JOURNAL OF MATHEMATICAL PHYSICS VOLUME 12, NUMBER 8 AUGUST 1971

The Sound Wave Boundary Value Problem in Kinetic Theory. I

T. G. RICHARDSON AND L. SIROVICH
Division of Applied Mathematics, Center for Fluid Dynamics, Brown University, Providence, Rhode Island 02912

(Received 3 August 1970)

A detailed investigation of the generation of sound waves from an oscillating piston is considered within the framework of kinetic theory. Two kinetic models appropriate to the problem are developed. Both of these are exactly solved by the Wiener–Hopf method. Under a certain special limit, gas dynamics is shown to hold. A variety of other special limits are also considered.

1. INTRODUCTION

An early investigation of the sound propagation problem by means of kinetic theory is to be found in the pioneering report of Wang Chang and Uhlenbeck.[1] In brief their method is based upon the expansion of the distribution function in terms of moments. Using this method, they showed that a series expansion for sound speed and attenuation rate in the frequency ω could be obtained. The leading terms of these yield the same results as the Euler, Navier-Stokes, and Burnett equations, etc. A numerical investigation based on this method and using an extremely large number of moments is to be found in the work of Pekeris and his co-workers.[2,3] However, these results proved to be very poor in the transition and high frequency limit when compared with the experiments of Greenspan[4] and Meyer and Sessler.[5] A discussion of these results is to be found in Sirovich and Thurber.[6] In that paper a method for investigating sound waves by means of fairly elaborate kinetic models is given. The results of their investigation showed extremely close agreement with the above-mentioned experiments.

In all of the above theoretical investigations, only the problem of sound propagation was considered. By this we mean that only the dispersion relation for a plane wave was analyzed. The boundary value problem corresponding to the experiments of Refs. 4 and 5 was not considered. This lead to a controversy (see Refs. 7 and 8) concerning the applicability of the plane wave description in the neighborhood of the oscillating wall. It was felt that in this region a free flow analysis would be more appropriate. The free flow analysis and certain experiments in their support are given in Refs. 9 and 10. However, even these new experiments provided data which fell more closely on the sound dispersion curves than on the free flow curves. It was shown in Ref. 7 that the free flow analysis is of very questionable value at high frequencies even within one mean free path of the wall.

The real issue can only be resolved by the solution of the exact boundary value problem. A certain amount of penetration into this problem has been made by Ostrowsky and Kleitman,[11] Weitzner,[12] Mason,[13,14] and Buckner and Ferziger.[15] Due to various analytical difficulties a number of restrictive assumptions had to be made in each of these investigations. To a certain extent the result of these studies was to raise more questions rather than to settle the above controversy. Notable among these new questions was the result in Ref. 11, that the falloff of a disturbance from an oscillating wall at large distances is $O[\exp(-x^{\frac{2}{3}})]$. This is in clear violation of the widely held view that gas dynamics is the valid theory at large distances, since this theory predicts a simple exponential falloff which is clearly recessive when compared with the result of Ref. 11. We mention in passing that this same peculiar falloff was also found in a study of shock wave structure by Lyubarskii.[16] The explanation for this behavior is due to the BGKW model (Bhatnagar, Gross, and Krook[17] and Welander[18]) or variations of it which were used in the above theoretical investigations, for, in this model, the collision frequency is a constant and fast molecules have unbounded free paths. This point is made in a recent paper.[19] It is also shown there that other fall-offs occur when the collision frequency is nonconstant. If we denote the collision frequency by $\nu(\xi)$ and if

$$\alpha = \lim_{\xi \to \infty} \ln \nu / (\ln \xi),$$

then in general one has a falloff $O[\exp(-x^{2/(3-\alpha)})]$. This has been demonstrated in a particular boundary value problem using a model equation (Sirovich and York[20]) and also for the full linearized Boltzmann equation (Richardson and Sirovich[21]).

The resolution of the boundary value problem can be made in terms of contributions due to the point spectrum (essentially sound propagation) and continuous spectrum (in which the collision frequency and boundary conditions play a major role). A major aim of our investigation is to better understand the interplay of these two effects. Specifically, we wish to know if and where the point and continuous spectra individually dominate. We also wish to take up the detailed effect of collision frequency. This last part will be taken up in the second part of this study, and in the present paper we will assume a constant collision frequency.

We are critical of the previous treatments in their choice of models and in their boundary conditions. For these reasons we will (in Sec. 2) formulate the problem and treat the boundary conditions with more than usual care. It is shown that previous studies, at least implicitly, assume that the flow under investigation is distant from the piston. We develop the problem under the sole assumption that the Mach number based on piston speed is small, and a simple transformation renders the analysis valid up to the wall.

In Sec. 3 we introduce two kinetic models for the sound problem. Both are developed with the goal of faithfully describing both the plane wave and continuous spectrum contributions to the solution. These models are exactly solved by essentially the Wiener–Hopf technique in Sec. 4.

Although our solutions are explicit, they do not yield to ready analysis. For low frequencies our solution implicitly settles the above-mentioned controversy in showing that the discrete spectrum dominates over the continuous spectrum in the neighborhood of the piston. A specific solution for all frequencies must await machine calculation. In the limit of low frequency oscillations a number of results can be obtained. For one thing, in this limit the neighborhood of the piston is dominated by the discrete spectrum, ie., the plane wave solution. The role of hydrodynamics also emerges. We show that in the limit of $\nu \to \infty$ with $x\omega/(RT)^{\frac{1}{2}}$ held fixed, hydrodynamical theory (the plane wave) dominates. Also of interest is the extent of the region in which the plane wave solution dominates the continuous spectra portion of the solution. The asymptotic extent of this is $x \ll \omega^{-6}$. Beyond this region the description is essentially nonhydrodynamic. A more detailed picture of this "crossover" phenomena is given in Sec. 5.

2. STATEMENT OF THE SOUND PROPAGATION PROBLEM

We begin our discussion with the Boltzmann equation

$$\left(\frac{\partial}{\partial t'} + \xi_1' \frac{\partial}{\partial \tilde{x}'}\right) F = JF, \tag{2.1}$$

where $F = F(\tilde{x}, \boldsymbol{\xi}', t')$ is the molecular distribution function, $\tilde{x}'$ is distance measured from the mean position of a sinusoidally oscillating piston, $\boldsymbol{\xi}'$ is the molecular velocity, and t' is the time. JF is the nonlinear Boltzmann collision operator or any particular model of it.

Equation (2.1) can be nondimensionalized with respect to a constant ν, representative of molecular collision frequency, a mean molecular speed $(RT_0)^{\frac{1}{2}}$, where T_0 is the mean gas temperature, and ρ_0, the mean density. The dimensionless variables are then defined by

$$\nu\tilde{x}'/(RT_0)^{\frac{1}{2}} = \tilde{x}, \quad \boldsymbol{\xi}'/(RT_0)^{\frac{1}{2}} = \boldsymbol{\xi}, \quad \nu t' = t. \tag{2.2}$$

Since boundary conditions are applied at the piston position $\tilde{x}' = x_p'(t)$, rather than the mean position $\tilde{x}' = 0$, we make a (noninertial) transformation which takes the piston position $x_p'(t)$ into the origin of a coordinate x';

$$\tilde{x}' - x_p'(t) = x', \quad \frac{d}{dt} x_p'(t) = u_p'(t),$$

where $u_p'(t)$ is the piston velocity. $\tilde{x}_p'$, u_p' are nondimensionalized by (2.2) and we write

$$x'\nu/(RT_0)^{\frac{1}{2}} = x, \quad u_p'(t)/(RT_0)^{\frac{1}{2}} = \epsilon u_p(t),$$
$$\nu x_p'(t)/(RT_0)^{\frac{1}{2}} = x_p(t),$$

where ϵ is chosen so as to make $u_p = O(1)$ and hence is in effect the piston Mach number. Finally by writing

$$F(\tilde{x}', \boldsymbol{\xi}', t') = f(x, \boldsymbol{\xi}, t),$$

(2.1) becomes

$$\left[\frac{\partial}{\partial t} + (\xi_1 + \epsilon u_p)\frac{\partial}{\partial x}\right] f = \frac{1}{\nu} Jf. \tag{2.3}$$

We impose the following boundary conditions on (2.3):

$$\int_{-\infty}^{\infty} (\xi_1 - \epsilon u_p) f(x = 0, \boldsymbol{\xi}, t)\, d\boldsymbol{\xi} = 0, \tag{2.4a}$$

$$f(x = 0, \boldsymbol{\xi}, t) = \rho_0 (2\pi R T_0)^{-\frac{3}{2}} \rho_p (T_p)^{-\frac{3}{2}} \times \exp\left[(\boldsymbol{\xi} - \epsilon u_p)^2 (2T_p)^{-1}\right]; \quad \xi_1 > \epsilon u_p(t). \tag{2.4b}$$

Equations (2.4) have been nondimensionalized according to (2.2) and

$$\rho_p = \rho_p'/\rho_0, \quad T_p = T_p'/T_0.$$

The first condition, (2.4a), states that there is no mass flow through the piston surface. For this to be true, it must be assumed that the "waiting time" of molecules on the surface is small compared to the period of oscillations of the piston. It should be mentioned that for very high frequency oscillations this might not be the case. (In the case of specular reflection at the piston there is no waiting time and the condition is exact.) The second equation, (2.4b), specifies that the molecules leave the piston diffusely, i.e., for $\xi_1 > \epsilon u_p$ the distribution function is Maxwellian centered at the piston speed and with a density and temperature ρ_p, T_p which must be determined. (Other "outgoing" distribution functions can be prescribed. We choose this since it seems to be the most realistic condition.) One of the unknown parameters is ultimately fixed by (2.4a). In general, to determine the other one, another boundary condition must be applied. (For example, the temperature of the piston can be given, or a heat flow condition supplied at the piston.) However, we will subsequently show that, for the model operators that we will consider, (2.4b) must satisfy a symmetry condition of the equation which effectively relates ρ_p and T_p, so that (2.4a) suffices for a complete determination. It will be seen that the method of solution holds in principle for any given outgoing distribution function. The distribution function for molecules striking the wall is of course determined with the full solution.

Equation (2.3) together with (2.4) forms a completely general framework for the sound propagation problem in a rarefied gas. The linearization is carried out by assuming that the Mach number ϵ is very much less than 1. We emphasize that this will be the sole "smallness" assumption which we make. This is in contrast to previous treatments which, at least implicitly, assume that the piston position x_p is small. This added generality is a direct consequence of the coordinate transformation.

