$$

in (2.1), we obtain

$$DG = (K - \nu)G,$$

where (2.6)

$$KG = \int \Omega^{\frac{1}{2}}(\boldsymbol{\xi})k(\boldsymbol{\xi}, \boldsymbol{\xi}_*)\Omega^{\frac{1}{2}}(\boldsymbol{\xi}_*)G(\boldsymbol{\xi}_*)\, d\boldsymbol{\xi}_*.$$

In this case one considers the Hilbert space H defined by the inner product

$$[F, G] = \int FG\, d\boldsymbol{\xi}. \tag{2.7}$$

In certain contexts a further transformation is useful. Set

$$G = \hat{G}\nu^{-\frac{1}{2}}$$

in (2.6) to obtain

$$D\hat{G} = (\nu\hat{K} - \nu)\hat{G}, \tag{2.8}$$

with

$$\hat{K}G = \int \frac{K(\boldsymbol{\xi}, \boldsymbol{\xi}_*)}{[\nu(\xi)\nu(\xi_*)]^{\frac{1}{2}}}\, G(\boldsymbol{\xi}_*)\, d\xi_*$$
$$= \int \left[\frac{\Omega(\xi)}{\nu(\xi)}\right]^{\frac{1}{2}} k(\boldsymbol{\xi}, \boldsymbol{\xi}_*)\left[\frac{\Omega(\xi_*)}{\nu(\xi_*)}\right]^{\frac{1}{2}} G(\boldsymbol{\xi}_*)\, d\boldsymbol{\xi}_*. \tag{2.9}$$

Similarly we set

$$g = \hat{g}\nu^{-\frac{1}{2}}$$

in (2.4); we obtain

$$D\hat{g} = (\nu\hat{k} - \nu)\hat{g},$$

with (2.10)

$$\hat{k}\hat{g} = \int \frac{\Omega_* k(\boldsymbol{\xi}, \boldsymbol{\xi}_*)}{[\nu(\xi)\nu(\xi_*)]^{\frac{1}{2}}}\, \hat{g}(\boldsymbol{\xi}_*)\, d\boldsymbol{\xi}_*.$$

For the last form the inner product

$$\langle f, g\rangle = \int \nu\Omega \bar{f} g\, d\boldsymbol{\xi} \tag{2.11}$$

and the resulting Hilbert Space $\hat{h}$ prove useful.

Hecke[20] first demonstrated that for rigid sphere molecules $\hat{K}$ is compact on H. Subsequent proofs of this have been given by Carleman[21] and Finkelstein.[22] A consequence of this is that

$$\nu^{-1}k$$

is compact with respect to $\hat{h}$. More recently, Dorfman[23] and Grad[7] have demonstrated a stronger result; namely that for rigid sphere molecules, K itself is compact on H. (And hence k is compact on h.) In the latter reference the compactness proof is extended to a somewhat more general class of molecules.

For rigid spheres the impact parameter has for a limit of integration the molecular diameter D. Hence from (2.2)

$$\nu = \int \Omega_* \, |\boldsymbol{\xi} - \boldsymbol{\xi}_*|\, d(b^2/2)\, d\epsilon\, d\boldsymbol{\xi}_*$$
$$= D^2\sqrt{2\pi}\left[e^{-\xi^2/2} + \left(\xi + \frac{1}{\xi}\right)\int_0^{\xi} e^{-x^2/2} dx\right]. \tag{2.12}$$

In general, most infinite-range potentials do not lead to convergent forms for ν. To obtain convergence, we can, for example, introduce a radial cutoff in the impact parameter.[24] If we denote the radial cutoff by D, it is clear that (2.12) is again obtained for the collision frequency. It remains an open question, however, as to whether K (or $\hat{K}$) under this condition is a compact operator.

We can generalize the above by introducing a velocity dependent cutoff $R(|\boldsymbol{\xi} - \boldsymbol{\xi}_*|)$, which leads to

$$\nu(\xi) = \pi \int \Omega_* \, |\boldsymbol{\xi} - \boldsymbol{\xi}_*|\, R^2(|\boldsymbol{\xi}_* - \boldsymbol{\xi}|)\, d\boldsymbol{\xi}_*. \tag{2.13}$$

As is clear, $\nu(\xi)$ is convergent under a wide set of conditions. Angular cutoff[7] is included in this category. This last assumption eliminates collisions which produce only grazing collisions. For example, for repulsive power law potentials

$$V = \kappa/r^p, \quad \kappa > 0,$$

we obtain from the first form of (2.3),

$$\nu(\xi) \equiv 2\pi A(\epsilon)\int \Omega_* \, |\boldsymbol{\xi}_* - \boldsymbol{\xi}|^{(p-4)/p}\, d\boldsymbol{\xi}_*. \tag{2.14}$$

