JOURNAL OF MATHEMATICAL PHYSICS VOLUME 11, NUMBER 4 APRIL 1970

Asymptotic Evaluation of Multidimensional Integrals*

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(Received 8 May 1969; Revised Manuscript Received 24 November 1969)

The asymptotic evaluation of a wide class of multidimensional integrals occurring in mathematical physics is considered. In this class are included integrals of the form

$$\frac{1}{(2\pi)^N}\int_{-\infty}^{\infty}\rho(\mathbf{k})\exp\left[i\mathbf{k}\cdot\mathbf{x}-\sigma(\mathbf{k})t\right]d\mathbf{k}.$$

A semiconstructive method is proven and certain classes of integrals are asymptotically evaluated. Examples involving problems in partial differential equations and a transport equation are given.

1. INTRODUCTION

It is often the case in mathematical physics that the resolution of a problem reduces itself to the evaluation of integrals. This is especially true in the case of linear problems. In spite of this, formidable problems still usually remain. Often the integrals one encounters do not have representations in terms of familiar or, for that matter, tabulated functions. In such cases one tries to take advantage of the presence of large parameters in the integrand.[1] Techniques for exploiting the presence of a single large parameter occurring in 1-dimensional integrands have been considered exhaustively in the literature.[2–4] These classical techniques have also proven successful in a number of cases involving multidimensional integrals,[5–9] but progress there has not been as great. Generally speaking, these methods represent the asymptotic evaluation in terms of an evaluation at the stationary point of a function. The location of the stationary point of this exponent is, of course, not part of the classical methods, and this part of the calculation usually proves impossible except when only elementary functions are involved.

In this paper, we develop a method for the asymptotic evaluation of integrals which avoids these restrictions and difficulties. We consider integrals over an arbitrary number of dimensions, containing a number of parameters. In order to do this, we naturally have to give up a certain amount of generality. We do this by focusing on integrals which are typical of a large class that occur in mathematical physics. As the reader will see, the restrictions placed on the integrand of the integrals under study are typically the case in physical problems containing a dissipative mechanism.

The method discussed in this paper has already proven successful in a number of problems in gas dynamics,[10–13] magnetohydrodynamics[14,15] and kinetic theory.[16,17] A general discussion for integrals over one dimension has already been given.[10]

2. STATEMENT OF THE MAIN RESULT

To begin with, we consider integrals of the form

$$I=\frac{1}{(2\pi)^N}\int_{-\infty}^{\infty}e^{i\mathbf{k}\cdot\mathbf{x}-\sigma(\mathbf{k})t}\rho(\mathbf{k})\,d\mathbf{k}. \tag{1}$$

[This is generalized below by Eq. (21).] Both $\mathbf{k}$ and $\mathbf{x}$ denote N-dimensional vectors and $d\mathbf{k}$ represents the N-space volume element. The integration may extend over any part of N space. The infinite limits of integration are indicated only for simplicity; other limits can be included in the support of ρ. The sole restriction in this regard is that the region of integration include the origin.

Without loss of generality we may take

$$\sigma(\mathbf{k})=0,\quad \mathbf{k}=0. \tag{2}$$

$\sigma(\mathbf{k})$ is said to be admissible if it satisfies the following five conditions:

(i) $\operatorname{Re}\sigma=\sigma_r\geq 0$,

(ii) $\sigma_r=0$, only if $\mathbf{k}=0$,

(iii) $\sigma\in C$,

(iv) in the neighborhood of the origin

$$\sigma=if(\mathbf{k})+g(\mathbf{k})+O(\mathbf{k}^3),$$

where f and g are real, continuous, and homogeneous degree one and two, respectively,

(v) $g=0$, only if $\mathbf{k}=0$.

Condition (i), which demands that $-\sigma_r$ have a global maximum, states that the system in question is stable, and condition (ii) then adds that it be dissipative. Condition (iv) is obtained if $\sigma\in C^3$, and is therefore somewhat weaker. That the first order is pure imaginary and the second pure real is often a direct consequence of the transformation properties of the equations governing the system.

For most purposes it suffices to place the following weak restriction on the function $\rho(\mathbf{k})$:

(vi) $|\rho|, \quad \int_{-\infty}^{\infty} |\rho| \, d\mathbf{k} < M < \infty.$

[Actually, as will be clear, (vi) is stronger than necessary, but we avoid such mathematical niceties.]

Main Result

If σ is admissible and ρ satisfies (vi), then I, as given by (1), can be written as

$$I = I_0 + O^*(t^{-\frac{1}{2}(N+1)}),$$

where

$$I_0 = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x} - if(\mathbf{k})t - g(\mathbf{k})t} \rho(\mathbf{k}) \, d\mathbf{k} \tag{3}$$

and $O^*(t^{-p})$ represents a quantity such that

$$\lim_{t\to\infty} t^{p-\delta} O^*(t^{-p}) = 0$$

for any small $\delta > 0$.

With the additional condition at the origin

(vii) $\rho = \rho_0 + O(\mathbf{k})$,

we obtain

$$I = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x} - if(\mathbf{k})t - g(\mathbf{k})t} d\mathbf{k}\rho_0 + O^*(t^{-\frac{1}{2}(N+1)})$$

or

$$I = I^0\rho_0 + O^*(t^{-\frac{1}{2}(N+1)}), \tag{4}$$

where I^0 is defined through (4). We prove (3) and (4), and an extension (21) and (22) in Sec. 3. In the remainder of this section we comment on certain aspects of the calculations involved in (3) and (4).

Before going further, it should be noted that the main result is, in a sense, only semiconstructive. The integral appearing in (3), and even the one in (4), cannot generally be carried out in terms of elementary functions. Even after taking into consideration the homogeneity requirements on f and g [see (iv)], we are left with an integral which cannot, in general, be carried out. In Sec. 4, however, we carry out the full integration in several important special cases.

So far, nothing has been said of the parameters $\mathbf{x}$. In fact, the error estimates in (3) and (4) are completely independent of $\mathbf{x}$. This is another aspect of the semiconstructive nature of the calculation. The region of validity in $\mathbf{x}$ space of the calculation is the restriction to those $\mathbf{x}$ such that the integral of (3) or (4) is large compared with the error estimate. Often this region becomes apparent only after the completion of the integration of the integrals in (3) or (4). We give explicit examples of this in Sec. 4.