Linearization follows from the following perturbation expansion:

$$f = f^0[1 + \epsilon\hat{g} + O(\epsilon^2)]$$

with f^0 the absolute Maxwellian,

$$f^0 = \frac{\rho_0}{(2\pi RT_0)^{\frac{3}{2}}} \exp\left[-\frac{\xi^2}{2}\right] = \frac{\rho_0}{(RT)^{\frac{3}{2}}}\,\Omega.$$

To $O(1)$, (2.3) is identically satisfied, and, to $O(\epsilon)$, we obtain

$$\left(\frac{\partial}{\partial t} + \xi_1\frac{\partial}{\partial x}\right)\hat{g} = \frac{1}{\nu}\frac{\partial}{\partial\epsilon}(Jf)\Big|_{\epsilon=0} = L\hat{g}. \quad (2.5)$$

Defining the inner product

$$(f, g) = \int_{-\infty}^{\infty} \Omega f^* g\, d\boldsymbol{\xi}, \quad (2.6)$$

where f^* is the complex conjugate of f, we have, for the hydrodynamical moments,

$$\rho' = \int_{-\infty}^{\infty} f\, d\boldsymbol{\xi}' = \rho_0 + \epsilon\rho_0\hat{\rho} + O(\epsilon^2),$$

$$\rho' u' = \int_{-\infty}^{\infty} \xi_1' f\, d\boldsymbol{\xi}' = \epsilon\rho_0(RT_0)^{\frac{1}{2}}\hat{u} + O(\epsilon^2),$$

$$\rho' RT' = \int_{-\infty}^{\infty} \{[(\xi_1' - u')^2 + \xi_2^2 + \xi_3^2]/3\} f\, d\boldsymbol{\xi}' = \rho_0 RT_0 + \epsilon\rho_0 RT_0(\hat{\rho} + \hat{T}) + O(\epsilon^2),$$

with the perturbed hydrodynamical quantities given by

$$\hat{\rho} = (1, \hat{g}), \quad \hat{u} = (\xi_1, \hat{g}), \quad \hat{T} = (\tfrac{1}{3}\xi^2 - 1, \hat{g}). \quad (2.7)$$

The boundary conditions (2.4) become

$$(\xi_1, \hat{g}(x=0)) = u_p(t) \quad (2.8a)$$

$$\hat{g}(x = 0; \boldsymbol{\xi}, t) = \hat{\alpha}_1(t) + \hat{\alpha}_2(t)(\tfrac{1}{2}\xi^2 - \tfrac{3}{2}) + \xi_1 u_p, \quad (2.8b)$$

where the unknown functions $\hat{\alpha}_1(t)$, $\hat{\alpha}_2(t)$ arise from

$$\rho_p = 1 + \epsilon\hat{\alpha}_1(t) + O(\epsilon^2),$$
$$T_p = 1 + \epsilon\hat{\alpha}_2(t) + O(\epsilon^2).$$

3. DISCUSSION OF KINETIC MODELS

The system (2.5), (2.6) is completed by specifying a particular collision operator L. We base our discussion on the well-known model operator of Bhatuagar, Gross, Krook,[17] and Welander,[18]

$$Jf = \hat{\nu}(f_0 - f). \quad (3.1)$$

Here f_0 is the local Maxwellian, $\hat{\nu} = \nu\rho'/\rho_0$, and ν is constant. Linearizing according to (2.5) becomes

$$L\hat{g} = -\hat{g} + \hat{\rho} + \xi_1\hat{u} + (\tfrac{1}{2}\xi^2 - \tfrac{3}{2})\hat{T}, \quad (3.2)$$

where we have used the definitions (2.7). We focus on the problem of an oscillating piston by taking $u_P = \exp(i\omega t)$. Collecting (2.5), (2.8), (3.2) and defining $\hat{g} = g(x, \xi)e^{i\omega t}$, $\hat{\rho} = \rho e^{i\omega t}$, $\hat{u} = ue^{i\omega t}$, $\hat{T} = Te^{i\omega t}$, $\hat{\alpha}_{1,2} = \alpha_{1,2}(\omega)e^{i\omega t}$, we have the boundary value problem defined by

$$\left(1 + i\omega + \xi_1\frac{\partial}{\partial x}\right)g = \rho + \xi_1 u + (\tfrac{1}{2}\xi^2 - \tfrac{3}{2})T, \quad (3.3)$$

$$[1, g(x=0)] = 1, \quad (3.4a)$$

$$g(x = 0, \boldsymbol{\xi}) = \alpha_1(\omega) + \alpha_2(\omega)(\tfrac{1}{2}\xi^2 - \tfrac{3}{2}) + \xi_1, \quad \xi_1 > 0. \quad (3.4b)$$

The restriction to $\partial/\partial t = i\omega$ is, of course, the case of steady state oscillations—which corresponds to asymptotically long times. As indicated, the parameters α_1, α_2, are functions of ω, and their determination is part of the problem. From (3.4a) we note that ϵ is the normalized velocity amplitude of the piston.

Equation (3.3) is seen to be an integro-differential equation for g. The mass conservation equation [taking the inner product (2.7) with respect to 1] is

$$i\omega\rho + \frac{\partial u}{\partial x} = 0. \qquad (3.5)$$

This implies that (3.3) has really only two independent moments of g on the right-hand side. As a result it may be reduced to two coupled integral equations. Unfortunately, no procedures for exact solution are known for this type of problem. Thus for the sound problem, if one seeks exact solutions (as we do here), it is necessary to approximate (3.3) in such a way that there is only a single moment of g present. (For low-speed shear problems this is exactly the case, which accounts for the amount of success which is met with in such problems.) Several such approximations have been introduced in connection with the sound propagation problem. (Buckner and Ferziger[15] present an interesting alternative. They make no assumption on the number of moments but instead replace the boundary with a known oscillating source.) In Refs. 11, 13, and 14 the "isothermal" model ($T = 0$) is considered:

$$\left(1 + i\omega + \xi_1 \frac{\partial}{\partial x}\right) g = \rho + \xi_1 u. \qquad (3.6)$$

It is of interest to note that (3.6) does not provide exact solutions to (3.3) with $T = 0$. To prove this, let us integrate (3.6) with respect to the first two moments to obtain

$$i\omega\rho + \frac{\partial u}{\partial x} = 0,$$

$$i\omega u + \frac{\partial \rho}{\partial x} = 0. \qquad (3.7)$$

To obtain the second, we have written

$$\frac{\partial}{\partial x}(\xi_1^2 - 1 + 1, g) = 3\frac{\partial T}{\partial x} + \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial x},$$

using (3.2), the fact that $g = g(x, \xi_1)$ only, and the isothermal assumption. Therefore, (3.7) and hence (3.6) permit plane wave solutions

$$e^{i\omega t - ikx}, \quad k = \omega,$$

and k is real. To see that this is impossible, we seek a plane wave solution of (3.6) directly. This then takes the form

$$(1 + i\omega - ik\xi_1)g = \rho + \xi_1 u, \qquad (3.8)$$

where ρ and u are now constants. Taking the inner product with respect to g^*, we obtain

$$(1 + i\omega)\|g\|^2 - ik(g, \xi_1 g) = |\rho|^2 + |u|^2, \qquad (3.9)$$

and $\|g\|^2 = (g, g)$. The real part of (3.9) is

$$\|g\|^2 = |\rho|^2 + |u|^2,$$

which implies that

$$g = \rho + \xi_1 u. \qquad (3.10)$$

Substituting (3.10) into (3.8) demonstrates that $k = \omega$ is impossible, hence a contradiction, and hence (3.7) does not admit a solution with $T = 0$.

Another type of model is due to Weitzner.[12] There g is assumed to depend only on ξ_1 so that one has the equation

$$\left(1 + i\omega + \xi_1 \frac{\partial}{\partial x}\right) g = \rho + \xi_1 u + \tfrac{1}{2}(\xi_1^2 - 1)(\xi_1^2 - 1, g). \qquad (3.11)$$

This too cannot produce a solution to (3.3). For, comparison of (3.11) with (3.3) shows that, in order for a solution of (3.11) to satisfy (3.3), the distribution can only be a function of ξ_1 in velocity space. However, imposing this on (3.3), we see that this is impossible unless $T = 0$. But this has been shown above to be incompatible with (3.3).

A related and somewhat more severe difficulty associated with (3.6) and (3.11) is that they lead to incorrect "sound speeds"; (3.6) produces sound waves travelling at $(\frac{3}{5})^{\frac{1}{2}}$ of the correct adiabatic speed and (3.11) $3/\sqrt{5}$ of the adiabatic speed. We now introduce two models which to some degree eliminate this shortcoming.

Adiabatic Model

First consider the "adiabatic" model defined by

$$\left(1 + i\omega + \xi_1 \frac{\partial}{\partial x}\right) g = \frac{\xi^2}{3}\rho + \xi_1 u. \qquad (3.12)$$

This equation can be obtained by assuming that $\partial Q/\partial x = 0$ in the energy equation of (3.3), namely in

$$i\omega T + \frac{2}{3}\frac{\partial u}{\partial x} + \frac{2}{3}\frac{\partial Q}{\partial x} = 0.$$

Then this and (3.5) give $\rho = \frac{3}{2}T$, hence (3.12) from (3.3). Note that (3.12) conserves mass and momentum but not energy just as (3.6). Because of the adiabatic assumption, the dispersion relation of (3.12) gives the adiabatic sound speed, $(\frac{5}{3})^{\frac{1}{2}}$ to lowest order in the frequency ω (Appendix A). However, again an exact solution of (3.12) does not yield an exact solution of the BGKW model.

Positive Wave Model

We now introduce a second model of (3.3). Our objective will be to construct a model which faithfully portrays the plane wave solutions of (3.3), but involves only a single moment of g. For convenience we write

$$\chi = [1, \xi_1, \xi^2/\sqrt{6} - (\tfrac{3}{2})^{\frac{1}{2}}],$$

and note that $(\chi_i, \chi_j) = \delta_{ij}$. Then (3.3) can be rewritten as

$$\left(1 + i\omega + \xi_1 \frac{\partial}{\partial x}\right) g = \boldsymbol{\chi} \cdot (\boldsymbol{\chi}, g). \quad (3.13)$$

Now we seek a plane wave solution of this, i.e., we assume

$$g = G(\boldsymbol{\xi}) e^{sx}, \quad (3.14)$$

to obtain

$$(1 + i\omega + s\xi_1) G = \boldsymbol{\chi} \cdot (\boldsymbol{\chi}, G) = \boldsymbol{\chi} \cdot \mathbf{a}. \quad (3.15)$$

In order that

$$G = \boldsymbol{\chi} \cdot \mathbf{a}/(1 + i\omega + s\xi_1) \quad (3.16)$$

be a solution of (3.15), it is necessary that

$$\det [\mathbf{1} - (\boldsymbol{\chi}, \boldsymbol{\chi}/(1 + i\omega + s\xi_1))] = 0. \quad (3.17)$$

Among the possible roots s which satisfy (3.17), we choose the one $s = s_0(\omega)$ which lies in the third quadrant. Then, by taking $\omega > 0$, this gives a wave propagating to the right (and decaying in the direction of propagation). Next let $\mathbf{a} = \mathbf{a}(s_0; \omega)$ denote the eigenvector corresponding to s_0,

$$[\mathbf{1} - (\boldsymbol{\chi}, \boldsymbol{\chi}/(1 + i\omega + s_0\xi_1))] \cdot \mathbf{a} = 0, \quad (3.18)$$

and for convenience we take $\mathbf{a}^* \cdot \mathbf{a} = 1$; as before the asterisk signifies the complex conjugate.