The constant $A(\epsilon)$ is given by

$$A(\epsilon) = \left[\frac{4K}{m}\right]^{2/p} \int_0^{(\pi/2)-\epsilon} \left|\beta \frac{d\beta}{d\theta}\right| d\theta,$$

and $\beta(\theta)$ implicitly by

$$\theta = \int_0^{\mu_0} \frac{d\mu}{1 - \mu^2 - (\mu/\beta)^p}$$

[μ_0 is such that $1 - \mu_0^2 - (\mu_0/\beta)^p = 0$]. It is easily shown that $A(\epsilon)$ diverges as $\epsilon \to 0$. Comparing (2.14) with (2.13) we obtain

$$R(|\boldsymbol{\xi} - \boldsymbol{\xi}_*|) = [A(\epsilon)]^{\frac{1}{2}}/|\boldsymbol{\xi}_* - \boldsymbol{\xi}|^{2/p}$$

for angular cutoff.

[20] E. Hecke, Math. Z. **12**, 272 (1922).
[21] T. Carleman, *Problèmes mathématiques dans la theoŕe cinétique des gas* (Almquist & Wiksells, Uppsala, 1957).
[22] L. Finkelstein, thesis, Hebrew University, Jerusalem, 1962.
[23] P. Dorfman, Proc. Natl. Acad. Sci. (U.S.) **50**, 805 (1963).

[24] All accepted demonstrations of the Boltzmann equation, including the modern hierarchy derivations assume, at least implicitly, a finite-range cutoff in the interaction potential. Whether the assumption of angular cutoff (Ref. 7) can be compatibly included in these derivations is not clear. It should be noted that the still open question of correctly terminating intermolecular effects is not a classical problem.

The principal reason for introducing the angular cutoff lies in the fact that $K(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ may then be shown to be completely continuous.[7] From the mathematical viewpoint, the efficacy of this assumption is therefore clear; however, its physical implications cast some doubt on it. For from (2.14), we see that the fast-moving molecules have almost no encounters. Another way of seeing this is by considering the free-path

$$l(\xi) = \xi/\nu(\xi).$$

For rigid spheres

$$\lim_{\xi\to\infty} \frac{\xi}{\nu} = \frac{1}{\pi D^2 n},$$

(since $\nu \sim n\pi D^2\xi$), whereas for angular cutoff molecules

$$\lim_{\xi\to\infty} \frac{\xi}{\nu} = \infty$$

(since in this case $\nu \sim \xi^{(p-4)/p}$).

3. KINETIC MODELS

In order to learn about the effect of velocity-dependent collision frequencies, it is natural to consider kinetic models exhibiting this effect. Ford[14] in considering the initial value problem introduces

$$Dg = Fg - \nu(\xi)g, \tag{3.1}$$

with

$$Fg = \nu(\xi)\int \Omega_*[1 + \boldsymbol{\xi}\cdot\boldsymbol{\xi}_* + \tfrac{1}{6}(\xi^2 - 3)(\xi_*^2 - 3)]g_*\, d\boldsymbol{\xi}_*.$$

One easily sees that

$$0 = (F - \nu)[1, \boldsymbol{\xi}, \xi^2] = (k - \nu)[1, \boldsymbol{\xi}, \xi^2].$$

Although F shares this property with the exact operator k, (3.1) does not preserve the conservation laws. This may be traced to the fact that F, unlike k, is not symmetric under the inner product (2.5).

In an independent paper, Cercignani[8] avoided this difficulty in a striking manner. (Other models have been introduced in neutron diffusion; these, however, do not satisfy all the conservation laws.) We now give a general derivation of kinetic models.

Since $\hat{K}$ has been shown to be compact in certain cases, it is natural to make use of standard methods for the approximation of $\hat{K}$. From the formal viewpoint, this is most elegantly done in terms of a finite dyadic expansion in the eigenfunctions of $\hat{K}$. However, except for $(1, \boldsymbol{\xi}, \xi^2)$ (which are eigenfunctions), nothing else is known of the eigenfunctions of $\hat{K}$. Therefore for practical reasons, it is more advantageous to expand in terms of known functions. We will also consider $\nu^{-1}k$, (which is compact with respect to $\hat{h}$) instead of $\hat{K}$.