In this same vein, we point out that it is conceivable that the integral terms in (3) or (4) are less than, or of the same order as, the error estimate for all values of $\mathbf{x}$. In such a case, the calculation as it stands only represents an estimate for the integral I.

At this point, we mention an essential difference between (3) and (4). In general, the modulus of the error estimates in (3) and (4) are quite different. The first form (3) only involves an expansion in terms of the scales of the underlying operator leading to (1), while (4) involves in addition an expansion of the data of the problem. In other words, (3) leads to a sharper result and, hence, may be used for significantly shorter times. As an illustration, in gas dynamics,[10] (3) is valid for times large compared to the mean time between molecular collision, while (4), in addition, requires that the time be large compared with the time taken by a sound wave to traverse the initial disturbance.

In the remainder of this section, we indicate how one can typically obtain the functions $f(\mathbf{k})$ and $g(\mathbf{k})$. For many problems of mathematical physics, this usually presents a simple calculation. The following remarks are only meant to be formal.

Let us consider a problem which may be considered as an initial-value problem. Consider

$$\frac{\partial v}{\partial t} = Lv. \tag{5}$$

L is a linear operator and v belongs, say, to a Hilbert space (perhaps finite). The problem, then, is to solve (5), subject to specified initial data

$$v\ (t = 0) = v^0. \tag{6}$$

Further, let us assume that the problem has already been Fourier transformed, i.e., L, v, and v^0 are to be regarded as functions of $\mathbf{k}$. Using formal manipulations and inverting the transformations leads to the following representation for the solution:

$$v = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}} e^{tL} v^0 \, d\mathbf{k}. \tag{7}$$

The representation of e^{tL} itself involves a number of problems, but, generally speaking, it can be represented in terms of the spectrum of L. Therefore, a typical term which arises out of the point spectrum of L has the form

$$\frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x} + \lambda(\mathbf{k})t} \rho(\mathbf{k}) \, d\mathbf{k}. \tag{8}$$

The function $\rho(\mathbf{k})$ is partly due to the operator and partly due to the initial data. The function $\lambda(\mathbf{k})$ is an eigenvalue of L, i.e., there exists a q such that

$$Lq = \lambda q.$$

The above integral is, of course, of the form I in Eq. (1). To employ the main result it is, of course, necessary to prove λ admissible. Aside from this, it is important to note the way in which $\lambda(\mathbf{k})$ arises. Now, although λ may be quite difficult to obtain, its expansion is in practice much simpler to obtain. Formally, one writes

$$L = L_0 + kL_1 + k^2L_2 + \cdots$$

and

$$\lambda = \lambda_0 + k\lambda_1 + k^2\lambda_2 + \cdots,$$

where the L_i (known) and the λ_i are homogeneous of degree zero in $\mathbf{k}$. A number of results and methods for such perturbation series for L are discussed in the literature.[18,19]

Finally, although it is not our intention here to consider the initial-value problem in any detail, one further point is worth mentioning. This has to do with the solution to (5) and (6), say, in the form (7). Suppose there exists a discrete eigenvalue of L, λ, such that its real part for $\mathbf{k} = 0$ is greater than any other part of the spectrum of L. Clearly, then, for $t \to \infty$ the major contribution to (7) is given by (8), and by our main result this has the form (3) or (4).

3. PROOF OF THE MAIN RESULT AND ITS GENERALIZATION

We require the following lemma in our proof.

Lemma: For $\sigma(\mathbf{k})$ admissible there exists a $g_0 > 0$ and an $\epsilon_1 > 0$, such that

$$\sigma_r - \hat{\epsilon}^2 g_0 \geq 0 \tag{9}$$

for all $|\mathbf{k}| > \hat{\epsilon}$ and any $\hat{\epsilon} > 0$ such that $\hat{\epsilon} \leq \epsilon_1$.

Proof: Since g is homogeneous of degree two

$$g(\mathbf{k}) = k^2 g(\mathbf{e}),$$

with

$$\mathbf{e} = \mathbf{k}/k. \tag{10}$$

From the continuity of g and condition (v) we can also write

$$g_M \geq g(\mathbf{e}) \geq g_m > 0,$$

with g_M and g_m the maximum and minimum, respectively.

From condition (iv),

$$\sigma_r - \tfrac{1}{2}k^2 g_m > \tfrac{1}{2}k^2 g_m + O(k^3).$$

Hence, there exists an $\epsilon_0 > 0$ such that

$$\sigma_r - \tfrac{1}{2}k^2 g_m > 0, \quad |k| < \epsilon_0. \tag{11}$$

In fact, let ϵ_0 be the maximum such value.

Next from the continuity of σ_r and the dissipative condition (ii), we have that σ_r is bounded away from zero if $|\mathbf{k}|$ is bounded away from zero. Therefore, for all $k_0 > 0$, we have

$$\sigma_r \geq G(k_0) = \inf_{|\mathbf{k}|>k_0} \sigma_r(\mathbf{k}) > 0. \tag{12}$$

Then there exists an $\hat{\epsilon}$ such that

$$0 < \hat{\epsilon} < \epsilon_0$$

and

$$G(k_0) \geq \tfrac{1}{2}\hat{\epsilon}^2.$$

For, if this were not true, there would exist a point set $\{k_i\}$ such that

$$\sigma(\mathbf{k}_i) < \tfrac{1}{2}g_m\epsilon_i^2,$$

where $\{\epsilon_i\}$ is a sequence converging to zero. From (11), $|k_i| \geq \epsilon_0$ for all i. But then this contradicts (12). Denote the largest such $\hat{\epsilon} \leq \epsilon_0$ by ϵ_1. Then from (11) we have

$$\sigma_r - \tfrac{1}{2}\hat{\epsilon}^2 g_m \geq 0$$

for all $|\mathbf{k}| > \hat{\epsilon}$ and $\hat{\epsilon} \leq \epsilon_1$. Setting $g_0 = \frac{1}{2}g_m$, we have proven the lemma.