Now consider the equation

$$\left(1 + i\omega + \xi_1 \frac{\partial}{\partial x}\right) g = \boldsymbol{\chi} \cdot \mathbf{a} v(x), \quad (3.19)$$

with

$$v(x) = (\mathbf{a} \cdot \boldsymbol{\chi}, g) \quad (3.20)$$

and $\mathbf{a}$ as defined above. A plane wave solution, (3.14), of (3.19) gives

$$G = \frac{\boldsymbol{\chi} \cdot \mathbf{a}}{1 + i\omega + s\xi_1} (\mathbf{a} \cdot \boldsymbol{\chi}, G). \quad (3.21)$$

The inner product of both sides with respect to $\boldsymbol{\chi} \cdot \mathbf{a}$ gives

$$(\mathbf{a} \cdot \boldsymbol{\chi}, G) = (\mathbf{a} \cdot \boldsymbol{\chi}\ G)\mathbf{a}^* \cdot \left(\boldsymbol{\chi}, \frac{\boldsymbol{\chi}}{1 + i\omega + s\xi_1}\right) \cdot \mathbf{a},$$

hence the dispersion relation

$$\mathbf{a}^* \cdot \left(\boldsymbol{\chi}, \frac{\boldsymbol{\chi}}{1 + i\omega + s\xi_1}\right) \cdot \mathbf{a} = 1. \quad (3.22)$$

Multiplying (3.18) on the left by $\mathbf{a}^*$ shows that $s = s_0(\omega)$ satisfies (3.22). In Appendix A it is shown that s_0 is the only solution of (3.22). By observing that $(\mathbf{a} \cdot \boldsymbol{\chi}, G)$ in (3.21) is merely a constant, it follows that (3.21) and (3.16) are identical up to a constant multiplier. Therefore, the model (3.19) has the same plane wave solution as the BGKW model (3.13).

The positive wave model (3.19) does not *a priori* satisfy the conservation equations. On the other hand, the plane wave solution does. This follows trivially from the fact that any solution of the BGKW equation satisfies the conservation equations. Hence, in any region in which the plane wave dominates, (3.19) does lead to the conservation laws. Now, although the positive wave model does not yield an exact solution of the BGKW model, it will have this property asymptotically. We will later show in what follows that the plane wave is dominant in one important region.

In the limit $\omega \to 0$, s_0 and $\mathbf{a}(s_0)$ take especially simple forms. This calculation follows from (3.17), (3.18) and yields

$$s_0 = -(\tfrac{3}{5})^{\frac{1}{2}} i\omega + O(\omega^2)$$
$$\mathbf{a}^\circ = (\tfrac{3}{10})^{\frac{1}{2}}, (\tfrac{1}{2})^{\frac{1}{2}}, (\tfrac{1}{5})^{\frac{1}{2}} + O(\omega).$$

(These are the same as would be obtained from the gasdynamic Euler equations.) Under this limit

$$\boldsymbol{\chi} \cdot \mathbf{a}^\circ = (1/\sqrt{2})(\xi_1 + \xi^2/\sqrt{15}) + O(\omega),$$

and the asymptotic positive wave model has the form

$$\left(1 + i\omega + \xi_1 \frac{\partial}{\partial x}\right) g = \frac{1}{\sqrt{2}}\left(\xi_1 + \frac{\xi^2}{\sqrt{15}}\right) w(x), \quad (3.23)$$

$$w(x) = (1/\sqrt{2})(\xi_1 + \xi^2/\sqrt{15}, g). \quad (3.24)$$

The positive wave model (3.19), (3.20) and its asymptotic form (3.23), (3.24) both yield to solution by the methods of the following section. However, since the exact calculation of $s_0(\omega)$ and $\mathbf{a}^\circ$ is difficult, all explicit calculations will refer to (3.23), (3.24).

Boundary Conditions

As stated after (2.4), the particular model operator that we choose, (3.12), (3.19), or (3.23), places a restriction on the form of (2.4b) or, in the linearized form, on (3.4b). To show this, consider (3.12) and (3.23) in the limit $\xi_1 \to 0$ for $x > 0$. As will be clear from the representation of the solution, $\partial g/\partial x$ exists for $x > 0$. Then, in the limit,

$$(1 + i\omega) g(x, \xi_1 = 0) = \begin{cases} [(\xi_2^2 + \xi_3^2)/3]\rho(x), & \text{adiabatic model,} \\ [(\xi_2^2 + \xi_3^2)/\sqrt{30}]w(x), & \text{positive wave model.} \end{cases}$$

The prescribed boundary value must also have this symmetry; hence

$$\alpha_1(\omega) = \tfrac{3}{2}\alpha_2(\omega)$$

and (3.4b) becomes

$$g_0 = g(x = 0, \boldsymbol{\xi}) = \alpha_2(\omega)\tfrac{1}{2}\xi^2 + \xi_1, \quad \xi_1 > 0. \quad (3.25)$$

Thus (3.12) or (3.23) together with (3.4a) and (3.25) form the complete boundary value problem. [A more complicated form than (3.25) results for the model (3.19).]

4. SOLUTION OF THE BOUNDARY VALUE PROBLEM

A variety of equivalent methods are available for solving the given model equations. An approach of some generality is the normal modes method used by Cercignani.[22] Another approach is that of Weitzner,[12] who uses transforms. In this paper we use an approach which relies on the reduction of the problem to a pure integral equation of the Wiener–Hopf type.

The equations to be solved are (3.12) [for which ρ can be eliminated by (3.5)] or (3.23) with boundary conditions (3.4a) and (3.25). These can be typically represented by the equation

$$\left(1 + i\omega + \xi_1 \frac{\partial}{\partial x}\right) g = f_1(\boldsymbol{\xi}) v(x) + f_2(\boldsymbol{\xi}) \frac{\partial v}{\partial x}. \quad (4.1)$$

v is taken as a generic moment, depending on the model, and we write

$$v(x) = (f_3(\boldsymbol{\xi}), g). \quad (4.2)$$

Thus f_1, f_2, and f_3 are known for each model. Our method of solution allows any number of derivatives of v to appear on the right-hand side of (4.1), but for simplicity we consider only one here. Integrating (4.1) gives the equation

$$\begin{aligned} g(x, \boldsymbol{\xi}) &= H(\xi_1) g_0(\boldsymbol{\xi}) \exp\left[-\frac{(1+i\omega)}{\xi_1} x\right] \\ &+ H(\xi_1) \int_0^x \frac{1}{\xi_1}\left[f_1(\boldsymbol{\xi}) v(s) + f_2(\boldsymbol{\xi}) \frac{\partial v}{\partial s}\right] \\ &\times \exp\left[-\frac{(1+i\omega)}{\xi_1}(x - s)\right] ds \\ &- H(-\xi_1) \int_x^\infty \frac{1}{\xi_1}\left[f_1(\boldsymbol{\xi}) v(s) + f_2(\boldsymbol{\xi}) \frac{\partial v}{\partial s}\right] \\ &\times \exp\left[-\frac{(1+i\omega)}{\xi_1}(x - s)\right] ds. \quad (4.3) \end{aligned}$$

H is the Heaviside function, g_0 is the given boundary value (3.25). Parts integrating terms in $\partial v/\partial s$ gives

$$\begin{aligned} g(x, \boldsymbol{\xi}) &= H(\xi_1)\{g_0(\boldsymbol{\xi}) - [f_2(\boldsymbol{\xi})/\xi_1] v(0)\} \\ &\times \exp\{-[(1+i\omega)/\xi_1] x\} + [f_2(\boldsymbol{\xi})/\xi_1] v(x) \\ &+ H(\xi_1) \int_0^x (1/\xi_1) \\ &\times \{f_1(\boldsymbol{\xi}) - (1+i\omega)[f_2(\boldsymbol{\xi})/\xi_1]\} v(s) \\ &\times \exp\{-[(1+i\omega)/\xi_1](x - s)\} ds \\ &- H(-\xi_1) \int_x^\infty (1/\xi_1) \\ &\times \{f_1(\boldsymbol{\xi}) - (1+i\omega)[f_2(\boldsymbol{\xi})/\xi_1]\} v(s) \\ &\times \exp\{-[(1+i\omega)/\xi_1](x - s)\} ds. \quad (4.4) \end{aligned}$$

Taking the inner product (4.2) of (4.4), one obtains the integral equation

$$v(x) = \int_0^\infty K(x - s) v(s)\, ds + f(x), \quad (4.5)$$

where (in the cases under study f_z is real)

$$\begin{aligned} K(x) &= (1/\gamma) \int_{-\infty}^\infty \Omega(\boldsymbol{\xi})[f_3(\boldsymbol{\xi})/\xi_1] \\ &\times \{f_1(\boldsymbol{\xi}) - (1+i\omega)[f_2(\boldsymbol{\xi})/\xi_1]\} \\ &\times \exp\{-[(1+i\omega)/\xi_1] x\} \\ &\times [H(\xi_1)H(x) - H(-\xi_1)H(-x)]\, d\boldsymbol{\xi}, \quad (4.6) \end{aligned}$$

$$\begin{aligned} f(x) &= (1/\gamma) \int_{-\infty}^\infty \Omega(\boldsymbol{\xi}) f_3(\boldsymbol{\xi}) H(\xi_1) \\ &\times \{g_0(\boldsymbol{\xi}) - [f_2(\boldsymbol{\xi})/\xi_1] v(0)\} \\ &\times \exp\{-[(1+i\omega)/\xi_1] x\}\, d\boldsymbol{\xi}, \quad (4.7) \end{aligned}$$

$$\gamma = 1 - (f_3(\boldsymbol{\xi}), [f_2(\boldsymbol{\xi})/\xi_1]). \quad (4.8)$$

In (4.5) redefine the functions as follows:

$$\begin{aligned} v(x) &= v(x), \quad x > 0, \qquad f(x) = f(x), \quad x > 0, \\ &= 0, \qquad x < 0, \qquad\qquad = 0, \qquad x < 0, \end{aligned}$$

and let

$$q(x) = \begin{cases} 0, & x > 0, \\ -\int_0^\infty K(x-s) v(s) ds, & x < 0. \end{cases}$$

Then (4.5) is extended to the integral equation

$$v(x) = \int_{-\infty}^\infty K(x - s) v(s)\, ds + f(x) + q(x). \quad (4.9)$$

The Fourier transform of (4.9) is taken using

$$g(k) = \int_{-\infty}^\infty e^{ikx} g(x)\, dx,$$

and the same functional notation is used for the transformed and untransformed function. (The argument signifies the variable under consideration.) This yields the Wiener–Hopf equation

$$v(k)[1 - K(k)] = f(k) + q(k), \quad (4.10)$$

where the transforms of (4.6) and (4.7) are

$$\begin{aligned} K(k) &= \frac{1}{\gamma} \int_{-\infty}^\infty \frac{\Omega(\boldsymbol{\xi}) f_3(\boldsymbol{\xi})}{1 + i\omega - ik\xi_1} \\ &\times \left(f_1(\boldsymbol{\xi}) - (1+i\omega)\frac{f_2(\boldsymbol{\xi})}{\xi_1}\right) d\boldsymbol{\xi}, \quad (4.11) \end{aligned}$$

$$\begin{aligned} f(k) &= \frac{1}{\gamma} \int_{-\infty}^\infty \frac{\Omega(\boldsymbol{\xi}) f_3(\boldsymbol{\xi}) H(\xi_1) \xi_1}{1 + i\omega - ik\xi_1} \\ &\times \left(g_0(\boldsymbol{\xi}) - \frac{f_2(\boldsymbol{\xi})}{\xi_1} v(0)\right) d\boldsymbol{\xi}. \quad (4.12) \end{aligned}$$

In (4.10), by construction, $v(k)$ is analytic for $\text{Im}\, k > 0$ and $q(k)$ is analytic for $\text{Im}\, k < 0$. From (4.11) it is seen that $1 - K(k)$ defines two different analytic functions for $\text{Im}\,(1 + i\omega/ik) \gtrless 0$, i.e., across the line $L = \{k \mid 1 + i\omega - ik\xi_1 = 0, -\infty < \xi_1 < \infty\}$. Each, of course, may be continued into the other half-plane. From (4.12), $f(k)$ defines an analytic function having a cut on the half-line $L_- = \{k \mid 1 + i\omega - ik\xi_1 = 0, \xi_1 > 0\}$ in the fourth quadrant. (L_+ will denote $L - L_-$ in the second quadrant.) Considering k real, we see that both f and K are continuous at $k = 0$ and analytic elsewhere. In addition $1 - K(k) = 1 + O(1/k)$, $f(k) = O(1/k)$ for $k \to \infty$. From Appendix A we know that $1 - K$ has no zeros on the real line for $\omega \neq 0$.