Denote by $\chi_n(\boldsymbol{\xi})$ the orthonormal set of polynomials (say generated by the Gram–Schmidt process) which form a basis in $\hat{h}$, i.e.,

$$\langle \chi_n, \chi_m \rangle = \delta_{nm}.$$

For convenience we take χ_i, $i = 1, 5$, to be constructed from $(1, \boldsymbol{\xi}, \xi^2)$. We formally write

$$g = \sum_{n=1}^{\infty} a_n \chi_n, \tag{3.2}$$

with

$$a_n = \langle \chi_n, g \rangle. \tag{3.3}$$

Next we define the projection operator

$$P_N = \sum_{n=1}^{N} \chi_n \langle \chi_n, \rangle. \tag{3.4}$$

Then considering (2.1) and (2.3), we define the kinetic models

$$\begin{aligned} Dg + \nu(\xi)g &= \nu P_N \nu^{-1} k P_N g \\ &= k_N g = \nu \sum_{m,n\le N} a_n \kappa_{nm} \chi_m, \end{aligned} \tag{3.5}$$

with

$$\kappa_{mn} = (\chi_m, k\chi_n).$$

(Note since k_N is finite-dimensional, it is compact with respect to $\hat{h}$ and h.)

It should be noted that κ_{mn} is defined in terms of the inner product in h, and not $\hat{h}$. This is a great convenience since the κ_{mn} are then closely related to the so-called bracket integrals for which there is a large literature.[3,25]

The symmetry of k_N under the inner product (2.5) now follows from the symmetry of k. For

$$\begin{aligned} (f, k_N g) &= \langle f, P_N \nu^{-1} k P_N g \rangle = \langle P_N f, \nu^{-1} k P_N g \rangle \\ &= (P_N f, k P_N g) = (k P_N f, P_N g) = \langle \nu^{-1} k P_N f, P_N g \rangle \\ &= (k_N f, g). \end{aligned}$$

From these, one easily obtains that $\nu - k_N$ is non-negative, and the remaining Boltzmann-like property

$$(\nu - k_N)(1, \boldsymbol{\xi}, \xi^2) = 0$$

has already been built into the models for $N \geq 5$.

For $N = 5$,

$$Dg = Cg = \nu(\xi) \sum_{i=1}^{5} \langle \chi_i, g \rangle \chi_i, \tag{3.6}$$

which is the model introduced and studied by Cercignani.[8,26]

[25] J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids* (John Wiley & Sons, Inc., New York, 1954).
[26] C. Cercignani, Ann. Phys. (N.Y.) **40**, 469 (1966).

For later purposes we exhibit the simplest nontrivial model; i.e., if $N = 1$, we have

$$Dg + \nu(\xi)g = \frac{\nu(\xi)}{\nu_0}\int \nu\Omega g \, d\boldsymbol{\xi},$$

$$\nu_0 = \int \nu\Omega \, d\xi. \tag{3.7}$$

This equation, which preserves only the continuity equation, is a generalization of the isosteric model discussed in I.

4. SPECTRUM OF THE LINEAR BOLTZMANN EQUATION AND MODELS

The purpose of this section is two-fold. On the one hand we wish to examine the continuous spectrum of the models of the last sections. Secondly we wish to compare this with the spectrum of the exact linear Boltzmann. The agreement of these should lend further support to the use of the models in the next two sections.

The spectral problem for the Boltzmann equation has already been considered by Ford[14] and Grad,[15] and for neutron diffusion by Corngold.[12] These previous discussions are, however, strictly formal. Below, we also give a mathematically rigorous discussion for the Boltzmann equation, as well as for the model equations. Although the rigorous analysis is more complete than the previous studies, it does not reveal any significant differences from these formal discussions.[12,14,15] Other discussions of the spectrum for certain kinetic models have appeared in the course of problem solving.[16–19]

To initiate the discussion we consider

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\xi}\cdot\frac{\partial}{\partial \mathbf{x}} + \nu\right)\tilde{G} = K\tilde{G}, \tag{4.1}$$

where we leave open the nature of ν and of the linear operator K. We formally write

$$\tilde{G} = e^{\sigma t + sx}G(\xi)$$

(simple arguments show that there is no loss of generality in assuming 1-dimensionality for either of the two problems discussed below),so that

$$(\sigma + s\xi_1 + \nu)G = KG.$$

The two main problems which naturally arise are: (1) the initial-value problem for which $s\ (= i\kappa)$ is pure imaginary, and (2) the boundary-value problem (actually steady-state oscillations) for which $\sigma\ (= i\omega)$ is pure imaginary.

We therefore write

$$\sigma G = (K - i\kappa\xi_1 - \nu)G = L_i G, \tag{4.2}$$

and

$$sG = \left(\frac{1}{\xi_1}K - \frac{i\omega + \nu}{\xi_1}\right)G = L_b G, \tag{4.3}$$

for the initial- and boundary-value problems, respectively. Our aim is then to discuss the spectrum of the operators L_i and L_b. Aspects of the discrete spectra are discussed in Secs. 5 and 6. The present discussion will be devoted to the continuous spectrum.

We will make repeated use of the following generalization of a theorem by Weyl, given by Kato in his recent book.[27]

Theorem: If T is a closed operator from a Banach space to itself, and A is compact, relative to T (i.e., A is T-compact), then T and $T + A$ have the same essential spectrum.

