independent elements over the reals or $3^n \times 3^n$ elements over the complexes. It is thus a basis for all $3^n \times 3^n$ matrices, in other words a complete matrix algebra. While this algebra satisfies no simple commutators or anticommutators, it satisfies the following relations:

$$\begin{aligned} Q_k\pi_l &= \omega\pi_l Q_k, & k \geq l, \\ \pi_k\pi_l &= \omega^{-1}\pi_l\pi_k, & k > l, \\ Q_k\pi_l &= \omega^{-1}\pi_l Q_k, & k < l, \\ Q_k Q_l &= \omega^{-1}Q_l Q_k, & k > l. \end{aligned}$$

Some of the triple terms such as $\pi_1\pi_2\pi_3$ equal $\pi_3\pi_2\pi_1$, but this is not a general property of the algebra. Thus, this algebra does not serve as a representation for parastatistics.

If we write the series $\pi_1, Q_1, \pi_2, Q_2, \pi_3, Q_3, \cdots, \pi_n, Q_n$, then, excluding the end factors, each entry has the same commutation relations with its neighbors; i.e., $\pi_1 Q_1 = \omega^{-1}Q_1\pi_1$, $Q_1\pi_2 = \omega^{-1}\pi_2 Q_1 \cdots$. This is essentially the mathematical structure behind the fact that the dual transformation is an automorphism. If we define a transformation partially by $D: \pi_1 \to Q_1 \to \pi_2 \to \cdots \pi_n \to Q_n \to \Phi(Q_n)$ such that $\Phi(U) = U^K$ and $D: \Lambda_1 A \to B \to \Lambda_1 A$, then $\Phi(U) = U^{-1}$ and $\Phi(Q_n) = \omega^{-1}\pi_1 U^{-2}$ complete the automorphism and preserve the commutation relations of Q_n with π_1 and π_n. The transformation given above serves the general q-component model. In the 3-component system, we have $\omega^{-1} = \omega^2$ and $U^{-2} = U$.

APPENDIX C

The critical energy E_c can be determined by assuming that the free energy is a differentiable function. This implies that f is differentiable with respect to its argument. With $kT\theta = 1$, we have at high- and low-temperatures, respectively,

$$E = -2\frac{\partial}{\partial\theta}\log u - \frac{f'(x/u)}{f(x/u)}\frac{\partial(x/u)}{\partial\theta}, \quad T \geq T_c,$$

$$E = -2\frac{\partial \log \alpha}{\partial\theta} - \frac{f'(\beta/\alpha)}{f(\beta/\alpha)}\frac{\partial(\beta/\alpha)}{\partial\theta}, \quad T \leq T_c.$$

The unknown function f'/f can be eliminated at the critical point to obtain

$$E_c = -\left[\frac{2\alpha'}{\alpha}\left(\frac{x}{u}\right)_c' - \frac{2u'}{u}\left(\frac{\beta}{\alpha}\right)_c'\right] \Big/ \left[\left(\frac{x}{u}\right)_c' - \left(\frac{\beta}{\alpha}\right)_c'\right].$$

The derivatives taken with respect to θ at the critical temperature satisfy $(x/u)_c' = -(\beta/\alpha)_c'$, so that

$$E_c = \epsilon_0 + \epsilon_1 - (\epsilon_1 - \epsilon_0)/\sqrt{q}.$$

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Effect of Molecular Collision Frequency on Solutions of the Linearized Boltzmann Equation*

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The linearized Boltzmann equation is considered for steady-state oscillations. Denoting molecular collision frequency by $\nu(\xi)$ and writing $\nu = O(\xi^\alpha)$ for ξ large, we show that solutions for $x \to \infty$ behave like $\exp(-x^{2/(3-\alpha)})$.This shows that the continuous spectra dominates hydrodynamics for all except the rigid sphere or radial cutoff case ($\alpha = 1$).