In order to solve (4.10) by the Wiener–Hopf method, it is necessary to construct a splitting into upper and lower analytic functions. This is accomplished in a standard way,[23] by introducing the functions

$$Q(k) = (1/2\pi i)\int_{-\infty}^{\infty} \log\,[1 - K(t)]\, dt/(t - k), \quad (4.13)$$

$$P(k) = (1/2\pi i)\int_{-\infty}^{\infty} f(t) \exp\,[Q^-(t)]\, dt/(t - k). \quad (4.14)$$

From the above mentioned properties of f and K, (4.13) and (4.14) exist. Again P and Q define two analytic functions for $\text{Im}\, k \gtrless 0$. Unless otherwise stated, the particular function being considered is determined by the value of the argument, i.e., assume no analytic continuation unless specifically stated. (Analytic continuation will be denoted by the superscript $\pm$.) In particular from the Plemelj formula,[23] as $\text{Im}\, k \to 0$ from above and below ($\pm$ respectively),

$$Q^+(k) - Q^-(k) = \log\,[1 - K(k)]. \quad (4.15)$$

Next consider the function

$$\begin{aligned} A(k) &= v(k) \exp\,[Q(k)] - P(k), \quad \text{Im}\, k > 0, \\ &= q(k) \exp\,[Q(k)] - P(k), \quad \text{Im}\, k < 0. \quad (4.16) \end{aligned}$$

A is analytic by construction in the domain of definition, and, as $\text{Im}\, k \to 0^\pm$, by (4.15),

$$\begin{aligned} A^+(k) - A^-(k) &= \exp\,[Q^-(k)]\{v^+(k) \\ &\quad \times \exp\,[Q^+(k) - Q^-(k)] \\ &\quad - [P^+(k) - P^-(k)] \\ &\quad \times \exp\,[-Q^-(k)] - q^-(k)\} \\ &= \exp\,[Q^-(k)]\{v(k)[1 - K(k)] \\ &\quad - f(k) - q(k)\} \\ &= 0, \end{aligned}$$

since the term in square brackets is zero from (4.10). Thus A is analytic for $\text{Im}\, k \gtrless 0$ and continuous for $\text{Im}\, k = 0$. Hence it is analytic everywhere. In the limit of k large, Q, P are both $O(1/k)$ so that (4.16) gives for $\text{Im}\, k > 0$

$$v(k) = P(k) \exp\,[-Q(k)]. \quad (4.17)$$

Taking the inverse Fourier transform, we have

$$v(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} P(k) \exp\,[-Q(k)]\, dk, \quad (4.18)$$

where the path of integration is parallel to and just above the real line.

At this point the problem is essentially solved since (4.18) can be put in (4.4) to give g for all $(x, \boldsymbol{\xi})$. However, it is natural to push the contour of (4.18) as far as possible into the lower half-plane since the solution for $x > 0$ is desired. In doing this, we also split the solution into contributions from the point and continuous spectra. First we continue P and Q into the lower half-plane by means of (4.15) and find

$$v(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} \frac{\exp\,[-Q(k)]}{1 - K(k)} \times \{P(k) + f(k) \exp\,[Q(k)]\}\, dk, \quad (4.19)$$

where now the contour is just below the real axis, passing through the origin, and P and Q are now defined for $\text{Im}\, k < 0$.

We note again that the line $L((1 + i\omega)/it, -\infty < t < \infty)$ is a cut for $K(k)$ and L_- a cut for $f(k)$. Also, for the models under study, $1 - K$ has a single root, k_0, in the lower half-plane (see Appendix A). Therefore, k_0 is a pole for $(1 - K)^{-1}$ and a branch point for $\ln\,(1 - K)$. The branch cut is taken between k_0 and ∞ as indicated in Fig. 1. (One may easily show that the origin is an essential singularity of K.)

Considering P, (4.14), and making use of the contours indicated in Fig. 1, we find (in the following we will take $\text{Im}\, k < 0$, although the final results do not depend on this)

$$P(k) = -f(k) \exp Q(k) + \frac{1}{2\pi i}\int_{L_-} \{f(t)\} \exp Q(t) \frac{dt}{t - k}, \quad (4.20)$$

where

$$\{f\} = f(t)_r - f(t)_l.$$

Here $f(t)_r$ and $f(t)_l$ signify that t approaches L from the right and left, as viewed in Fig. 1.

Using the same contours for Q, (4.13), we obtain

$$Q(k) = -\ln\,(1 - K(k)) + \frac{1}{2\pi i}\int_{L_-} \{\ln\,[1 - K(t)]\} \frac{dt}{t - k} + \ln\,(k_0 - k). \quad (4.21)$$

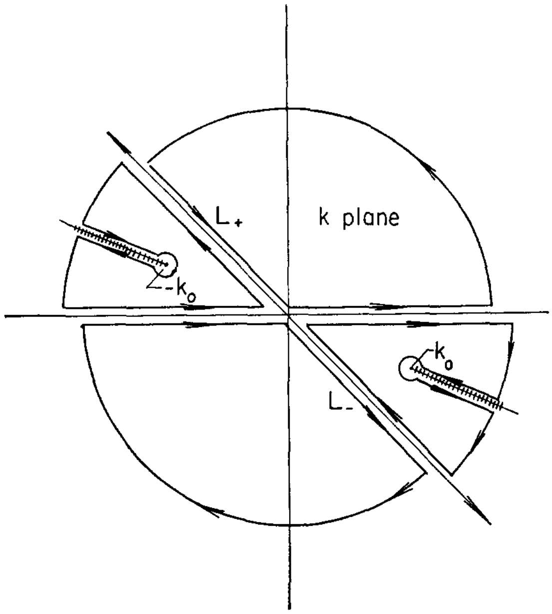

FIG. 1. Complex k plane.

Then, defining

$$X(k) = \exp\left(\frac{1}{2\pi i}\int_{L-}\{\ln(1-K)\}\frac{dt}{t-k}\right), \quad (4.22)$$

we have from (4.21)

$$\exp(-Q)/(1-K) = [X(k)(k_0-k)]^{-1}. \quad (4.23)$$

On defining

$$N(k) = -\frac{1}{2\pi i}\int_{L-}\{f(t)\}\exp(Q(t))\frac{dt}{t-k}, \quad (4.24)$$

(4.19) becomes

$$v(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\frac{N(k)}{(k-k_0)X(k)}\,dk. \quad (4.25)$$

N and X are analytic except on L_- and the pole is explicit. Therefore, the continuation of the contour down into the lower half-plane yields

$$v(x) = -i[N(k_0)/X(k_0)]e^{-ik_0x} + (2\pi)^{-1} \times \int_{L-}[e^{-ikx}/(k-k_0)]\{N(k)/X(k)\}\,dk, \quad (4.26)$$

where the boldface curly brackets again indicate the jump of the enclosed function across L_-.

We defer discussion of the solution to the next section and now specialize the above results to the adiabatic and positive wave models.

Adiabatic Model

For this model

$$f_1(\xi) = f_3(\xi) = \xi_1,\quad f_2(\xi) = -(1/3i\omega)\xi^2,$$

so that, from (4.8),

$$\gamma = (1+i\omega)/i\omega.$$

Therefore, from (4.11) and (4.12) we have

$$K(k) = \frac{i\omega}{1+i\omega}\int_{-\infty}^{\infty}\frac{\Omega(t)t^2\,dt}{1+i\omega-ikt} + \frac{1}{3}\int_{-\infty}^{\infty}\frac{\Omega(t)(t^2+2)}{1+i\omega-ikt}\,dt, \quad (4.27)$$

$$f(k) = \int_0^{\infty}\frac{\Omega(t)t}{1+i\omega-ikt}\left(\frac{i\omega}{1+i\omega}tg_0(t) + \frac{(t^2+2)}{3(1+i\omega)}\right)dt. \quad (4.28)$$

Here we have integrated out the ξ_2 and ξ_3 variables and in (4.28)

$$g_0(\xi_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Omega(\xi_2)\Omega(\xi_3)g_0(\xi)\,d\xi_2\,d\xi_3 = \xi_1 + \alpha_2(\omega)(1+\tfrac{1}{2}\xi_1^2).$$

In these, $\Omega(x) = (2\pi)^{-\frac{1}{2}}\exp(-x^2/2)$ is the one-dimensional Gaussian. It is clear from (4.27) that

$$K(-k) = K(k). \quad (4.29)$$

In particular, if t lies on the cut L of the function K,

$$K(t)_r = K(-t)_l.$$

From this and an obvious change of variable, we obtain

$$X(-k) = \exp\left(-\frac{1}{2\pi i}\int_{L+}\{\ln(1-K(t))\}\frac{dt}{t-k}\right). \quad (4.30)$$

L_+ is the path indicated in Fig. 1 extending from the origin to ∞.