Actually, we are not justified in identifying the continuous spectrum with the essential spectrum. For example, a dense set of discrete eigenvalues and a continuous spectrum show up in the same way in the essential spectrum. Under the circumstances this seems unlikely and we will refer to the essential spectrum as the continuous spectrum. In all cases we assume that $\nu(\xi)$ is real, positive, and continuous for $\boldsymbol{\xi}\epsilon R^3$, and also that K is compact on H. (We could consider h equally well.)

We start our discussion by considering

$$1/[\nu(\xi) + i\kappa\xi_1].$$

This by the assumptions on ν, is bounded and hence defined on all H. Therefore it is closed, and hence its inverse

$$T_i = \nu(\xi) + i\kappa\xi_1 \tag{4.4}$$

is also closed. By the Weyl–Kato theorem, T_i and L_i have the same continuous spectrum.

Next consider

$$P = \xi_1/[\nu(\xi) - i\omega].$$

Since P is defined on all functions of H which vanish outside a set of finite measure, its domain $D(P)$ is dense in H. Hence its adjoint

$$P^* = \xi_1/[\nu(\xi) + i\omega]$$

exists and is closed. From this it follows that

$$(P^*)^{-1} = T_b = [i\omega + \nu(\xi)]/\xi_1 \tag{4.5}$$

is closed.

[27] T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, New York, 1966). For the result quoted see Theorem 5.35. For definitions and other results used in this section, see Chap. IV of this book.

We next wish to show that $(1/\xi_1)K$ is T_b compact. To this end it is convenient to introduce the Banach space B, defined by the graph norm

$$\|u\|_T = \{\|u\|^2 + \|T_b u\|^2\}^{\frac{1}{2}},$$

where $\|\ \|$ refers to the norm in H. The compactness of K in H is not sufficient to induce the compactness of $(1/\xi_1)K$ on bounded sets of B. [For example, if $\phi \notin D(\xi_1^{-1})$, then $\phi\mathfrak{L}$, with $\mathfrak{L}$ a linear functional, is compact and $\xi_1^{-1}\phi\mathfrak{L}$ does not even exist.] The additional property of K which we use is that for $u = \xi_1 w$ with $w \in H$,

$$K\xi_1 w = \xi_1 y, \tag{4.6}$$

with $\|y\| \in H$. Since K is compact, this is only a condition at the origin. This property of K is an immediate consequence of the isotropy of the Boltzmann equation (Sec. 3) and its models (Sec. 4).

Let M denote a bounded set of functions of B, i.e., $\phi \in M$ implies

$$\|\phi\|_T < C,$$

when C is a constant. It is immediate that M is bounded in H. Hence we may choose a sequence $\{\phi_i\} \in M$ such that K maps it into a convergent sequence. Since $\|\phi_i\|_{T_b} < C$, it follows that we can write

$$\phi_i = \xi_1 \psi_i,$$

with ψ_i such that $\psi_i \in H$ (since $\|T_b\phi\|$ exists). Now consider

$$\left\| \frac{1}{\xi_1} K(\phi_i - \phi_j) \right\|^2 = \left[\frac{1}{\xi_1} K(\phi_i - \phi_j), \frac{1}{\xi_1} K(\phi_i - \phi_j) \right].$$

But by the isotropy condition (4.6), we have

$$K\phi_i = K\xi_i\psi_i = \xi_1 y_i,$$

with $y_i \in H$ is $\psi_i \in H$. Hence

$$\left\| \frac{1}{\xi_1} K(\phi_i - \phi_j) \right\|^2 = \|y_i - y_j\|^2.$$

The vanishing of the right-hand side as $i, j \to \infty$ is a consequence of

$$\lim_{i,j\to\infty} \|K(\phi_i - \phi_j)\| = 0,$$

and the boundedness of $\|y_i\|$. Hence K is T_b-compact and by the Weyl–Kato theorem, T_b and L_b have the same continuous spectrum.

It only remains for us to identify the above ν and K with the various possibilities open to us. There are three cases to which the above analysis immediately applies. (1) Rigid-sphere molecules, since K is compact.[20] (2) Intermolecular potentials with angle cutoff, again since K is compact.[7] (3) The models of Sec. 3, since K is finite-dimensional and hence compact. It is highly plausible that the above discussion also applies to intermolecular potentials with radial cutoff. But thus far no compactness proof for K has been given.[28]

Since T_i and T_b are multiplicative operators their continuous spectra are given by their ranges when these are regarded as functions. Various situations are sketched in Figs. 1–6. The discussion of these has been given in the introduction and we do not repeat it.