Proof of the Main Result

From condition (iv) we have

$$\lim_{|\mathbf{k}|\to 0} \left| \frac{\sigma - if - g}{k^3} \right| = c' < \infty,$$

which may be zero. In any case, we set

$$c = 1 + c'.$$

There exists an $\epsilon_2 > 0$ such that

$$|\sigma - if - g| \leq c\,|\mathbf{k}|^3 \tag{13}$$

for

$$|\mathbf{k}| \leq \epsilon_2.$$

Next, we choose

$$\epsilon_3 = \min(\epsilon_1, \epsilon_2),$$

where ϵ_1 is the same as that of the lemma. Then, for $\epsilon < \epsilon_3$ we decompose the integral (1) as follows:

$$\begin{aligned}(2\pi)^N I &= \int_{|\mathbf{k}|\geq\epsilon} e^{i\mathbf{k}\cdot\mathbf{x}-\sigma(\mathbf{k})t}\rho(\mathbf{k})\,d\mathbf{k} \\ &+ \int_{|\mathbf{k}|<\epsilon} (e^{i\mathbf{k}\cdot\mathbf{x}-\sigma(\mathbf{k})t} - e^{i\mathbf{k}\cdot\mathbf{x}-ift-gt})\rho(\mathbf{k})\,d\mathbf{k} \\ &+ \int_{|\mathbf{k}|<\epsilon} e^{i\mathbf{k}\cdot\mathbf{x}-if(\mathbf{k})t-g(\mathbf{k})t}\rho(\mathbf{k})\,d\mathbf{k} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Then, since

$$|(2\pi)^N I - I_3| \leq |I_1| + |I_2|,$$

we need only estimate the first two integrals. Using (9) of the lemma, we easily have

$$|I_1| < \int_{|\mathbf{k}|\geq\epsilon} e^{-\sigma_r t}\,|\rho|\,d\mathbf{k}$$
$$= e^{-g_0\epsilon^2 t}\int_{|\mathbf{k}|\geq\epsilon} e^{-(\sigma_r - g_0\epsilon^2)t}\,|\rho|\,d\mathbf{k} \leq Me^{-g_0\epsilon^2 t}. \quad (14)$$

Next, writing

$$I_2 = \int_{|\mathbf{k}|<\epsilon} e^{i\mathbf{k}\cdot\mathbf{x} - ift - gt}(e^{(g+if-\sigma)t} - 1)\rho\,d\mathbf{k}$$

and taking absolute values, we have

$$|I_2| < M\int_{|\mathbf{k}|<\epsilon} |e^{(g+if-\sigma)t} - 1|\,d\mathbf{k}.$$

Using a well-known inequality and Eq. (13), we obtain

$$|I_2| \leq Mc\Omega_N t\int_0^\epsilon e^{ck^3 t}k^{N+2}\,dk$$
$$= \frac{e^{c\epsilon^3 t}Mc\Omega_N t\epsilon^{N+3}}{N+3} = K\epsilon^{N+3}te^{c\epsilon^3 t}, \quad (15)$$

where Ω_N is the surface area of the unit sphere in N-dimensions.[20]

We now set

$$\epsilon = t^{-\frac{1}{2}(1-\delta)}, \quad (16)$$

where $\delta > 0$ is small and

$$t > \epsilon_0^{-2/(1-\delta)}. \quad (17)$$

With this choice of ϵ, Eq. (13) becomes

$$|I_1| < Me^{-g_0 t^\delta}$$

and (15) becomes[21]

$$|I_2| \leq Ke^{c/(t^{\frac{1}{2}} - \frac{3}{2}\delta)}/t^{\frac{1}{2}(N+1) - \frac{1}{2}\delta(N+1)} = O^*(t^{-\frac{1}{2}(N+1)}). \quad (18)$$

This proves the main result (3), since the extension of the limits of integration to ∞ in I adds an asymptotically small contribution.

It is clear from the above proof that condition (vii) immediately leads to (4). In fact, it seems that a many-term expansion of $\rho(k)$ leads to an asymptotic expansion. There would be no value in this, since, if $\rho = O(\mathbf{k})$, a simple estimate on I shows that $I_1 = O^*(t^{-\frac{1}{2}(N+1)})$, i.e., it is of the same order as already neglected terms. Therefore, if $\rho(\mathbf{k})$ satisfies (vii), Eq. (4) is obtained, i.e.,

$$I = I^0\rho_0 + O^*(t^{-\frac{1}{2}(N+1)}),$$

where I^0 is the same as defined through (4):

$$I^0(\mathbf{x}, t) = \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x} - ift - gt}\,d\mathbf{k}. \quad (19)$$

For reasons which are discussed in the next section, it is sometimes best not to use the expansion of $\rho(\mathbf{k})$ even if ρ satisfies (vii). In these cases, we can write, instead of (3),

$$I = I^0 * \rho(\mathbf{x}) + O^*(t^{-\frac{1}{2}(N+1)}), \quad (20)$$

where[22]

$$\rho(\mathbf{x}) = \int_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{x}}\rho(\mathbf{k})\,d\mathbf{k}.$$

The asterisk in the first term of (20) denotes the N-dimensional spatial convolution product.

An examination of the proof of the main result given in this section shows that it depends in no essential way on the form $e^{i\mathbf{k}\cdot\mathbf{x}}$, in which the vector $\mathbf{x}$ appears. In fact, if this exponential is replaced by any function $F(\mathbf{x}, \mathbf{k})$ which is uniformly bounded, no alteration in the proof is necessary. Hence, writing

(viii) $|F(\mathbf{x}, \mathbf{k})| < \infty$ uniformly,

we extend our main result.

Extension of the Main Result

Consider the integral

$$I' = \frac{1}{(2\pi)^N}\int_{-\infty}^{\infty} F(\mathbf{x}, \mathbf{k})e^{-\sigma(\mathbf{k})t}\rho(\mathbf{k})\,d\mathbf{k}, \quad (21)$$

with σ admissible, ρ satisfying (vi), and F satisfying (viii). Then, for large t, we have

$$I' = \int_{-\infty}^{\infty} F(\mathbf{x}, \mathbf{k})e^{-if(\mathbf{k})t - g(\mathbf{k})t}\rho(\mathbf{k})\,d\mathbf{k} + O^*(t^{-\frac{1}{2}(N+1)}). \quad (22)$$

If ρ satisfies (vii), an expression similar to (4) may also be written.

It is clear that the above generalization is quite extensive, and we have chosen to focus attention on Fourier transform type integrals (1) only because of their natural importance.