From (4.29) it is clear that $-k_0$ is a root of the dispersion relation $1-K=0$ if k_0 is a root. Therefore, as indicated in Fig. 1, $-k_0$ is a branch point of $\ln[1-K(t)]$. On making use of the contours indicated in the upper half-plane of Fig. 1, we obtain in analogy with (4.22),

$$\exp[Q(k)] = -[X(-k)(k+k_0)]^{-1}, \quad (4.31)$$

and combining (4.22) and (4.31) gives

$$X(k)X(-k) = [1-K(k)]/(k^2-k_0^2). \quad (4.32)$$

Inserting (4.31) into (4.24) gives

$$N(k) = \frac{1}{2\pi i}\int_{L-}\frac{\{f(t)\}\,dt}{X(-t)(t+k_0)(t-k)}. \quad (4.33)$$

Positive Wave Model

We have

$$f_1(\xi) = f_3(\xi) = 2^{-\frac{1}{2}}(\xi_1 + \xi^2/\sqrt{15}),\quad f_2 = 0,$$

and, from (4.8), $\gamma = 1$. From (4.11) and (4.12),

$$K(k) = \tfrac{1}{2}\int_{-\infty}^{\infty}\Omega(\xi)\left(\xi_1 + \frac{\xi^2}{\sqrt{15}}\right)^2 \frac{d\xi}{1 + i\omega - ik\xi_1}, \quad (4.34)$$

$$f(k) = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\Omega(\xi)H(\xi_1)\left(\xi_1 + \frac{\xi^2}{\sqrt{15}}\right) \times \xi_1 g_0(\xi)\frac{d\xi}{1 + i\omega - ik\xi_1}. \quad (4.35)$$

In analogy with (4.30) and (4.31),

$$\exp Q(k) = 1/Y(k),$$

$$Y(k) = \exp\left(\frac{-1}{2\pi i}\int_{L+}\log\frac{1 - K(t)_r}{1 - K(t)_l}\frac{dt}{t - k}\right) \quad (4.36)$$

since there is now no root to $1 - K = 0$ in the upper half-plane (see Appendix B). From (4.23)

$$X(k)Y(k) = (1 - K)/(k_0 - k). \quad (4.37)$$

Finally, substitution into (4.24) gives

$$N(k) = \frac{-1}{2\pi i}\int_{L-}\frac{\{f(t)\}}{Y(t)}\frac{dt}{t - k}. \quad (4.38)$$

Alternate Representation

The above results can be put in a somewhat simpler form with the transformation

$$z = (1 + i\omega)/ik.$$

This transformation takes L into the real axis [and L_- into $(0, \infty)$]. Under this transformation we also have $\{f\} \to -\langle f\rangle$, where

$$\langle f(t)\rangle = f(t)^+ - f(t)^-.$$

Here, as usual, f^+ and f^- signify the limits of $f(z)$ as z approaches the real axis from above and below, respectively. Functions of z will in general be denoted by tildes, e.g.,

$$\tilde{K}(z) = K((1 + i\omega)/iz)$$

Adiabatic model: From (4.29)

$$\tilde{K}(z) = -\frac{i\omega z}{(1 + i\omega)^2}\int_{-\infty}^{\infty}\Omega(t)t^2\frac{dt}{t - z} - \frac{z}{3(1 + i\omega)}\int_{-\infty}^{\infty}\frac{\Omega(t)(t^2 + 2)}{t - z}\,dt. \quad (4.39)$$

Also from (4.28),

$$\tilde{f}(z) = \int_0^{\infty}\Omega(t)t(i\omega t g_0(t) + \tfrac{1}{3}(t^2 + 2))\frac{dt}{t - z} = \frac{(1 + i\omega)^2}{z}f\left(\frac{1 + i\omega}{iz}\right). \quad (4.40)$$

Next we define

$$\tilde{X}(z) = -\frac{(1 + i\omega)^2}{\omega z_0 z}X\left(\frac{1 + i\omega}{iz}\right). \quad (4.41)$$

Then, using contours of the type shown on Fig. 1, one can show

$$\tilde{X}(z) = \frac{1}{z}\exp\left(\frac{1}{2\pi i}\int_0^{\infty}\frac{\langle\ln[1 - \tilde{K}(\tau)]\rangle}{\tau - z}\,d\tau\right) \quad (4.42)$$

and also

$$\tilde{X}(-z) = -\frac{1}{z}\exp\left(\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{\langle\ln[1 - \tilde{K}(\tau)]\rangle}{\tau - z}\,d\tau\right). \quad (4.43)$$

Then from (4.32)

$$\tilde{X}(z)\tilde{X}(-z) = -\frac{1 - \tilde{K}(z)}{z^2 - z_0^2}\left(\frac{1 + i\omega}{i\omega}\right)^2. \quad (4.44)$$

We see from (4.42) and (4.43) that $\tilde{X}(z)$ and $\tilde{X}(-z)$ have cuts, respectively, on the positive and negative real axis. Therefore, from (4.48) and the Plemelj formulas, we may write

$$\tilde{X}(z) = -\frac{1}{2\pi i}\left(\frac{1 + i\omega}{i\omega}\right)^2\int_0^{\infty}\frac{\langle 1 - \tilde{K}(t)\rangle}{\tilde{X}(-t)(t^2 - z_0^2)}\frac{dt}{t - z}. \quad (4.45)$$

From (4.33) and (4.40) we find

$$\tilde{N}(z) = \frac{1}{2\pi i}\int_0^{\infty}\frac{\langle\tilde{f}(t)\rangle\,dt}{\tilde{X}(-t)(t_0 + z_0)(t - z)} = -\frac{i\omega(1 + i\omega)}{z}N\left(\frac{1 + i\omega}{iz}\right). \quad (4.46)$$

The various jump quantities are now easily obtained:

$$\langle\tilde{f}(t)\rangle = 2\pi i\Omega(t)t(i\omega t g_0(t) + \tfrac{1}{3}(t^2 + 2)), \quad (4.47)$$

$$\langle 1 - \tilde{K}(t)\rangle = 2\pi i\Omega(t)t\left(\frac{i\omega t^2}{(1 + i\omega)^2} + \frac{t^2 + 2}{3(1 + i\omega)}\right). \quad (4.48)$$

Returning to (4.26), we first recognize that $v(x) = u(x)$, the macroscopic velocity, for the adiabatic model, and, then inserting (4.41) and (4.46), we find

$$u(x) = \frac{1 + i\omega}{(i\omega)^2 z_0}\frac{\tilde{N}(z_0)}{\tilde{X}(z_0)}\exp\left(-\frac{(1 + i\omega)x}{z_0}\right) - \frac{1}{2\pi i}\int_0^{\infty}\frac{\exp[-(1 + i\omega)x/z]}{z - z_0}\frac{1 + i\omega}{(i\omega)^2 z} \times\left\langle\frac{\tilde{N}(z)}{\tilde{X}(z)}\right\rangle dz, \quad (4.49)$$

where

$$\left\langle\frac{\tilde{N}}{\tilde{X}}\right\rangle = \frac{1}{\tilde{X}^+\tilde{X}^-}(\tilde{N}^+(\tilde{X}^- - \tilde{X}^+) + \tilde{X}^+(\tilde{N}^+ - \tilde{N}^-)). \quad (4.50)$$

Then from (4.46) and (4.47)

$$\tilde{N}^+ - \tilde{N}^- = \frac{\Omega(z)z}{\tilde{X}(-z)(z+z_0)}[i\omega z g_0(z) + \tfrac{1}{3}(z^2+2)], \quad (4.51)$$

and from (4.45) and (4.48)

$$\tilde{X}^- - \tilde{X}^+ = \frac{\Omega(z)z}{\tilde{X}(-z)(z^2-z_0^2)}\left(\frac{z^2}{i\omega} + \frac{1+i\omega}{(i\omega)^2}\frac{z^2+2}{3}\right). \quad (4.52)$$

Positive wave model: In analogy with the above treatment, we write

$$\tilde{N}(z) = \frac{1}{2\pi i}\int_0^\infty \frac{\langle \tilde{f}(t)\rangle}{Y(t)}\frac{dt}{t-z} \quad (4.53)$$

$$\tilde{X}(z) = \frac{1}{z}\exp\frac{1}{2\pi i}\int_0^\infty \log\frac{1-\tilde{K}(t)^+}{1-\tilde{K}(t)^-}\frac{dt}{t-z} \quad (4.54)$$

$$\tilde{Y}(z) = \exp\frac{1}{2\pi i}\int_{-\infty}^0 \log\frac{1-\tilde{K}(t)^+}{1-\tilde{K}(t)^-}\frac{dt}{t-z}, \quad (4.55)$$

with

$$\tilde{f}(z) = 2^{\frac{1}{2}}\int_0^\infty \frac{\Omega(t)t}{t-z} \times \left[\tfrac{1}{30}(t^4 + 2(15)^{\frac{1}{2}}t^3 + 19t^2 + 4(15)^{\frac{1}{2}}t + 8) + \frac{1}{2(15)^{\frac{1}{2}}}\left(\frac{\alpha_2(\omega)}{2} - \frac{1}{(15)^{\frac{1}{2}}}\right) \times (t^4 + (15)^{\frac{1}{2}}t^3 + 4t^2 + 2(15)^{\frac{1}{2}}t + 8)\right]dt, \quad (4.56)$$

$$1 - \tilde{K}(z) = 1 + \frac{z}{30(1+i\omega)}\int_{-\infty}^\infty \frac{\Omega(t)}{t-z}(t^4 + 2(15)^{\frac{1}{2}}t^3 + 19t^2 + 4(15)^{\frac{1}{2}}t + 8)\,dt. \quad (4.57)$$

The jumps are now

$$\langle \tilde{f}(z)\rangle = 2\pi i 2^{\frac{1}{2}}\Omega(z)z \times \left[\tfrac{1}{30}(z^4 + 2(15)^{\frac{1}{2}}z^3 + 19z^2 + 4(15)^{\frac{1}{2}}z + 8) + \frac{1}{2(15)^{\frac{1}{2}}}\left(\frac{\alpha_2(\omega)}{2} - \frac{1}{(15)^{\frac{1}{2}}}\right) \times (z^4 + (15)^{\frac{1}{2}}z^3 + 4z^2 + 2(15)^{\frac{1}{2}}z + 8)\right], \quad (4.58)$$

$$\langle 1 - \tilde{K}(z)\rangle = 2\pi i\frac{\Omega(z)z}{30(1+i\omega)}(z^4 + 2(15)^{\frac{1}{2}}z^3 + 19z^2 + 4(15)^{\frac{1}{2}}z + 8). \quad (4.59)$$

The identity analogous to (4.44) is

$$\tilde{X}(z)\tilde{Y}(z) = -\frac{1-\tilde{K}(z)}{z-z_0}\frac{1+i\omega}{i\omega}, \quad (4.60)$$

and then

$$\tilde{X}(z) = -\frac{1}{2\pi i}\frac{1+i\omega}{i\omega}\int_0^\infty \frac{\langle 1-\tilde{K}(t)\rangle}{Y(t)(t-z_0)}\frac{dt}{t-z}. \quad (4.61)$$

Finally, inverting the transform (4.53), we have

$$w(x) = +\frac{1+i\omega}{i\omega z_0}\frac{\tilde{N}(z_0)}{\tilde{X}(z_0)}\exp\left(-\frac{(1+i\omega)x}{z_0}\right) - \frac{1}{2\pi i}\int_0^\infty \frac{\exp[-(1+i\omega)x/z]}{z-z_0}\frac{1+i\omega}{i\omega z} \times \left\langle\frac{\tilde{N}(z)}{\tilde{X}(z)}\right\rangle dz, \quad (4.62)$$

with $\langle N/X\rangle$ calculated as in (4.50) with

$$\tilde{N}^+ - \tilde{N}^- = \langle \tilde{f}(z)\rangle / \tilde{Y}(z)$$

and

$$\tilde{X}^- - \tilde{X}^+ = \frac{1+i\omega}{i\omega}\frac{\langle 1-\tilde{K}(z)\rangle}{\tilde{Y}(z)(z-z_0)}. \quad (4.63)$$

To study a hydrodynamic moment of g, it is, of course, necessary to put (4.62) back into (4.3) and take the appropriate moment. Our solution in the z variable for both models has the same form as is found in Refs. 12 and 22, while in the k variable it is similar to that of Ref. 11.