5. LOW-FREQUENCY EXPANSIONS (CONSTANT COLLISION-FREQUENCY MODELS)

Using the notation of I, we first consider models of the form

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + 1\right) g = kg = \sum_{m,n \le N} a_n \beta_{mn} \psi_m, \tag{5.1}$$

where

$$a_m = (\psi_m, g), \tag{5.2}$$

and then β_{mn} are constants. A plane wave is of the form

$$g = e^{\sigma t + sx} G(\boldsymbol{\xi}), \tag{5.3}$$

where σ and s are in general complex.[1] One then easily shows that a necessary and sufficient condition for a plane wave is that

$$G = \frac{\sum_{m,n \le N} \alpha_n \beta_{nm} \psi_m}{\sigma + s\xi_1 + 1},$$

with

$$\alpha_r = (\psi_r, G),$$

such that

$$(\mathbf{1}s - \mathbf{C})\boldsymbol{\alpha} = 0, \tag{5.4}$$

where

$$C_{mn} = s \sum_{r \le N} \beta_{nr} \left(\psi_m, \frac{\psi_r}{1 + \sigma + s\xi_1} \right). \tag{5.5}$$

(The purpose of the factor of s will be seen shortly.) Equation (5.4) therefore implies that σ and s must satisfy

$$D = \det(\mathbf{1}s - \mathbf{C}) = 0. \tag{5.6}$$

This in turn implies a functional dependence of σ on s (or vice versa), and we shall focus attention on $\sigma = \sigma(s)$. In I it is shown, by direct construction, that $\sigma(s)$ is not analytic for $N = 1$ in (5.5). We now consider the general case.

The entries of $\mathbf{C}$, (5.5), are composed of terms of the type

$$C_{ij} = s \int \frac{\bar{\psi}_i \psi_j \Omega}{1 + \sigma + s\xi_1} d\boldsymbol{\xi}.$$

[28] For an interesting paper which casts doubt on this conjecture, see C. Cercignani, Phys. Fluids 10, 2097 (1967).

Since the ψ_i are polynomials, each such term may be reduced to the form[29]

$$C_{ij}(z) = p_{ij}(z) + g_{ij}(z)F(z),$$

where p_{ij} and g_{ij} are polynomials and

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{-(x^2/2)}}{(2\pi)^{\frac{1}{2}}} \frac{dx}{x-z}, \quad z = -(1+\sigma)/s. \tag{5.7}$$

We can therefore expand (5.6) in the form

$$D = \sum_{n=0}^{N} A_n(z, s)[F(z)]^n = 0, \tag{5.8}$$

where the A_n are polynomials in z and s.

Our primary interest is in the roots of D for which[30]

$$\sigma = s = 0, \tag{5.9}$$

which correspond to the hydrodynamical modes of propagation. (In Sec. 7 other branches are briefly discussed.) In this case $z \to \infty$, and we easily have that in this limit

$$F(z) \sim -\frac{1}{z}\left(1 + \frac{1}{z^2} + \frac{1\cdot 3}{z^4} + \frac{1\cdot 3\cdot 5}{z^6} + \cdots\right). \tag{5.10}$$

Actually (5.7) defines two different analytic functions $F^+(z)$, for $\operatorname{Im} z > 0$, and $F^-(z)$, for $\operatorname{Im} z < 0$. Then a careful argument[29] shows (5.10) to be valid in a region larger than the half-plane given by $-\pi/4 < \arg z < 5\pi/4$ for F^+, and for $-5\pi/4 < \arg z < \pi/4$ for F^-.

If (5.10) is substituted into (5.8), the asymptotic expansion for $\sigma(s)$,

$$\sigma = b_1 s + b_2 s^2 + \cdots,$$

can be obtained. The explicit form for the b_r's has been discussed previously,[2,29,31] and in I we have shown that for $N = 1$ in (5.1), this expansion is divergent.

We now demonstrate that $\sigma(s)$ is not analytic for arbitrary N. To prove this we assume the contrary, i.e., that $\sigma = \sigma(s)$ is analytic in some small neighborhood of the origin of the s plane. Then, in the deleted neighborhood of the origin,

$$z(s) = -[1 + \sigma(s)]/s$$

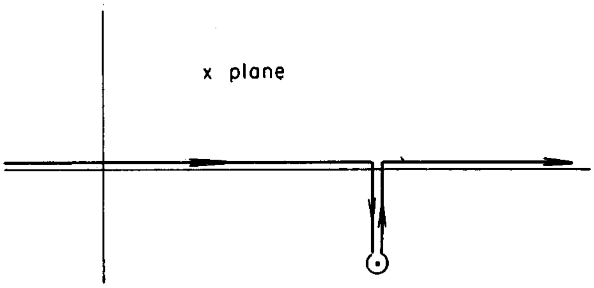

FIG. 7. Path of integration for $F^+(z)$, when $\operatorname{Im} z < 0$.