4. SPECIAL CASES AND EXTENSIONS

Case (1):

$$\sigma \sim i\boldsymbol{\alpha}\cdot\mathbf{k} + \sum_{i,j=1}^{N}\beta_{ij}k_i k_j$$

in the neighborhood of the origin: If, in addition to being admissible, we have also that σ has two continuous derivatives at the origin, then we may conclude that

$$f = \boldsymbol{\alpha} \cdot \mathbf{k},$$
$$g = \sum_{i,j=1}^{N} \beta_{ij} k_i k_j,$$

where $\boldsymbol{\alpha}$ is a real constant vector and $\boldsymbol{\beta}$ is a real, symmetric, positive-definite matrix of order N. In this case, I^0 takes the form

$$I^0 = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot(\mathbf{x}-\boldsymbol{\alpha}t) - \mathbf{k}\cdot\boldsymbol{\beta}\cdot\mathbf{k}t} \, d\mathbf{k}, \tag{23}$$

which may be integrated immediately and gives

$$I^0 = (4\pi t)^{-\frac{1}{2}N} e^{-(\mathbf{x}-\boldsymbol{\alpha}t)\cdot\boldsymbol{\beta}^{-1}\cdot(\mathbf{x}-\boldsymbol{\alpha}t)/4t} / (\det \boldsymbol{\beta})^{\frac{1}{2}}. \tag{24}$$

In the above forms, the dot product denotes the inner product in Euclidean N space.

Now, having the form (24) for I^0, we can, in this case, give a precise characterization to the region in $\mathbf{x}$ space for which the asymptotic approximation is valid. Writing, for example,

$$I = I^0 \rho_0 + O^*(t^{-\frac{1}{2}(N+1)}),$$

we clearly have that $\mathbf{x}$ must be such that

$$(\mathbf{x} - \boldsymbol{\alpha}t) \cdot \boldsymbol{\beta}^{-1} \cdot (\mathbf{x} - \boldsymbol{\alpha}t) = o(t \ln t). \tag{25}$$

Using the properties of $\boldsymbol{\beta}$, a cruder estimate is that $|\mathbf{x} - \boldsymbol{\alpha}t| = o((t \ln t)^{\frac{1}{2}})$. Outside these regions we have the estimate that $I = O^*(t^{-\frac{1}{2}(N+1)})$.

Case (2): 1-Dimensional Integrals: For $N = 1$, $\mathbf{k} = k$ and the admissibility condition (iv) is clearly equivalent to σ having two derivatives at the origin. In this case I^0 [Eq. (24)]has the form

$$I^0 (N = 1) = e^{-(x-\alpha t)^2/4\beta t} / (4\pi\beta t)^{\frac{1}{2}}. \tag{26}$$

The range of validity is still given by (25). In terms of the integral[22]

$$\mathfrak{J} = \frac{1}{2\pi} \int e^{-\sigma(k)t + ikx} \, dk,$$

we can write

$$\mathfrak{J} = I^0 + O^*(t^{-1}). \tag{27}$$

We now consider the next term in the asymptotic development of $\mathfrak{J}$ or I. To accomplish this, we assume that $\sigma(k)$ satisfies

$$(\text{iv}'){:}\ \sigma = i\alpha k + \beta k^2 - i\gamma k^3 + O(k^4)$$

instead of (iv). ($\beta > 0$ and α and γ real.) The condition (13) is now replaced by

$$|\sigma - i\alpha k - \beta k^2 + i\gamma k^3| < ck^4, \quad |k| < \epsilon_2. \tag{28}$$

Also, instead of condition (vii), we now let $\rho(k)$ be such that

$$(\text{vii}'){:}\ \rho = \rho_0 + \rho_1 k + O(k^2).$$

Then, using (iv′) and repeating an argument analogous to that given in the previous section, we can directly prove the following:

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-\alpha t) - \beta k^2 t + i\gamma k^3 t} \rho(k) \, dk + O^*(t^{-\frac{3}{2}}). \tag{29}$$

Equivalently, instead of (27) we can write

$$\begin{aligned} \mathfrak{J} &= \mathfrak{J}^0 + O^*(t^{-\frac{3}{2}}) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-\alpha t) - \beta k^2 t + i\gamma k^3 t} \, dk + O^*(t^{-\frac{3}{2}}). \end{aligned} \tag{30}$$

Finally, with the additional requirement (vii′) on $\rho(k)$, we can write

$$I = \left(\rho_0 - \frac{i}{\alpha} \rho_1 \frac{\partial}{\partial x}\right) \mathfrak{J}^0 + O^*(t^{-\frac{3}{2}}), \tag{31}$$

where $\mathfrak{J}^0$ is defined through (30). On setting

$$k = \eta - i\beta/3\gamma$$

in $\mathfrak{J}^0$, we can reduce it to a standard representation of the Airy function Ai (x), and we obtain[23]

$$\mathfrak{J}^0 = \frac{e^{\beta(x-\alpha t)/3\gamma + 2\beta^3 t/27\gamma^2}}{\pi(3\gamma t)^{\frac{1}{3}}} \operatorname{Ai}\left(\frac{x - \alpha t}{(3\gamma t)^{\frac{1}{3}}} + \frac{\beta^2 t^{\frac{2}{3}}}{(3\gamma)^{\frac{4}{3}}}\right). \tag{32}$$

This, in turn, may be expanded for t large and we obtain[24]

$$\mathfrak{J}^0 = \frac{e^{-(x-\alpha t)^2/4\beta t}}{(4\pi\beta t)^{\frac{1}{2}}} \left[1 - \frac{3(x - \alpha t)}{4\beta^2 t} + \frac{(x - \alpha t)^3}{8\beta^3 t^2} + \cdots\right]. \tag{33}$$

It is also clear that the range of validity is only marginally extended. That is, the form in (33) holds in the basically parabolic region

$$(x - \alpha t)^2 = o(t \ln t),$$

and outside this region we have the estimate $O^*(t^{-\frac{3}{2}})$.

The further development for, say, $\mathfrak{J}$ may be continued in this way. Further differentiability conditions on σ at the origin have to be assumed, and their series development substituted. It is clear that the exact evaluation of $\mathfrak{J}^0$ given by (32) was fortuitous and that the integrals in the general case cannot be expected to have known forms. However, as (33) already indicates, such an evaluation is not really necessary and a direct (second) asymptotic analysis of $\mathfrak{J}^0$ for $t \to \infty$ could have to be performed. This is also true in the general case, although we do not pursue this line of study.