5. ANALYSIS OF THE SOLUTION

Determination of $g(x=0, \xi_1<0)$

Adiabatic model: As a first step in the study of the solution (4.49) we calculate the boundary value of the distribution for incoming molecules by evaluating (4.4) at $x=0$ for $\xi_1<0$ with the appropriate f_1, f_2, and f_3. This gives

$$g(x=0, \xi_1<0) = -\int_0^\infty \left[1 + \frac{1+i\omega}{3i\omega}\frac{\xi^2}{\xi_1^2}\right]u(s)\exp\left[\frac{(1+i\omega)s}{\xi_1}\right]ds - \frac{1}{3i\omega}\frac{\xi^2}{\xi_1}. \quad (5.1)$$

We demonstrate that (5.1) is bounded in the limit $\xi_1\to 0$. By Watson's Lemma we have

$$\int_0^\infty u(s)\exp\left(\frac{(1+i\omega)s}{\xi_1}\right)ds = -\frac{\xi_1}{1+i\omega}u(0) + \frac{\xi_1^2}{(1+i\omega)^2}u'(0) + O(\xi_1^2), \quad (5.2)$$

where we use $u(x) = u(0) + u'(0)x + O(x)$ which is demonstrated in Appendix B. Note that, unlike shear problems, $u'(0)$ exists—however, this is already signaled by the continuity equation (3.5). Using (5.2) in (5.1), we find, since $u(0) = 1$, that

$$\lim_{\xi_1 \to 0^-} g(x = 0, \boldsymbol{\xi}) = -\frac{(\xi_2^2 + \xi_3^2)}{3i\omega(1 + i\omega)} u'(0). \quad (5.3)$$

Positive wave model: Here we find that from (4.3)

$$g(x = 0, \xi_1 < 0) = -\frac{1}{\sqrt{2}} \int_0^\infty \left[1 + \frac{\xi^2}{\sqrt{15}\xi_1}\right] w(s) \times \exp\left[\frac{(1 + i\omega)s}{\xi_1}\right] ds \quad (5.4)$$

and in the limit for ξ_1 small, by Watson's lemma,

$$\lim_{\xi_1 \to 0^-} g(x = 0, \boldsymbol{\xi}) = \frac{(\xi_2^2 + \xi_3^2)}{\sqrt{30}\,(1 + i\omega)} w(0). \quad (5.5)$$

Comparing (5.3) and (5.5) with (3.25), we see that the distribution will not be continuous at $\xi_1 = 0$ unless

$$\begin{aligned} \alpha_2(\omega) &= \tfrac{2}{3}[u'(0)/i\omega(1 + i\omega)], \quad \text{adiabatic}, \\ &= (2/\sqrt{30})[w(0)/(1 + i\omega)], \quad \text{positive wave}. \end{aligned} \quad (5.6)$$

However, α_2 is fixed by (3.4a), and (5.6) is not satisfied. Hence the distribution function is discontinuous at the wall.

Low Frequency Oscillations

Adiabatic model: We will consider the behavior of the solution (4.49) as $\omega \to 0$. The expansion of z_0 in this case is given by (A.2). Now the point spectrum contribution to (4.49) is

$$u_p(x) = \frac{1 + i\omega}{(i\omega)^2 z_0} \frac{\tilde{N}(z_0)}{\tilde{X}(z_0)} e^{-(1+i\omega)x/z_0}. \quad (5.7)$$

$\tilde{N}(z_0)$ is given by (4.40, 4.44), and using (A.2) we find to first order

$$\tilde{N}(z_0) = -\frac{1}{z_0^2} \int_0^\infty \frac{\Omega(t)t}{\tilde{X}(-t, \omega = 0)} \frac{(t^2 + 2)}{3} dt + O(\omega^3) \quad (5.8)$$

[where $1/z^2 = -(3/5)\omega^2 + O(\omega^3)$]. The dependence of $\tilde{X}$ on ω is indicated and from (4.41, 4.43) we observe that it is well behaved at $\omega = 0$. Similarly $\tilde{X}$ is expanded using (4.45), (4.49), and we find

$$\tilde{X}(z_0) = -\frac{1}{z_0^3 (i\omega)^2} \int_0^\infty \frac{\Omega(t)t}{X(-t, \omega = 0)} \times \frac{(t^2 + 2)}{3} dt + O(\omega^2). \quad (5.9)$$

Then (5.8), (5.9) give in (5.7)

$$u_p(x) = [1 + O(\omega)] e^{-(1+i\omega)x/z_0}. \quad (5.10)$$

The continuous spectrum contribution to (4.49) is given by

$$u_c(x) = -\frac{1}{2\pi i} \int_0^\infty \frac{e^{-(1+i\omega)x/z_0}}{z - z_0} \frac{1 + i\omega}{(i\omega)^2 z} \left\langle \frac{\tilde{N}(z)}{\tilde{X}(z)} \right\rangle dz. \quad (5.11)$$

From (4.40), (4.44) we find that to first order

$$\tilde{X}(z) = \frac{1}{z_0} \int_0^\infty \frac{\Omega(t)t}{\tilde{X}(-t, 0)} \frac{(t^2 + 2)}{3} \frac{dt}{t - z} + O(\omega^2), \quad (5.12)$$

and

$$\tilde{X}(z) = \frac{1}{z_0^2 (i\omega)^2} \int_0^\infty \frac{\Omega(t)t}{\tilde{X}(-t, 0)} \frac{(t^2 + 2)}{3} \frac{dt}{t - z} + O(\omega). \quad (5.13)$$

Therefore

$$\frac{1}{z - z_0} \frac{1 + i\omega}{(i\omega)^2 z} \frac{\tilde{N}(z)}{\tilde{X}(z)} = -\frac{1}{z} + O(\omega). \quad (5.14)$$

If we now take the jump of (5.14), it is seen to be of $O(\omega)$ since the $O(1)$ term is analytic across the real axis [while the $O(\omega)$ term is not]. Hence in the limit, for a fixed x,

$$u_c(x) = O(\omega). \quad (5.15)$$

Thus (5.10), (5.15) and the expansion of z_0 give

$$u(x, t) = [1 + O(\omega)] \exp\{i\omega[t - (\tfrac{3}{5})^{\frac{1}{2}}x] - \tfrac{2}{5}(\tfrac{3}{5})^{\frac{1}{2}}\omega^2 x + O(\omega^3 x)\} + O(\omega). \quad (5.16)$$

Hence, for any fixed value of x and as $\omega \to 0$, the flow is governed not only by the plane wave (the point spectrum) but actually by hydrodynamics, since the amplitude of the plane wave is unity, i.e., $u(0) = 1$ in the normalization.

Positive wave model: For this model, since we are interested in the velocity moment u, it is necessary to substitute the solution for w, (4.63), into (4.3) and solve for u. Hence u will have a representation such as (4.63) in terms of the point and continuous spectra, and we again write

$$u = u_p(x) + u_c(x).$$

After some calculation, the contribution from the point spectrum is found to be

$$u_p(x) = -\frac{\tilde{N}(z_0)}{i\omega \tilde{X}(z_0)} \int_{-\infty}^\infty \frac{\Omega(\xi_1)}{\sqrt{2}} \times \left(\xi_1 + \frac{\xi_1^2 + 2}{\sqrt{15}}\right) \frac{\xi_1\, d\xi_1}{\xi_1 - z_0} e^{-(1+i\omega)x/z_0}, \quad (5.17)$$

and that of the continuous spectrum is

$$
\begin{aligned}
u_c(x) = \int_0^\infty & \Omega(\xi_1)\xi_1[\xi_1 + \tfrac{1}{2}\alpha_2(\omega)(\xi_1^2 + 2)] \\
& \times \exp\left(-\frac{(1+i\omega)x}{\xi_1}\right) d\xi_1 \\
& + \frac{1}{i\omega}\int_0^\infty \frac{\Omega(\xi_1)}{\sqrt{2}}\left(\xi_1 + \frac{\xi_1^2+2}{\sqrt{15}}\right)\xi_1 e^{-(1+i\omega)x/\xi_1} \\
& \times \left(-\frac{1}{2\pi i}\int_0^\infty \frac{1}{(z-z_0)(\xi_1 - z)}\left\langle\frac{\tilde{N}(z)}{\tilde{X}(z)}\right\rangle dz \right. \\
& \left. + \frac{\tilde{N}(z_0)}{\tilde{X}(z_0)}\frac{1}{(\xi_1 - z_0)}\right) d\xi_1 \\
& + \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{\Omega(\xi_1)}{\sqrt{2}}\left(\xi_1 + \frac{\xi_1^2+2}{\sqrt{15}}\right)\frac{\xi_1}{i\omega} \\
& \times \int_0^\infty \frac{e^{-(1+i\omega)x/z}}{(z-z_0)(\xi_1 - z)}\left\langle\frac{\tilde{N}(z)}{\tilde{X}(z)}\right\rangle dz. \qquad (5.18)
\end{aligned}
$$

First consider the pole contribution for ω small. The ξ_1 integral behaves asymptotically as $-1/\sqrt{2}\, z_0 + O(\omega^2)$. The expansions of $\tilde{N}(z_0)$ and $\tilde{X}(z_0)$ follow from (4.53), (4.56) and (4.59), (4.61) respectively. We find

$$
\begin{aligned}
\tilde{N}(z_0) = -\frac{2^{\frac{1}{2}}}{z_0}\int_0^\infty & \frac{\Omega(t)t}{\tilde{Y}(t)} \\
& \times \left[\tfrac{1}{30}(t^4 + 2(15)^{\frac{1}{2}}t^3 + 19t^2 + 4(15)^{\frac{1}{2}}t + 8)\right. \\
& + \frac{1}{2(15)^{\frac{1}{2}}}\left(\frac{\alpha_2(0)}{2} - \frac{1}{15^{\frac{1}{2}}}\right)(t^4 + (15)^{\frac{1}{2}}t^3 + 4t^2 \\
& \left. + 2(15)^{\frac{1}{2}}t + 8)\right]\frac{dt}{t-z} + O(\omega^2), \qquad (5.19)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{X}(z_0) = -\frac{1}{i\omega z_0^2}\int_0^\infty & \frac{\Omega(t)t}{\tilde{Y}(t)}\tfrac{1}{30}(t^4 + 2(15)^{\frac{1}{2}}t^3 \\
& + 19t^2 + 4(15)^{\frac{1}{2}}t + 8)\frac{dt}{t-z} + O(\omega^2). \qquad (5.20)
\end{aligned}
$$

Unlike the adiabatic model (5.17), $\alpha_2(0)$ occurs to first order in $\tilde{N}(z_0)$ [(5.19)]. Therefore, we must determine it to fix u_p and u_c. The condition for its determination is (3.4a), and so in the limit we have

$$1 = \lim_{\omega\to 0}\,[u_p(x=0) + u_c(x=0)].$$

As is easily seen by (5.17), (5.18), this is a linear equation in $\alpha_2(0)$. However, it is unnecessary to solve for it, since we observe that if

$$\alpha_2(0) = 2/\sqrt{15},$$

by (5.19), (5.20)

$$\tilde{N}(z_0)/\tilde{X}(z_0) = 2^{\frac{1}{2}}i\omega z_0 + O(\omega),$$

so that (5.17) to first order at $x = 0$ is

$$u_p(x=0) = 1 + O(\omega). \qquad (5.21)$$

Also we see that

$$[\tilde{N}(z)/\tilde{X}(z)] = O(\omega).$$

The remaining calculation is analogous to that of (5.12)–(5.14) and need not be repeated. All contributions to u_c are seen to be $O(\omega)$. Therefore, altogether, we have for this model

$$
\begin{aligned}
u(x,t) = [1 + O(\omega)]\exp\{& i\omega(t - (\tfrac{3}{5})^{\frac{1}{2}}x) \\
& - \tfrac{1}{5}(\tfrac{3}{5})^{\frac{1}{2}}\omega^2 x + O(\omega^3 x)\} + O(\omega). \qquad (5.22)
\end{aligned}
$$

Again for any fixed x and as $\omega \to 0$, not only does the plane wave dominate, but also hydrodynamics, since the amplitude is unity.