is analytic, and since $F^+(z)$ [and $F^-(z)$] is entire, $F^+[z(s)]$ is also analytic in the deleted neighborhood of the origin in the s plane. On first restricting attention to s such that $\operatorname{Im} z > 0$, we have by hypothesis that

$$D(F^+(z(s)); z(s), s) = 0. \tag{5.11}$$

But since this holds for a continuous point set, it is an identity in s and by analytic continuity holds in the entire neighborhood of the origin. For s such that $\operatorname{Im} z < 0$, the analytically continued form of $F^+(z)$ must be used. Clearly, this is obtained by choosing the path of integration for (5.7) as shown in Fig. 7. From this we immediately have

$$F^+(z) = F^-(z) + i(2\pi)^{\frac{1}{2}} e^{-z^2/2}.$$

Substituting (5.12) into (5.11) [and hence into (5.8)], and taking $s = -i\epsilon$ with $\epsilon \to 0$, we have a contradiction. For by (5.10), $F^-(z)$ is bounded as $z \to \infty$, and dividing (5.8) by $(F^+)^N$, the contradiction is shown.

Therefore, $\sigma = \sigma(s)$ is not analytic in the neighborhood of $s = 0$. [Hence $s = s(\sigma)$ is not analytic for $\sigma \sim 0$.]

6. LOW-FREQUENCY EXPANSIONS (VELOCITY-DEPENDENT COLLISION FREQUENCY MODELS)

A similar discussion now follows for velocity-dependent collision frequencies. For simplicity we start our discussion with the simple model, (3.7), already introduced in Sec. 3,

$$\left[\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + \nu(\xi)\right] g = \frac{\nu}{\nu_0} \int \Omega \nu(\xi) g \, d\boldsymbol{\xi} = \frac{\nu}{\nu_0} \hat{\rho}. \tag{6.1}$$

We make the following assumptions on $\nu(\xi)$, all of which are in keeping with its interpretation of being a collision frequency:

$$\nu(\xi) \geq \nu(\xi = 0) = 1 > 0$$

and for ξ large

$$\nu(\xi) \sim \xi^\alpha, \quad 0 \leq \alpha \leq 1 \tag{6.2}$$

[29] The closed form for arbitrary $C_{ij}(z)$ is given explicitly in L. Sirovich and J. K. Thurber in *Rarefied Gasdynamics*, J. J. de Leeuw, Ed. (Academic Press Inc., New York, 1965), Vol. 1, p. 21.

[30] That (5.9) satisfies (5.8) follows from the properties of the β_{nm}. Actually, this follows from the collisional invariants of the collision integral.

[31] L. Sirovich and J. K. Thurber in *Rarefied Gasdynamics*, J. A. Laurmann, Ed. (Academic Press Inc., New York, 1963), Vol. 1, p. 159.

(the first condition is merely a normalization). We shall also assume that $\nu(\xi)$ may be piecewise analytically continued in the complex plane.

Repeating the discussion which led to (5.6), we now arrive at the dispersion relation

$$\pi(\sigma, s) = 1 - \frac{1}{\nu_0}\int_{-\infty}^{\infty} \frac{\Omega\nu^2(\xi)\, d\boldsymbol{\xi}}{\sigma + s\xi_1 + \nu(\xi)} = 0, \quad (6.3)$$

and we seek $\sigma = \sigma(s)$ such that $\sigma(0) = 0$. [It is clear that since (6.1) preserves continuity, $\sigma = s = 0$ is a root of (6.3).] The formal expansion

$$\sigma = b_1 s + b_2 s^2 + \cdots$$

may be obtained directly from (6.3). For we may write

$$\begin{aligned}\hat{\mathcal{F}} &= \int_{-\infty}^{\infty} \frac{\nu}{\nu_0}\frac{\Omega\nu}{\sigma + s\xi_1 + \nu}\, d\boldsymbol{\xi} \\ &= \int_{-\infty}^{\infty} \frac{\nu}{\nu_0}\frac{\Omega}{1 + (\sigma + \xi_1 s)/\nu}\, d\boldsymbol{\xi} \\ &= \int_{-\infty}^{\infty} \frac{\nu}{\nu_0}\Big[1 - \frac{\sigma + s}{\nu} \pm \cdots + \Big(-\frac{\sigma + s\xi_1}{\nu}\Big)^n \\ &\quad + \frac{[-(\sigma + s\xi_1)/\nu]^{n+1}}{1 + (s\xi_1 + \sigma)/\nu}\Big]\Omega\, d\boldsymbol{\xi} \\ &= 1 - \frac{\sigma}{\nu_0}\int_{-\infty}^{\infty}\Omega\, d\boldsymbol{\xi} - \frac{s}{\nu_0}\int_{-\infty}^{\infty}\xi_1\Omega\, d\boldsymbol{\xi} + \cdots \\ &\quad + \int\Big(-\frac{\sigma + s\xi_1}{\nu}\Big)^n\frac{\nu}{\nu_0}\,\Omega\, d\boldsymbol{\xi} + O[(|\sigma| + |s|)^{n+1}].\end{aligned} \quad (6.4)$$