The same remarks are also valid in the N-dimensional case. In general, the estimate $O^*(t^{-\frac{1}{2}(N+1)})$ may be improved upon by assuming further differentiability conditions on σ, and the development of I may be obtained. Since this is straightforward and, perhaps, of only limited value, we do not pursue it further.

Finally, we remark on the distinction between using the development of $\rho(k)$ directly in I and, on the other hand, leaving this intact and only developing $\mathfrak{J}$ in

$$I = \mathfrak{J}(x, t) * \rho(x).$$

This distinction is important and even crucial in certain problems. Referring back to the formal problem posed by (5)–(8) in Sec. 2, we recognize that, in expanding in small k, two distinct expansions are in play. There is, of course, the expansion of the underlying operator, but there is also the expansion of the data of the problem. In general, this involves two entirely different time scales. For example, in problems involving gas dynamics, the expansion of the operator is tantamount to considering times large compared to the time between molecular collisions (which is extremely small under ordinary conditions). If the initial data is also expanded, the circumstances become more involved and the time it takes a soundwave to traverse the data comes into play. This latter quantity can be quite large, and the utility of the resulting asymptotic development becomes quite limited. These remarks manifest themselves in the modulus of the error term $O^*(t^{-\frac{1}{2}(N+1)})$. The constant that is implicit in this symbol can be radically different under the two different expansions. This is already clear in (31), where the presence of ρ_1 can signal that t must be extremely large for the development to be valid. As a practical rule, one may say that only that portion of $\rho(k)$ arising from the underlying operator should be expanded and that expanding the remaining portion can badly inhibit the usefulness of the asymptotic development.

The above remarks are applicable without modification to the N-space case.

Case (3): $\sigma = \sigma(|\mathbf{k}|)$. In a number of applications (see, e.g., Refs. 12 and 13), due to the isotropy of the underlying equations, an admissible σ is a function of only $k = |\mathbf{k}|$. Although σ is not differentiable in this case, the admissibility condition immediately leads to

$$\sigma = i\alpha k + \beta k^2 + O(k^3)$$

with α real and $\beta > 0$. As shown in the previous discussion, when $N = 1$, the calculation is straightforward. This important case, however, in more than one dimension is far from trivial, and we now consider the case $N \geq 2$ in some detail.

We first note that the estimate of the error term may be greatly improved. To accomplish this, we can start with I itself, (1) or, alternatively, we may consider

$$\mathfrak{J} = \frac{1}{(2\pi)^N}\int e^{-\sigma(k)t+i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}, \tag{34}$$

since

$$I = \mathfrak{J}(\mathbf{x}, t) * \rho(\mathbf{x}).$$

The limits of integration in (34) have been purposely left out since, if convergence problems appear with infinite limits, we may take the limits of integration in (34) to be finite without loss of generality. [We have already demonstrated through (14) that the contribution from outside the neighborhood of the origin is exponentially small in time, uniformly in $\mathbf{x}$.[22]]

The integration of (34) can be carried out most easily by introducing spherical coordinates in N-space. Integrating over all angles but the polar angle yields

$$\mathfrak{J} = \frac{\Omega_{N-1}}{(2\pi)^N}\int_0^\pi d\theta e^{ikr\cos\theta}\sin^{N-2}\theta\, d\theta \int_{k\geq 0} e^{-\sigma(k)t}k^{N-1}\, dk,$$

where

$$\Omega_k = 2\pi^{\frac{1}{2}k}/\Gamma(\tfrac{1}{2}k) \tag{35}$$

is the area of unit sphere in k space. The remaining angular integration can be carried in terms of Bessel functions and yields,[25]

$$\mathfrak{J} = \frac{r^{1-\frac{1}{2}N}}{(2\pi)^{\frac{1}{2}N}}\int_{k\geq 0} J_{\frac{1}{2}N-1}(kr)k^{\frac{1}{2}N}e^{-\sigma(k)t}\, dk.$$

We now focus attention on

$$\mathfrak{Z} = r^{\frac{1}{2}N-1}\mathfrak{J} = (2\pi)^{-\frac{1}{2}N}\int_{k\leq 0} J_{\frac{1}{2}N-1}(kr)k^{\frac{1}{2}N}e^{-\sigma(k)t}\, dk. \tag{36}$$

As mentioned before,[22] the upper limit of integration may be taken to be finite if convergence difficulties appear with infinite limits of integration.

Since

$$|J_\nu(x)| \leq 1, \quad \nu \geq 0,$$

the integral $\mathfrak{Z}$ clearly falls under the hypothesis (viii) of the extension of the main result and we may apply (22). The modification due to the presence of $k^{\frac{1}{2}N}$ is, of course, of no consequence. Therefore, writing

$$\mathfrak{Z}_0 = (2\pi)^{-\frac{1}{2}N}\int_0^\infty J_{\frac{1}{2}N-1}(kr)e^{-i\alpha kt-\beta k^2 t}k^{\frac{1}{2}N}\, dk \tag{37}$$

and using the arguments leading to (18), we can easily show that

$$\mathfrak{Z} = \mathfrak{Z}_0 + O^*(t^{-\frac{1}{2}N-1}). \tag{38}$$

As preparation for the evaluation of $\mathfrak{J}_0$, we first express the Hankel expansion of the Bessel function[26]:

$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[\cos\left(x - \tfrac{1}{4}(2\nu+1)\pi\right) \times \left(\sum_{m=0}^{P}(-1)^m(\nu, 2m)(2x)^{-2m} + O(|x|^{-2P-2})\right) + \sin\left(x - \tfrac{1}{4}(2\nu+1)\pi\right) \times \left(\sum_{m=0}^{Q}(-1)^m(\nu, 2m+1)(2x)^{-2m-1} + O(|x|^{-2Q-3})\right)\right]. \quad (39)$$