1. INTRODUCTION

It is a well-accepted fact and certainly no surprise that the behavior of a gas near a wall is described by kinetic theory equations and not simply by fluid mechanical ones (at least under most conditions). Several investigations[1-3] have indicated that flow far from a boundary is also a nonhydrodynamical regime. We are speaking, in particular, of the $O(\exp(-x^{2/3}))$ falloff at large distances predicted in sound propagation[2] and shock structure[1] for constant molecular collision frequencies and a somewhat more complicated but still nonhydrodynamical falloff for velocity dependent collision frequencies recently shown in Refs. 3 and 4.

The purpose of this investigation is to demonstrate the effect of intermolecular collision frequency on solutions of the linearized Boltzmann equation. We will carry out our calculations in connection with a problem in steady-state oscillations. However, the results we obtain should be equally applicable to other problems and to transport equations more general than the Boltzmann equation. In view of the somewhat heavy calculations, we give below an outline of our results and discuss them here rather than in the next section.

A first result is that the collision frequency $\nu(\xi)$ for high speed molecules has a direct effect on the nature of the solution. In fact, writing

$$\nu(\xi) = O(\xi^\alpha) \tag{1.1}$$

for ξ large (for noncharged particles $0 \leq \alpha \leq 1$), we find, for example, that the density ρ is given by

$$\rho = O(e^{-x^{2/(3-\alpha)}}). \tag{1.2}$$

This result has already been shown in an approximate method for certain kinetic models.[3,4] Our demonstration is given within the framework of the linearized Boltzmann equation itself.

For oscillations of frequency ω, hydrodynamics predicts that the falloff of oscillations at large distances is given by

$$\rho = O(e^{-k(\omega)x}) \tag{1.3}$$

(where $\operatorname{Re} k > 0$, $\omega \neq 0$). From this we see that hydrodynamic theory at large distances is small compared to the kinetic theory prediction except when $\alpha = 1$. Hence the region at infinity is a kinetic theory boundary layer.

The region near a wall is intuitively, at least, a free-flow regime; however, the region at infinity, although containing particles not having undergone collisions, is not a free-flow regime. We determine (1.2) by a stationary exponent calculation of an integral. Examination of this shows that, for a fixed but large value of x, the main contribution in the calculation comes from particles whose speed is $\xi = O(x^{1/(3-\alpha)})$. Now the free path of a particle moving with a speed ξ is ξ/ν; therefore, for fast particles, the free path $\lambda = O(\xi^{1-\alpha})$ by (1.1). Hence the main contribution to the evaluation (1.2) comes from particles having a free path

$$\lambda = O(x^{(1-\alpha)/(3-\alpha)}).$$

Therefore, these particles for all values of α have undergone many collisions.

This raises an interesting point with regard to the Chapman–Enskog procedure. There, it will be recalled, it is assumed that spatial derivatives are slowly varying with respect to the mean free path. It is, of course, clear that the procedure is not uniform in the velocity since the combination $\boldsymbol{\xi} \cdot \partial/\partial \mathbf{x}$ occurs. Our present investigation therefore demonstrates that, for $\alpha < 1$ and for certain regimes, $\boldsymbol{\xi}$ in this combination cannot be regarded as small.

As a next point, we take up the question of where classical hydrodynamics is valid. For low-frequency phenomena, $k(\omega)$ in (1.3) is $O(\omega^2)$. Comparing this with (1.2), we can say that in the low-frequency limit at least hydrodynamic theory is valid for

$$x \ll \omega^{-2[(3-\alpha)/(1-\alpha)]}.$$

Therefore, only for $\alpha = 1$ (effectively rigid sphere molecules) does the hydrodynamic region extend to infinity.