From this we immediately obtain

$$\sigma = \frac{s^2}{\nu_0}\int_{-\infty}^{\infty}\frac{\Omega\xi_1^2}{\nu}\, d\boldsymbol{\xi} + O(s^4). \quad (6.5)$$

For the further investigation of $\sigma(s)$, we first reduce (6.4). Introducing spherical coordinates, we have, after some straightforward manipulations,

$$\begin{aligned}\hat{\mathcal{F}} &= \frac{1}{s}\mathcal{F} \\ &= \frac{1}{\nu_0 s(2\pi)^{\frac{1}{2}}}\int_0^{\infty} x e^{-x^2/2}\nu^2(x)\ln\left[\frac{\sigma + sx + \nu(x)}{\sigma - sx + \nu(x)}\right] dx,\end{aligned} \quad (6.6)$$

and the dispersion relation (6.3) is

$$s - \mathcal{F} = 0. \quad (6.7)$$

As was the case with (5.7), more than one analytic function is defined by (6.6). Now, however, the different functions are determined by the location of the branch points of the logarithm in (6.6). For σ and s small, $\sigma + sx + \nu(x)$ has a branch point x_+ such that

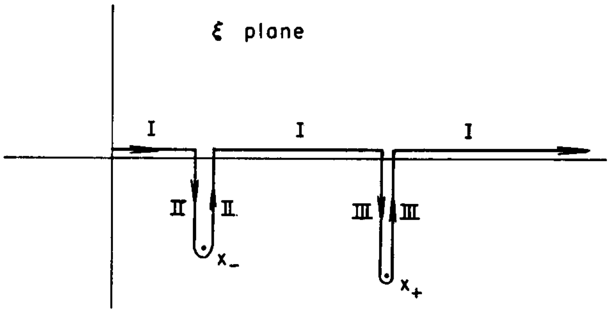

FIG. 8. Path of integration for $\mathcal{F}$.

$$x_+ \sim (-s)^{-1/(1-\alpha)}, \quad (6.8)$$

and $\sigma - sx + \nu(x)$ a branch point x_-, such that

$$x_- \sim s^{-1/(1-\alpha)}. \quad (6.9)$$

In addition to (6.8) and (6.9) there are branch points which are a distance, at most, $O(1)$ from the origin.

Case 1: $\alpha < 1$

In this case the additional branch points are immaterial as we shall see.

Assume that $\sigma = \sigma(s)$ is analytic in a neighborhood of the origin. Inserting this into the dispersion relation (6.7), we have

$$0 = s - \frac{1}{\nu_0(2\pi)^{\frac{1}{2}}} \times \int_0^{\infty} x e^{-x^2/2}\nu^2(x)\ln\left[\frac{\sigma(s) + sx + \nu(x)}{\sigma(s) - sx + \nu(x)}\right] dx. \quad (6.10)$$

Let us suppose that s in (6.10) is such that both x_+ and x_- are in the upper half-plane.[32] We will denote the integral in (6.10) by $\mathcal{F}^+$. Then since $\sigma(s)$ is analytic by assumption, and $\mathcal{F}^+(\sigma, s)$ is analytic, (6.10) is an identity in s, and by continuation it holds in a neighborhood of the origin, $s = 0$. If the branch points pass through the positive-real axis, the path of integration shown in Fig. 8 must be used for the integral in (6.10). [That both x_+ and x_- can be made to pass through $\xi > 0$ is clear from (6.8) and (6.9).] In this case, i.e., when x_- and x_+ are in the lower half-plane, as shown in the figure, we immediately have that

$$\mathcal{F}^+(\sigma(s), s) = \mathcal{F}^-(\sigma(s), s) + i\frac{(2\pi)^{\frac{1}{2}}}{\nu_0}\int_{\mathrm{Re}\, x_+}^{x_+} x\nu^2(x)e^{-x^2/2}\, dx + i\frac{(2\pi)^{\frac{1}{2}}}{\nu_0}\int_{\mathrm{Re}\, x_-}^{x_-} x\nu^2(x)e^{-x^2/2}\, dx, \quad (6.11)$$

[32] The configuration of x_+ and x_- depends on α, e.g., if $\alpha = 0$ then $x_+ = -x_-$ and the branch points are in opposite quadrants. Our subsequent discussion is uneffected by the actual configuration of x_+ and x_- and a choice is solely made for the purpose of illustration.

where $\mathcal{F}^-$ denotes the integral in (6.10) when x_+ and x_- are in the lower half-plane. (As pointed out in Footnote 32, the actual choice of a configuration of x_+ and x_- is immaterial.) But $\mathcal{F}^-$ is bounded as is shown by (6.4), whereas simple estimates show that the last two integrals in (6.11) can be made unbounded by allowing $s \to 0$. (Since $x_+ \neq x_-$, these two integrals do not identically cancel.) Hence we are lead to a contradiction and $\sigma(s)$ is not analytic [similarly, $s(\sigma)$ is not analytic].