If ν is of half-odd-integer order ($N = 3, 5, \cdots$), these series are known to terminate with

$$P = [\tfrac{1}{4}(2\nu - 1)] \geq 0, \qquad Q = [\tfrac{1}{4}(2\nu - 3)] \geq 0, \quad (40)$$

i.e., with the limits (40), the finite expansions in (39) are exact. In this case, N odd, the integration may be carried out explicitly and, in fact, if $N = 3$, then

$$\mathfrak{J}_0\,(N=3) = [16r^{\frac{1}{2}}(\beta t\pi)^{\frac{3}{2}}]^{-1} \times [(r+\alpha t)e^{-(r+\alpha t)^2/4\beta t}\operatorname{erfc}(i(r+\alpha t)/2(\beta t)^{\frac{1}{2}}) + (r-\alpha t)e^{-(r-\alpha t)^2/4\beta t}\operatorname{erfc}(i(\alpha t - r)/2(\beta t)^{\frac{1}{2}})]. \quad (41)$$

The argument in the first expression of the bracket is large, and, on performing the required asymptotic expansion, we find

$$\mathfrak{J}_0\,(N=3) = \frac{r - \alpha t}{16r^{\frac{1}{2}}(\beta t\pi)^{\frac{3}{2}}}\, e^{-(r-\alpha t)^2/4\beta t}\operatorname{erfc}\left(\frac{i(\alpha t - r)}{2(\beta t)^{\frac{1}{2}}}\right) + O(t^{-\frac{5}{2}}). \quad (42)$$

[In this last expression, r should be regarded as being $r \geq O(t)$. For r small the entire expression (36) will be shown below to be of negligible order.]

The general case for N odd may be obtained, but we do not give it, since it is tedious to express and, as we will shortly see, it carries already neglected orders. For N even, no explicit integration seems to be available.[27] At this point of the analysis, we abandon the search for an explicit calculation of (37), and perform a second asymptotic analysis. As will be seen, this is at no expense to the $O^*(t^{-\frac{1}{2}N-1})$ estimate, and we find an explicit calculation independently of the dimension N.

The asymptotic analysis of $\mathfrak{J}_0$ (37), under the condition $r = o(t)$, is fairly straightforward and we merely quote the result:

$$\mathfrak{J} \sim \frac{(-i)^N\Gamma(N-1)}{\Gamma(\frac{1}{2}N)\pi^{\frac{1}{2}N}2^{N-1}\alpha^N}\frac{r^{\frac{1}{2}N-1}}{t^N}, \quad r = o(t).$$

It is, therefore, clear that, for all $N \geq 2$, this is already small, compared with neglected terms. In what follows, therefore, we may restrict attention to

$$r \geq O(t).$$

Using this and returning to $\mathfrak{J}$ in (34) and (36), we see the superiority of the estimate (38) over the estimate given by (4).

As a first step in our evaluation of $\mathfrak{J}_0$, we demonstrate that

$$\hat{\mathfrak{J}}_0 = \frac{1}{(2\pi)^{\frac{1}{2}N}}\int_0^{O(t^{-p})} J_{\frac{1}{2}N-1}(kr)k^{\frac{1}{2}N}e^{-i\alpha tk-\beta k^2 t}\,dk,$$

with p such that $O(t^{-p}) = O^*(t^{-1})$ is of an already neglected order. To avoid carrying unimportant constants in our estimates, we consider instead $O(\hat{\mathfrak{J}}_0)$. Then, clearly,

$$O(\hat{\mathfrak{J}}_0) \leq \int_0^{t^{-p}} J_{\frac{1}{2}N-1}(kr)k^{\frac{1}{2}N}\,dk.$$

The integral on the right may be explicitly evaluated[28]:

$$O(\hat{\mathfrak{J}}_0) \leq r^{-\frac{1}{2}N-1}(r/t^p)^{\frac{1}{2}N}J_{\frac{1}{2}N}(r/t^p).$$

In view of the fact that $r \geq O(t)$ and $p < 1$, we can asymptotically evaluate the Bessel function and find

$$O(\hat{\mathfrak{J}}_0) \leq (r^{1+\frac{1}{2}}t^{\frac{1}{2}p(N-1)})^{-1},$$

which is, clearly, of an already neglected order. We next consider

$$\mathfrak{J}_0 - \hat{\mathfrak{J}}_0 = \frac{1}{(2\pi)^{\frac{1}{2}N}}\int_{O(t^{-p})}^{\infty} J_{\frac{1}{2}N-1}(kr)k^{\frac{1}{2}N}e^{-i\alpha kt-\beta k^2 t}\,dk.$$

From the limits of integration and the condition on r, kr is large, and we therefore write

$$J_{\frac{1}{2}N-1}(kr) = (2/\pi kr)^{\frac{1}{2}}\cos\left(kr - \tfrac{1}{4}(N-1)\pi\right) + O(|kr|^{-\frac{3}{2}}).$$

Hence, we consider

$$A = \frac{1}{(2\pi)^{\frac{1}{2}N}}\int_{O(t-p)}^{\infty}\left[J_{\frac{1}{2}N-1}(kr) - \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\cos\left(kr - \tfrac{1}{4}(N-1)\pi\right)\right]k^{\frac{1}{2}N}e^{-i\alpha kt-\beta k^2 t}\,dk.$$

Proceeding as before, we have

$$O(A) \leq \frac{1}{r^{\frac{3}{2}}}\int_{t^{-p}}^{\infty}k^{\frac{1}{2}(N-3)}e^{-\beta tk^2}\,dk \leq \frac{1}{r^{\frac{3}{2}}t^{\frac{1}{4}(N-1)}} \leq O(t^{-\frac{1}{4}N-\frac{5}{4}}),$$

which is also of a neglected order. Finally, it only remains for us to consider

$$B = \frac{1}{(2\pi)^{\frac{1}{2}N}} \int_0^{O(t^{-p})} k^{\frac{1}{2}N} \left(\frac{2}{\pi k r}\right)^{\frac{1}{2}} \times \cos(kr - \tfrac{1}{4}(N-1)\pi) e^{-i\alpha k t - \beta k^2 t} dk,$$

from which we directly obtain

$$O(B) \le [r^{\frac{1}{2}} t^{\frac{1}{2}p(N+1)}]^{-1}$$

and which again is of negligible order. We have, therefore, demonstrated that

$$\mathfrak{Z}_0 = \frac{2}{r^{\frac{1}{2}}(2\pi)^{\frac{1}{2}(N+1)}} \int_0^\infty k^{\frac{1}{2}(N-1)} e^{-\beta k^2 t - i\alpha k t} \times \cos(kr - \tfrac{1}{4}(N-1)\pi)\, dk + O^*(t^{-\frac{1}{2}N-1}). \quad (43)$$