Limit of x Large

Adiabatic model: For x large, the contribution from the point spectrum is clear and is $O(e^{-k_0 x})$ where $\mathrm{Re}\, k_0(\omega) > 0$ for $\omega \neq 0$, i.e., the decay is exponential. Now consider the continuous spectrum. From (5.11) and (4.50)–(4.52) this is

$$
\begin{aligned}
u_c(x) = -\frac{1+i\omega}{(i\omega)^2(2\pi)^{\frac{1}{2}}} & \\
\times \int_0^\infty & \frac{\exp[-(1+i\omega)x/z - z^2/2]}{\tilde{X}^+(z)\tilde{X}^-(z)\tilde{X}(-z)} \\
& \times \frac{z^2+2}{3(z^2 - z_0^2)}h(z)\,dz, \qquad (5.23)
\end{aligned}
$$

where

$$
\begin{aligned}
h(z) = \tilde{X}^+(z)&[1 + i\omega 3zg_0(z)/(z^2+2)] \\
& + \{\tilde{N}^+(z)/[(i\omega)^2(z - z_0)]\}[1 + i\omega 2(2z^2+1)/ \\
& (z^2+2)]. \qquad (5.24)
\end{aligned}
$$

The integral is of the form

$$I(x) = \int_0^\infty e^{-(1+i\omega)x/z - z^2/2}q(z)\,dz.$$

Set $z = (1+i\omega)^{\frac{1}{3}}x^{\frac{1}{3}}w$ and take the principal branch of the cube root. The path can be taken as the real axis again and

$$
\begin{aligned}
I(x) = \int_0^\infty \exp & \left[-(1+i\omega)^{\frac{2}{3}}x^{\frac{2}{3}}\left(\frac{1}{w} + \frac{w^2}{2}\right)\right] \\
& \times q[z(w)]\frac{dz}{dw}\,dw.
\end{aligned}
$$

The exponential has a maximum at $w = 1$, and by Laplace's formula, for x large, $I(x) \sim (2\pi/3)^{\frac{1}{2}} \times \exp[-\frac{3}{2}z_m^2]q(z_m)$, where

$$z_m = (1+i\omega)^{\frac{1}{3}}x^{\frac{1}{3}}. \qquad (5.25)$$

Therefore, (5.23) becomes

$$u_c(x) \sim -\frac{1+i\omega}{(i\omega)^2\sqrt{3}} \times \exp[-\tfrac{3}{2}z_m^2] \frac{h(z_m)}{\tilde{X}^+(z_m)\tilde{X}^-(z_m)\tilde{X}(-z_m)} \times \frac{z_m^2+2}{3(z_m^2-z_0^2)}.$$

If we keep all first-order terms for z_m large (x large), we find

$$u_c(x) \sim \frac{1+i\omega}{(i\omega)^2 3\sqrt{3}} e^{-\frac{3}{2}z_m{}^2} \frac{z_m^5}{z_m^2 - z_0^2}\left(\frac{1}{z_m}[1+\tfrac{3}{2}i\omega z_m \alpha_2(\omega)] - \frac{\tilde{N}^+(\infty)}{z_m(i\omega)^2(z_m-z_0)}(1+4i\omega)\right), \quad (5.26)$$

with

$$\tilde{N}^+(\infty) = \int_0^\infty \frac{\Omega(t)t}{\tilde{X}(-t)} \frac{[i\omega t g_0(t) + \frac{1}{3}(t^2+2)]\,dt}{(t+z_0)}. \quad (5.27)$$

Thus in the limit of large x the continuous spectrum, being $O(\exp[-\frac{3}{2}x^{\frac{2}{3}}])$ by (5.25), (5.26), dominates the point spectrum. This result was first obtained in Ref. 11 and in another context in Ref. 16. If one takes x large as the hydrodynamic limit, then this result contradicts hydrodynamic theory which predicts that the point spectrum is dominant. Shortly we give a proper definition of what is the hydrodynamic limit, and this resolves the contradiction.

For the positive wave model, the same type of argument may be given and a result similar to (5.26) can be obtained. In view of the similarity a separate analysis does not seem warranted.

Hydrodynamic limit: We now define the hydrodynamic limit to be

$$\frac{x'\omega'}{(RT_0)^{\frac{1}{2}}} \text{ fixed}, \quad \nu \to \infty. \quad (5.28)$$

Here x' is physical distance from the walls, $(RT_0)^{\frac{1}{2}}/\omega'$ is the wavelength of the sound disturbance, and ν is the collision frequency. In dimensionless variables, the normalization (2.2), we have that (5.28) is equivalent to

$$x\omega \text{ fixed}, \quad \omega \to 0. \quad (5.29)$$

The consequences of this limit given below should serve as sufficient motivation for making this definition.

Let us consider the exponential behavior of the point and continuous spectra under this limit. The continuous spectrum for both models has the factor

$$\exp[-\tfrac{3}{2}x^{\frac{2}{3}}] = \exp[-\tfrac{3}{2}(\omega x)^{\frac{2}{3}}(1/\omega)^{\frac{2}{3}}] \sim \exp[-\tfrac{3}{2}(1/\omega)^{\frac{2}{3}}]. \quad (5.30)$$

The point spectrum, on the other hand, is, by taking the real part to first order,

$$\exp[-\tfrac{3}{5}(\tfrac{2}{3})^{\frac{1}{2}}\omega^2 x] = \exp[-\tfrac{2}{5}(\tfrac{3}{5})^{\frac{1}{2}}(\omega x)\omega] = O(1). \quad (5.31)$$

Clearly the point spectrum dominates and the hydrodynamic solution (5.16) or (5.22) results. It is of interest to consider the sequence of limits

$$\omega^n x \text{ fixed}, \quad \omega \to 0, \quad (5.32)$$

for $n = 1, 2, 3, \cdots$. By (5.30), (5.31) it is apparent that the point spectrum will still dominate for $n - 2 < 2n/3$, i.e., $n < 6$. Hence hydrodynamics results as long as

$$x \ll 1/\omega^6. \quad (5.33)$$

Otherwise the continuous spectrum will appear as a boundary layer at infinity at the order (5.33).

We refer to the change of dominance in the point and continuous spectra as "crossover." Therefore, condition (5.33) can be referred to as asymptotic crossover, that is, for $x \leq O(1/\omega^6)$ the continuous spectrum certainly dominates. It seems worthwhile having a more precise criteria for crossover since conceivably this might be measurable in an experiment. The lead term of u_p for the adiabatic model, from (5.16), is

$$u_p(x) \sim \exp[-(\tfrac{3}{5})^{\frac{1}{2}}i\omega x - \tfrac{2}{5}(\tfrac{3}{5})^{\frac{1}{2}}\omega^2 x]. \quad (5.34)$$

The continuous spectrum is calculated from (5.25), (5.26), and we find, keeping all first-order terms,

$$u_c(x) \sim -\frac{x^{\frac{5}{3}}\exp(-\frac{3}{2}x^{\frac{2}{3}})}{3^{\frac{1}{2}}(5+3\omega^2 x^{\frac{2}{3}})}\left(\frac{1}{x^{\frac{1}{3}}}[1+\tfrac{3}{2}i\omega x^{\frac{1}{3}}\alpha_2(0)] - \frac{1}{x^{\frac{1}{3}}}[(\tfrac{3}{5})^{\frac{1}{2}}i\omega x^{\frac{1}{3}} + I]\right). \quad (5.35)$$

To evaluate $\tilde{N}^+(\infty)$, (5.27), in the limit of ω small, we have used the fact that

$$\int_0^\infty \frac{\Omega(t)t}{X(-t,0)}(t^2+2)\,dt = -5, \quad (5.36)$$

which follows by expanding (4.44) for z large and ω small. The explicit crossover is obtained from the match of (5.34) and (5.35). Although $\alpha_2(0)$ is not known, its explicit form is not important since it will be $O(1)$.

ACKNOWLEDGMENT

The support of the Office of Naval Research, Contract No. NONR 562(39), is gratefully acknowledged.

APPENDIX A: THE DISPERSION RELATIONS

Adiabatic Model

The dispersion relation is defined to be $1 - K = 0$ and is of a class that has been exhaustively studied in Refs. 24 and 25. In particular the roots come in pairs. For ω small, we find

$$-ik = \pm(\tfrac{3}{5})^{\frac{1}{2}} i\omega[1 - \tfrac{2}{5}i\omega + \tfrac{7}{25}(i\omega)^2 + O(\omega^2)], \quad \text{(A1)}$$

or, in the z variable,

$$1/z = \mp(\tfrac{3}{5})^{\frac{1}{2}} i\omega[1 - \tfrac{7}{5}i\omega + \tfrac{42}{25}(i\omega)^2 + O(\omega^3)]. \quad \text{(A2)}$$

We denote the root in the fourth quadrant by $+k_0$ (or $+z_0$). Using the methods of Ref. 24, one can show that there are in fact only the two roots as given above. Waves to the right and left are produced, moving at the adiabatic speed to lowest order and decaying in the direction of propagation. It is also known[25] that there exists a $\omega_0 > 0$ such that no roots exists for $\omega > \omega_0$. By arguments similar to (3.8)–(3.10) it is possible to show that there is no root of $\mathrm{Im}\, k = 0$, $\omega \neq 0$.