2. STEADY-STATE OSCILLATIONS

The problem of steady-state oscillations in a half-space has been discussed at length; the equation governing this is

$$\left(\nu(\xi) + i\omega + \xi_1 \frac{\partial}{\partial x}\right) g(x, \boldsymbol{\xi}) = Kg, \tag{2.1}$$

where ω is the frequency of oscillation and the linearized Boltzmann operator has been split into the difference $K - \nu$, ν being the molecular collision frequency. A discussion of the spectra of the operator $(i\omega + \xi_1(\partial/\partial x) + \nu - K)$ has been given in Ref. 5. In general, the spectra consists of point spectra and a two-dimensional region of continuous spectra. For (2.1), as is well known, only the outgoing distribution g $(\xi_1 > 0)$ is in any way specified at $x = 0$. We will, however, not really consider any specific boundary-value problem. Formally we will regard the solution as known and then, from this, seek properties of it.

Regarding the right-hand side of (2.1) as known, we formally integrate and find

$$\begin{aligned} g(x, \boldsymbol{\xi}) = {} & H(\xi_1) g_o(\boldsymbol{\xi}) \exp\left(\frac{-(\nu + i\omega)x}{\xi_1}\right) \\ & + H(\xi_1) \int_0^x \frac{1}{\xi_1} (Kg)(s, \boldsymbol{\xi}) \\ & \times \exp\left(\frac{-(\nu + i\omega)(x - s)}{\xi_1}\right) ds \\ & - H(-\xi_1) \int_x^\infty \frac{1}{\xi_1} (Kg)(s, \boldsymbol{\xi}) \\ & \times \exp\left(\frac{-(\nu + i\omega)(x - s)}{\xi_1}\right) ds, \end{aligned} \tag{2.2}$$

where H is the Heaviside function and $g_o(\boldsymbol{\xi}) = g(x = 0, \xi_1 > 0)$. Then, combining the last two terms of (2.2), we obtain

$$g(x, \boldsymbol{\xi}) = H(\xi_1) g_o(\boldsymbol{\xi}) \exp\left(\frac{-(\nu + i\omega)x}{\xi_1}\right) + \left(H(\xi_1)\int_0^x + H(-\xi_1)\int_0^\infty\right) \frac{1}{|\xi_1|}(Kg)(s, \boldsymbol{\xi}) \times \exp\left(\frac{-(\nu + i\omega)|x - s|}{|\xi_1|}\right) ds$$
$$\equiv (Bg_o)(x, \boldsymbol{\xi}) + (WKg)(x, \boldsymbol{\xi}), \tag{2.3}$$

where B and W represent the linear operators defined through (2.3). Note that (2.3) reduces the problem to one in integral equations; we will, however, not pursue this line of investigation here. We now solve for a moment of g, say the density

$$\rho(x) = (1, g)(x) \equiv \int_{-\infty}^{\infty} 1 \cdot g(x, \boldsymbol{\xi})(2\pi)^{-\frac{3}{2}} \exp(-\tfrac{1}{2}\xi^2)\, d\boldsymbol{\xi},$$

so that

$$\rho(x) = (1, Bg_o)(x) + (1, WKg)(x). \tag{2.4}$$

It is known that the term $(1, WKg)(x)$ contributes point spectra as well as continuous spectra to the solution. $(1, Bg_o)(x)$ contributes only continuous spectra. Here we examine the latter effect on $\rho(x)$ as x becomes large.

As will be clear, our discussion will apply to both normal and transverse oscillations, but for the sake of simplicity we assume purely normal oscillations in which case $g = g(\xi, \xi_1)$. Then we write the integral as

$$(1, Bg_o)(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \int_0^1 \xi^2 g_o(\xi, \mu) \times \exp\left(-\frac{\xi^2}{2} - \frac{[\nu(\xi) + i\omega]x}{\xi\mu}\right) d\xi\, d\mu,$$

where μ is the cosine of the polar angle and $\xi = |\boldsymbol{\xi}|$. Considering first the μ integration, it is clear that the exponential term involving μ is a maximum when $\mu = 1$, since $c = [\nu(\xi) + i\omega]/\xi$ is such that

$$|\arg c| < \tfrac{1}{2}\pi - \delta, \ \delta > 0.$$

Under the transformation $\mu = 1/(1 + p)$ this part of the calculation is reduced to the Laplace integral

$$e^{-xc} \int_0^\infty e^{-xcp} g_o\left(\xi, \frac{1}{1 + p}\right) \frac{dp}{(1 + p)^2}.$$