The resulting integral may now be carried out in terms of confluent hypergeometric functions[29] and, due to their special form in our case, these may in turn be written as parabolic cylinder functions.[30] Choosing these latter forms, we find that

$$\mathfrak{Z}_0 = \frac{\Gamma(\frac{1}{2}(N+1))\Gamma(\frac{1}{4}(N+3))2^{\frac{1}{4}(N+1)}}{r^{\frac{1}{2}}(2\pi)^{\frac{1}{2}(N+1)} 2(\beta t)^{\frac{1}{4}(N+1)}\Gamma(\frac{1}{2})} \times \left[e^{-\frac{1}{4}i(N-1)\pi - (r-\alpha t)^2/8\beta t} D_{-\frac{1}{2}(N+1)}\left(\frac{i(\alpha t - r)}{(2\beta t)^{\frac{1}{2}}}\right) + e^{\frac{1}{4}i(N-1)\pi - (r+\alpha t)^2/8\beta t} D_{-\frac{1}{2}(N+1)}\left(\frac{i(r+\alpha t)}{(2\beta t)^{\frac{1}{2}}}\right)\right] + O^*(t^{-\frac{1}{2}N-1}). \quad (44)$$

If N is set equal to 3 in (44), we get our previous result (42). Noting that the argument of the second term is large, and using the asymptotic estimate[31]

$$D_\nu(Z) = e^{-\frac{1}{4}Z^2} Z^\nu (1 + O(Z^{-2})), \quad |\arg Z| \le \tfrac{3}{4}\pi,$$

we conclude that this term is of negligible order. Therefore, we finally have

$$\mathfrak{Z} = (r^{\frac{1}{2}N-1})\mathfrak{J} = \frac{\Gamma(\frac{1}{2}(N+1))\Gamma(\frac{1}{4}(N+3)) e^{-\frac{1}{4}i(N-1)\pi - (r-\alpha t)^2/8\beta t}}{2\Gamma(\frac{1}{2}) r^{\frac{1}{2}} (2\pi^2 \beta t)^{\frac{1}{4}(N+1)}} \times D_{-\frac{1}{2}(N+1)}\left(\frac{i(\alpha t - r)}{(2\beta t)^{\frac{1}{2}}}\right) + O^*(t^{-\frac{1}{2}N-1}), \quad (45)$$

where we have used (38) to replace $\mathfrak{Z}_0$ by $\mathfrak{Z}$. In the interest of completeness, we note that, if N is odd, $\frac{1}{2}(N+1)$ is integer and the following formula is of the value[32] (n integer)

$$D_{-n-1}(Z) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{(-1)^n}{n!} e^{-\frac{1}{4}Z^2} \frac{d^n}{dZ^n}\left[e^{\frac{1}{2}Z^2} \operatorname{erfc}\left(\frac{Z}{\sqrt{2}}\right)\right].$$

For N even, the following may be useful[32]:

$$D_{-\frac{1}{2}}(Z) = (2\pi)^{-\frac{1}{2}} Z^{\frac{3}{2}} [K_{\frac{3}{4}}(\tfrac{1}{4}Z^2) - K_{\frac{1}{4}}(\tfrac{1}{4}Z^2)]$$

and

$$\nu D_{\nu-1}(Z) = e^{-\frac{1}{4}Z^2} \frac{d}{dZ}[e^{\frac{1}{4}Z^2} D_\nu(Z)],$$

where the K_μ refer to the modified Bessel functions.

We once again note from (45) that the estimate (38) is superior to (3) and (4) (except in the case $N = 2$, when it gives the same result). Also, note that the range of validity may be obtained from (45). Without going into details, we further note that, since $D_\nu = O(1)$ in the neighborhood of the origin, (45) is valid for at least

$$|r - \alpha t| = O(t^{\frac{1}{2}}).$$

Before ending this section, we add a cautionary example. Consider the following integral:

$$f(t) = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-t(1-e^{ik})}\, dk.$$

In this case, $\sigma = 1 - e^{ik}$ in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ certainly satisfies all the admissibility conditions. Then, applying the main result, we obtain

$$f(t) = (2\pi/t)^{\frac{1}{2}} e^{-\frac{1}{2}t} + O^*(t^{-1}).$$

Hence, the result of the asymptotic analysis is less than the error estimate. This signals the failure of the main result for this integral, as it should, since standard methods show that $f(t) \sim 2\pi i e^{-t}$.

5. APPLICATIONS

We consider three applications in the following. These have been chosen to demonstrate the range of the main result, rather than for their physical importance. Applications to a number of specific physical problems have already been cited.[10–17]

Although in each problem below a mathematically rigorous analysis may be given, our discussion is only meant to be formal.

Problem 1: Consider the following initial-value problem[33]:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 - \mu\nabla^2 \frac{\partial}{\partial t}\right) s(\mathbf{x}, t) = 0,$$

$$s(t = 0) = \delta(\mathbf{x}), \quad \frac{\partial s(t=0)}{\partial t} = \mu \nabla^2 \delta(\mathbf{x}), \quad (46)$$

with the constant $\mu > 0$. The partial differential equation (46) is probably the simplest one demonstrating wave propagation and diffusion. Introducing the Fourier transform

$$s(\mathbf{k}, t) = \int_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{x}} s(\mathbf{x}, t)\, d\mathbf{x},$$

we see that the solution to (46) is

$$s(\mathbf{x}, t) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}}(e^{\sigma+t} + e^{\sigma-t})\, dk,$$

where

$$\sigma^{\pm} = \tfrac{1}{2}[-\mu k^2 \pm (\mu^2 k^4 - 4k^2)^{\frac{1}{2}}].$$

Both σ^+ and σ^- satisfy the conditions of admissibility and, at the origin,

$$\sigma^{\pm} = \pm ik - \tfrac{1}{2}\mu k^2 + O(k^3).$$

The main result is, therefore, applicable. In particular, the results of Case (2) of the previous section apply. On using (45), we have, for $N \geq 2$,

$$s(\mathbf{x}, t) = \frac{\Gamma(\frac{1}{4}(N+1))\Gamma(\frac{1}{4}(N+3))e^{-\frac{1}{4}i(N-1)\pi-(r-t)^2/4\mu t}}{2r^{\frac{1}{2}(N+1)}\Gamma(\frac{1}{2})(\pi^2\mu t)^{\frac{1}{4}(N+1)}} \times D_{-\frac{1}{2}(N+1)}\left(\frac{i(t-r)}{(\mu t)^{\frac{1}{2}}}\right) + O^*(t^{-\frac{3}{4}N}),$$

and from (26), for $N = 1$,

$$s(x, t) = e^{-(x-t)^2/2\mu t}/(2\pi\mu t)^{\frac{1}{2}} + e^{-(x+t)^2/2\mu t}/(2\pi\mu t)^{\frac{1}{2}} + O^*(t^{-1}).$$