Case 2: $\alpha = 1$

In this case the previous argument fails since x_+ and x_- are no longer in the neighborhood of infinity. In this case a direct estimate of the terms in (4.4) shows that the infinite expansion converges for $(|\sigma| + |s|) < 1$ and hence $\mathcal{F}(\sigma, s)$ is analytic in the neighborhood of $\sigma = 0$, $s = 0$. Furthermore $\partial\mathcal{F}(0, 0)/\partial\sigma \neq 0$, and hence by the implicit function theorem, $\sigma(s)$ is analytic for $s \sim 0$. [Since $\partial\mathcal{F}(0, 0)/\partial s = 0$ and $\partial\mathcal{F}(0, 0)/\partial s^2 \neq 0$, s is an analytic function of $\sigma^{\frac{1}{2}}$.] We mention in passing that if $\alpha > 1$, analyticity is again obtained; however, such values of α seem to be unphysical.

Therefore, we have proven that for plane wave propagation as described by (6.1), $\sigma(s)$ is not analytic for $0 \leq \alpha < 1$, and is analytic for $\alpha = 1$.

Finally we point out that exactly the same formalism as was used in Sec. 5 applies to the general velocity-dependent collision frequency models discussed in Sec. 3. Further, it is clear that the analysis of this section applies directly to these general models. Therefore, for the models discussed in Sec. 3, $\sigma(s)$ is not analytic for collision frequencies such that

$$\nu(\xi) = 0(\xi^\alpha) \quad \text{as} \quad \xi \to \infty \quad \text{for} \quad 0 \leq \alpha < 1,$$

and analytic when $\alpha = 1$.

7. DISCUSSION

It has been pointed out that modes of propagation other than the hydrodynamical ones exist.[2,33,34] For example, if we set $s = 0$ in (5.6), we obtain N real nonpositive values of σ. Each of these yield a new branch $\sigma(s)$. Then provided $\sigma(0) \neq -1$, the same arguments used in Secs. 5 and 6 again apply, and $\sigma(s)$ is not analytic for $\alpha < 1$, and analytic for $\alpha = 1$. [And similarly for $s = s(\sigma)$.]

In I we showed that for the $N = 1$ model, (5.6), that $\sigma(s)$ is not analytic by the actual construction of the series for $\sigma = \sigma(s)$. In Secs. 5 and 6 we show that $\sigma = \sigma(s)$ is not analytic for $\alpha < 1$ by nonconstructive means. This leaves a gap in our discussion, for the asymptotic expansion of $\sigma(s)$ may still be convergent. [For example, if $D(\sigma, s) = \sigma - s - e^{-1/|s|}$, then $D(0, 0) = 0$, and $D(\sigma, s) = 0$ has the asymptotic root $\sigma \sim s$, whereas the actual root is certainly not analytic.] In view of the construction given in I, however, it is unlikely that this is actually the case.

Throughout our discussion we have only considered collision frequencies which are bounded away from zero. This assumption is of course based on the discussion in Sec. 2 of the actual forms which the collision frequency can take. For completeness, and since they have appeared in neutron diffusion, we briefly comment on velocity-dependent collision frequencies which can vanish. In this case, the continuous spectrum of the initial-value problem touches the origin. This immediately signals the non-analyticity of $\sigma(s)$ [or $s(\sigma)$]. It is interesting to note that this can now arise because of the slow-moving molecules [e.g., if $\nu(\xi) = 0(\xi)$ for ξ small] instead of the fast molecules as was the case in Secs. 5 and 6.

An interesting connection with the Chapman–Enskog procedure may be pointed out. In the present context the Chapman–Enskog procedure may be understood to be a series expansion for the distribution function in ik or in $i\omega$. Under mild conditions on the data of a problem, it is seen that this expansion is convergent if the continuous spectrum does not pass through the origin (for either the boundary- or initial-value problems). Otherwise it is at most an asymptotic expansion. It may even fail to be asymptotic if the continuous spectrum dominates the discrete modes. This, for example, is the case for forced oscillations when $\nu = 0(\xi^\alpha)$, $\alpha < 1$.

ACKNOWLEDGMENT

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[33] L. Sirovich, Phys. Fluids 6, 218 (1963).
[34] L. Sirovich, Phys. Fluids 6, 1428 (1963).