Watson's lemma applies, giving the μ integral asymptotically equal to $(x \to \infty)$

$$e^{-xc} g_o(\xi, 1)[(xc)^{-1} + O(x^{-2}c^{-2})].$$

Thus, we are left with

$$(1, Bg_o)(x) \sim \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \frac{\xi^3 g_o(\xi, 1)}{x[\nu(\xi) + i\omega]} \times \exp\left(-\frac{\xi^2}{2} - \frac{[\nu(\xi) + i\omega]x}{\xi}\right) d\xi. \tag{2.5}$$

The stationary points of the exponential function

$$f(\xi; x) = -\tfrac{1}{2}\xi^2 - [\nu(\xi) + i\omega]x/\xi$$

are given by

$$-\xi_0 - \frac{\nu'(\xi)_0}{\xi_0} x + \frac{\nu(\xi)_0 + i\omega}{\xi^2_0} x = 0 \tag{2.6}$$

and the second derivative is

$$-f_{\xi\xi}(\xi_0; x) = 3 + \frac{\nu''(\xi_0)}{\xi_0} x. \tag{2.7}$$

Take ξ to be a complex variable and expand $g_o(\xi, 1)$ about the solution ξ_o of (2.6); then, with (2.7) to lowest order, the method of steepest descent gives

$$(1, Bg_o) \sim \frac{e^{f(\xi_o; x)}}{[-f_{\xi\xi}(\xi_o; x)]^{\frac{1}{2}}} \frac{g_o(\xi_o, 1)\xi_o^3}{x[\nu(\xi_o) + i\omega]}. \tag{2.8}$$

To make (2.8) specific, we suppose that, for ξ large, ν has the asymptotic expansion

$$\nu(\xi) = k\xi^\alpha + \alpha V + O(\xi^{-\beta}),$$

with $0 \le \alpha \le 1$ and k, V, and β positive constants. The case $\alpha = 1$ corresponds to rigid sphere molecules or radial cutoff. $\alpha = 0$ gives a constant collision frequency as in the Krook model equation. Then the solution of (2.6) is, for x large, given by

$$\begin{aligned} \xi_o &= [(k + i\omega)x]^{\frac{1}{3}} + O(x^{(1-\beta)/3}), & \alpha &= 0, \\ &= [(1 - \alpha)kx]^{1/(3-\alpha)} + O(x^{(1-\alpha)/(3-\alpha)}), & 0 &< \alpha < 1, \\ &= [(V + i\omega)x]^{\frac{1}{3}} + O(x^{(1-\beta)/3}), & \alpha &= 1. \end{aligned} \tag{2.9}$$

Then for (2.7) we have

$$\begin{aligned} f_{\xi\xi}(\xi_o; x) &= -3 + O(x^{-\beta/3}), & \alpha &= 0, \\ &= -3 + \alpha + O(x^{-\alpha/(3-\alpha)}), & 0 &< \alpha < 1, \\ &= -3 + O(x^{-\beta/3}), & \alpha &= 1 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} f(\xi_o; x) &= -\tfrac{3}{2}[(k + i\omega)x]^{\frac{2}{3}} + O(x^{(2-\beta)/3}), & \alpha &= 0, \\ &= -\tfrac{1}{2}(3 - \alpha)(1 - \alpha)^{(\alpha-1)/(3-\alpha)}(kx)^{2/(3-\alpha)} + O(x^{(2-\alpha)/(3-\alpha)}), & 0 &< \alpha < 1, \\ &= -(kx) + O(x^{\frac{2}{3}}), & \alpha &= 1. \end{aligned} \tag{2.11}$$