Positive Wave Model

The dispersion relation for this model, from (4.34), is

$$1 - \tfrac{1}{30}\int_{-\infty}^{\infty} \Omega(t)[t^4 + 2(15)^{\frac{1}{2}}t^3 + 19t^2 + 4(15)^{\frac{1}{2}}t + 8]\, dt/(1 + i\omega - ikt) = 0, \quad \text{(A3)}$$

or (4.58) in the z variable. For ω small we find the root in the fourth quadrant

$$+ik = \tfrac{3}{5}i\omega[1 - \tfrac{1}{5}i\omega + O(\omega^2)]. \quad \text{(A4)}$$

This dispersion relation lacks the symmetry present in the adiabatic model. We want to show that (A4) is the only root of (A3). To carry this out, we study wave solutions of (3.23), (3.24),

$$(1 + i\omega + s\xi_1)g = (1/\sqrt{2})(\xi_1 + \xi^2/\sqrt{15})w, \quad \text{(A5)}$$

$$w = 1/\sqrt{2}(\xi_1 + \xi^2/\sqrt{15}, g). \quad \text{(A6)}$$

Taking the inner product of (A5) with respect to g^* and separating real and imaginary parts, we find

$$\omega + s_i(g, \xi_1 g) = 0, \quad \text{(A7)}$$

$$1 + s_r(g, \xi_1 g) = \|w\|^2. \quad \text{(A8)}$$

Here s_r and s_i are the real and imaginary parts of s, $(g, g) = \|g\|^2$, and for convenience we have taken $\|g\| = 1$. Hence, in particular $\|w\| \leq 1$. Equations (A7), (A8) give

$$\omega/(1 - \|w\|^2) = s_i/s_r. \quad \text{(A9)}$$

By assuming $\omega \geq 0$ for definiteness, (A9) implies that admissible roots lie in the first or third quadrant. Now consider (A5) for the limiting case $\omega = 0$. The complex conjugate of (A5) is

$$(1 - w)g^* + s^*\xi_1 g^* = (w^*/\sqrt{2})(\xi_1 + \xi^2/\sqrt{15}). \quad \text{(A10)}$$

Then (A10) multiplied by (A5), for $\omega = 0$, gives

$$|g|^2 + |s|^2\xi_1^2|g|^2 + 2s_r\xi_1|g|^2 = |w|^2/2(\xi_1 + \xi^2/\sqrt{15})^2,$$

and, taking the inner product of this with respect to 1, we have

$$1 + |s|^2\|\xi_1 g\|^2 + 2s_r(g, \xi_1 g) = \|w\|^2. \quad \text{(A11)}$$

First suppose that $(g, \xi_1 g) = 0$. Then (A11) is satisfied only if $s = 0$ and $\|w\| = 1$ (or, trivially, $\|\xi_1 g\|^2 = 0$, which implies $g = 0$). Secondly, if $(g, \xi_1 g) \neq 0$, by (A7) $s_i = 0$ for $\omega = 0$. It remains to show that $s_r = 0$ in that case. To do this, define

$$f(z) = \int_{-\infty}^{\infty} \frac{\Omega(t)\, dt}{t - z}.$$

For z above and below the real axis ($\pm$ respectively) the representation

$$f^{\pm}(z) = e^{-z^2/2}\left[\pm\frac{\pi i}{\sqrt{2\pi}} - \int_0^z e^{t^2/2}\, dt\right] \quad \text{(A12)}$$

is known.[24] Then the dispersion relation associated with (A5) is just (4.58) in the z variable, setting $s = -ik = (1 + i\omega)/z$. This can be reduced in terms of (A12) to

$$30(1 + i\omega) + z[f^{\pm}(z)\{z^4 + 2(15)^{\frac{1}{2}}z^3 + 19z^2 + 4(15)^{\frac{1}{2}}z + 8\} + z^3 + 2(15)^{\frac{1}{2}}z^2 + 20z + 6(15)^{\frac{1}{2}}] = 0. \quad \text{(A13)}$$

Now as $\omega \to 0$, $z_1 \to s_i/|s|^2$. Thus any root $z(\omega)$ of (A13) approaches a real value since we have shown that $s_i = 0$ when $\omega = 0$, and by continuity this is true in the limit, i.e., $s_i \to 0$ as $\omega \to 0$. By (A12) the imaginary part of (A13), for $\omega = 0$, is found to be

$$\pi/(2\pi)^{\frac{1}{2}}e^{-z^2/2}zP(z), \quad \text{(A14)}$$

where

$$P(z) = z^4 + 2(15)^{\frac{1}{2}}z^3 + 19z^2 + 4(15)^{\frac{1}{2}}z + 8. \quad \text{(A15)}$$

(A14) is zero if $z = 0$, but then the real part of (A13) is nonzero. We must check for zeros of $P(z)$ for z real. From (4.37) and (A3), P is seen to be the integral of a nonnegative function and therefore nonnegative. The stationary points of (A15) are found to be at $-(15)^{\frac{1}{2}}/2$, $-(15)^{\frac{1}{2}}/2 \pm (7)^{\frac{1}{2}}/2$, and at the minima (A15) is positive. Therefore, (A14) is never zero unless $z_r \to \infty$, which implies $s_r = 0$. Hence we conclude that when $\omega = 0$, $s = 0$. Therefore, at least for ω small, (A4) is the only root of (A3).

APPENDIX B: BOUNDEDNESS OF $u'(0)$ FOR THE ADIABATIC MODEL

That $u'(0)$ ought to exist follows from the continuity equation (3.5), since it exists if and only if $\rho(0)$ does, and the latter must be bounded by physical considerations. Differentiating (4.49) and evaluating at $x = 0$, we get

$$u'(0) = -\frac{(1+i\omega)^2}{(i\omega)^2 z_0^2}\frac{\tilde{N}(z_0)}{\tilde{X}(z_0)} + \frac{(1+i\omega)^2}{(i\omega)^2 2\pi i}\int_0^\infty \frac{1}{(z-z_0)}\cdot\frac{1}{z^2}\left\langle\frac{\tilde{N}(z)}{\tilde{X}(z)}\right\rangle dz. \quad \text{(B1)}$$

By (4.50)–(4.52) we see that

$$\langle \tilde{N}(z)/\tilde{X}(z)\rangle = O(z), \quad \text{(B2)}$$

as $z \to 0$. Therefore, the integrand of the integral in (B1) is $O(1/z)$ for z small and the integral does not appear to exist. We demonstrate that (B2) is in fact $O(z^2)$ so that (B1) does exist.

We begin by Fourier-transforming the continuity equation (3.5) in the z variable, (4.38), to give

$$u(k(z)) = -z/(1+i\omega) + [zi\omega/(1+i\omega)]\rho(k(z)). \quad \text{(B3)}$$

By (4.39)

$$u(k(z)) = \frac{z}{(z-z_0)(i\omega)^2}\langle \tilde{N}(z)/\tilde{X}(z)\rangle, \quad \text{(B4)}$$

and in the limit of z small, by (5.2), (B4) or (B3) is

$$u(k(z)) = -z/(1+i\omega) + O(z). \quad \text{(B5)}$$

Therefore, we must have $\rho(k(z)) = O(z)$ for (B3) to be consistent with (B5). This suggests that we write

$$\rho(k(z)) = z[M(z) + R(z)]/(z-z_0)\tilde{X}(z), \quad \text{(B6)}$$

where we suppose

$$M(z) = \int_0^\infty \frac{m(t)}{\tilde{X}(-t)(t+k_0)}\frac{dt}{t-k}. \quad \text{(B7)}$$

$m(t)$ is to be determined, and we require that M be bounded for z small, consistent with (B3). R is analytic in the plane. The form (B6) only says that ρ has the same singular behavior as u which follows from (B3). Therefore, (B3), (B4), and (B6) give

$$\frac{z}{(z-z_0)(i\omega)^2}\frac{\tilde{N}(z)}{\tilde{X}(z)} = -\frac{z}{1+i\omega} + \frac{z^2 i\omega}{1+i\omega}\frac{[M(z)+R(z)]}{(z-z_0)\tilde{X}(z)}. \quad \text{(B8)}$$

Evaluating the jump of (B8) across the positive real axis, one finds that

$$m(t) = \Omega(t)t\left(g_0(t) - \frac{t^2}{1+i\omega}\right)\frac{1+i\omega}{(i\omega)^2}, \quad \text{(B9)}$$

so that by (B2)

$$M(t) = \frac{1+i\omega}{(i\omega)^2}\int_0^\infty \frac{\Omega(t)t}{\tilde{X}(-t)(t+z_0)} \times \left(g_0(t) - \frac{t^2}{1+i\omega}\right)\frac{dt}{t-z}. \quad \text{(B10)}$$

To determine R, observe that in (B8) the left-hand side is $O(1)$ for z large, $(z-z_0)X(z) = O(1)$ so that $R = O(1/z)$. Hence it is zero everywhere. Also note that by (B10), M is bounded for z small. We thus have with this construction of M that

$$\frac{\tilde{N}(z)}{\tilde{X}(z)} = (i\omega)^2\frac{(z-z_0)}{1+i\omega} + (i\omega)^3 z\frac{M(z)}{\tilde{X}(z)}. \quad \text{(B11)}$$

Now the jump in M is $O(z)$ for z small so that (B2) is actually $O(z^2)$. Therefore, the continuous spectrum integral exists and $u'(0)$ is bounded.

[1] C. S. Wang Chang and G. E. Uhlenbeck, Univ. Mich. Eng. Res. Inst. ONR Contr. N60nr-23222 (1952).
[2] C. L. Pekeris, Z. Alterman, and L. Finklestein, "Integral and Intro-Differential Equations of P.I.C.C," *Symposium on the Numerical Treatment of Ordinary Differential Equations* (Birkhauser-Verlag, Basel, 1960), pp. 388–98.
[3] C. L. Pekeris, Z. Alterman, L. Finkelstein, and K. Frankowski, Phys. Fluids **5**, 1608 (1962).
[4] M. Greenspan, J. Acoust. Soc. **28**, 644 (1956).
[5] E. Meyer and G. Sessler, Z. Physik **149**, 15 (1957).
[6] L. Sirovich and J. K. Thurber, J. Acoust. Soc. **37**, 329 (1965).
[7] L. Sirovich and J. K. Thurber, J. Acoust. Soc. **38**, 478 (1965).
[8] G. Maidanik and H. L. Fox, J. Acoust. Soc. **38**, 477 (1965).
[9] G. Maidanik, H. L. Fox, and M. Heckl, Phys. Fluids **8**, 259 (1965).
[10] G. Maidanik and M. Heckl, Phys. Fluids **8**, 266 (1965).
[11] H. S. Ostrowsky and D. J. Kleitman, Nuovo Cimento **44B**, 49 (1966).
[12] H. Weitzner, in *Rarefied Gas Dynamics*, edited by J. H. de Leeuw (Academic, New York, 1965), Vol. I, p. 1.
[13] R. J. Mason, in Ref. 12, Vol. I, p. 48.
[14] R. J. Mason, in *Rarefied Gas Dynamics*, edited by C. L. Brundin (Academic, New York, 1967), Vol. I, p. 395.
[15] J. K. Buckner and J. H. Ferziger, Phys. Fluids **9**, 2315 (1966).
[16] G. Ya. Lyubarskii, Zh. Eksp. Teor. Fiz **40**, 1050 (1961) [Sov. Phys. JETP **13**, 740 (1961)].
[17] P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).
[18] P. Welander, Archiv Fysik **7**, 507 (1954).
[19] L. Sirovich and J. K. Thurber, J. Math. Phys. **10**, 239 (1969).
[20] L. Sirovich and H. York, Phys. Fluids **13**, 1731 (1970).
[21] T. G. Richardson and L. Sirovich, J. Math. Phys., **12**, 450 (1971).
[22] C. Cercignani, "Elementary Solutions of Linearized Kinetic Models and Boundary Value Problems in the Kinetic Theory of Gases," Div. of Appl. Math. and Div. of Eng. Technical Report, Brown University (1965).
[23] G. F. Carrier, M. Krook, and C. E. Pearson, *Functions of a Complex Variable* (McGraw-Hill, New York, 1966).
[24] L. Sirovich and J. K. Thurber, NYU Inst. Math. Sci. Report NYO-9757, MF-18; AEC Rand D. Rept. Contr. AT(30-1)-1480 (1961).
[25] L. Sirovich and J. K. Thurber, J. Math. Phys. **8**, 888 (1967).