$f(\xi_o; x)$ can be written compactly as

$$f(\xi_o; x) = -[kx + i\omega x\delta_{\alpha 0}]^{2/(3-\alpha)} \times \tfrac{1}{2}(3-\alpha)(1-\alpha)^{(\alpha-1)/(3-\alpha)} + o(x^{2/(3-\alpha)}), \quad 0 \leq \alpha \leq 1, \quad (2.12)$$

where the order term is more precisely given by (2.11). It can be shown by using (2.9)–(2.12) that paths in the complex plane can always be found such that the method of steepest descent is valid, and that (2.8) is then given as

$$(1, Bg_o) \sim \frac{g_o(\xi_o, 1)}{[3-\alpha+\delta_{\alpha 1}]^{\frac{1}{2}}} \left[1 - \alpha + \delta_{\alpha 1} \frac{(V+i\omega)^{\frac{2}{3}}}{kx^{\frac{1}{3}}}\right] \times \exp\{[kx + i\omega x\delta_{\alpha 0}]^{2/(3-\alpha)} \times \tfrac{1}{2}(3-\alpha)(1-\alpha)^{(\alpha-1)/(3-\alpha)} + o(x^{2/(3-\alpha)})\}. \quad (2.13)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

Thus we observe that $(1, Bg_o)(x)$ behaves essentially as $\exp(-ax^{2/(3-\alpha)})$, $a > 0$, for x large. This result, for $\alpha = 0$, was first obtained in Refs. 1 and 2 using the Krook equation. For $\alpha > 0$ the result was found in Refs. 3 and 4 by using an approximate method for kinetic models. Here we have demonstrated that this result holds in general for the linearized Boltzmann equation.

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Stochastic Theory of Quantum Mechanics for Particles with Spin

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The stochastic theory of quantum mechanics is further developed to include the problem of extended rigid particles, thus allowing the introduction of spin. It is demonstrated that the stochastic equations for the system's center of mass give rise to a generalized Schrödinger equation for integral or half-integral spin; in the particular case of spin $\frac{1}{2}$, upon elimination of the internal variables, Pauli's equation is obtained. A formal simplified relativistic extension of the theory is worked out and shown to lead to Dirac's equation in the case of spin $\frac{1}{2}$ and for a gyromagnetic ratio equal to 2; in the case of arbitrary spin, the theory gives an equation of the Feynman–Gell-Mann type.

I. INTRODUCTION

In a series of papers[1] we have proposed an elementary theory for a classical particle subject to a random interaction with its surroundings. In (I) it was shown that such a stochastic theory contains as a particularly simple case the quantum mechanics of a nonrelativistic spinless particle under the action of an external potential. In (II), (III), and (IV) we demonstrated that the theory applies also to more general situations, as, for example, the electromagnetic case, and to a system of interacting particles; in particular, the two-body problem was studied more closely. The aim of this paper is to extend the theory to particles with spin. With this purpose and following the line of thought presented in the aforementioned papers, we shall consider our stochastic particle as a spinning rigid body. We are aware of the fact that such a model is not fashionable, due to its inherent difficulties,[2] and that more abstract and formal procedures, which assign no classical analog to the spin variable, are being preferred. Nevertheless, the introduction of spinning rigid bodies has given lately a series of interesting results, both in the study of electron spin in particular[3,4] and in connection with some attempts to understand the nature of the quantum numbers of elementary particles; two representative examples of such attempts are given in Refs. 5 and 6.

To start with, we consider a system of stochastic particles and apply to them the methods previously developed. Upon introduction of the constraints defining a rotating rigid body, the theory gives a Schrödinger equation, except for some additional terms due both to the particle's extension and to its spin. The amplitude ψ is now a function of the center-of-mass