Problem 2: Consider the following transport equation[34,35]:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\xi}\cdot\nabla + \nu(\xi)\right)f = \nu(\xi)\left(\int(2\pi)^{-\frac{3}{2}}e^{-\frac{1}{2}\xi^2}f\nu\, d\xi\right)\Big/\left(\int(2\pi)^{-\frac{3}{2}}e^{-\frac{1}{2}\xi^2}\nu\, d\xi\right) = (\nu/\nu_1)\hat{\mathbf{p}}(\mathbf{x}, t) = Kf. \tag{47}$$

The collision frequency ν is a positive, monotonically increasing function of the magnitude of the molecular velocity $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$. We attempt to solve (47) in an unbounded domain and subject to the initial data

$$f(t = 0) = \delta(\mathbf{x}). \tag{48}$$

We first consider the Fourier-transformed problem

$$\left(\frac{\partial}{\partial t} + i\mathbf{k}\cdot\boldsymbol{\xi} + \nu - K\right)f = 0,$$
$$f(t = 0) = 1. \tag{49}$$

Next, writing

$$L = -i\mathbf{k}\cdot\boldsymbol{\xi} - \nu + K,$$

we write the Eq. (49) as

$$\frac{df}{dt} = Lf.$$

Following the formalism given in Sec. 2, the solution to Eqs. (49) is

$$f(\mathbf{k}, \boldsymbol{\xi}, t) = e^{tL}. \tag{50}$$

The operator $\nu^{-1}K$ is clearly a projector and, hence, $\nu - K$ is nonpositive. Further, we may prove that L has just one eigenvalue $\lambda(\mathbf{k})$ and that it satisfies all the admissibility conditions. (For $\mathbf{k}$ sufficiently large this eigenvalue may disappear.) In addition, the operator L has a continuous spectrum which covers a 2-dimensional region to the left of

$$\mathrm{Re}\,\sigma = \nu(0)$$

in the complex σ plane.[35] (For ν constant, this region degenerates to a single line $\mathrm{Re}\,\sigma = -\nu$.) Denoting this region by $C(k)$ and an element of area in the complex σ plane by ds, we can write (50) in the form

$$f(\mathbf{k}, \boldsymbol{\xi}, t) = e^{\lambda(\mathbf{k})t}g_0(\xi, k) + \int_{C(\mathbf{k})} e^{\sigma t}g(\sigma, \boldsymbol{\xi}, \mathbf{k})\, ds, \tag{51}$$

where the eigenfunction g_0 and the "improper eigenfunction" g are still to be determined. The eigenvalue determination leads to

$$(\lambda + i\mathbf{k}\cdot\boldsymbol{\xi} + \nu)g_0 = Kg_0.$$

On using the perturbation analysis outlined in Sec. 2, we easily find

$$\lambda = -\alpha k^2 + O(k^4),$$

with

$$\alpha = \frac{1}{3}\left(\int e^{-\frac{1}{2}\xi^2}\nu^{-2}\, d\boldsymbol{\xi}\right)\Big/\left(\int e^{-\frac{1}{2}\xi^2}\nu^{-1}\, d\boldsymbol{\xi}\right)$$

and

$$g_0 = \beta + O(k), \quad \beta = \text{const.}$$

Since $C(k)$ lies to the left of $\mathrm{Re}\,\sigma = -\nu(0)$, the contribution from the continuous spectrum is asymptotically small when compared with the point spectrum contribution. Therefore, for large times, we may neglect the integral term on the right of (51). On inverting the Fourier transforms and making use of the main result, we find

$$f(\mathbf{x}, \boldsymbol{\xi}, t) \sim \frac{\beta}{(2\pi)^3}\int_{-\infty}^{\infty} e^{-\alpha k^2 t + i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}$$

and, from the evaluation given in (24),

$$f(\mathbf{x}, \boldsymbol{\xi}, t) \sim \beta e^{-r^2/4\alpha t}/(4\alpha\pi t)^{\frac{3}{2}}.$$

Finally, to calculate the constant β, we note that (47) leads to the continuity equation and, hence, the total number of particles at the initial instant is conserved. Carrying out the required integration, we find that

$\beta = 1$ and, therefore,

$$f(\mathbf{x}, \boldsymbol{\xi}, t) \sim e^{-r^2/4\alpha t}/(4\alpha\pi t)^{\frac{3}{2}}.$$

Problem 3: Consider the following 3 problems:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}, \quad w\,(t = 0) = \delta(x), \qquad \text{(52a)}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) w = \delta(t)\delta(x), \qquad \text{(52b)}$$

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial t} = 0, \quad \frac{\partial w}{\partial t}(t = 0) = \delta(x),$$
$$w\,(t = 0) = 0. \qquad \text{(52c)}$$

Using transform techniques, we can easily analyze each of these and make them fall under the hypothesis leading to the main result. In fact, for $t \to \infty$, each problem leads to the same asymptotic result:

$$w = e^{-x^2 4t}/(4\pi t)^{\frac{1}{2}} + O^*(t^{-1}).$$

[Problem (52b), of course, should be considered in the complete (x, t) plane; for $t < 0$, however, the solution is exponentially small.]

Each of the problems (52) can, of course, be exactly solved; however, this is not the point. Equations (52) represent the three basic types of partial differential equations of second order. It is, of course, amusing that all three have the same asymptotic solution, but of more importance is the fact that the main result and the methods associated with it can be used independently of the type of partial differential equation.

ACKNOWLEDGMENT

The final version of the manuscript was prepared while the author was a Visiting Professor at the Institue Henri Poincaré of the University of Paris. He wishes to take this opportunity to express his gratitude for the invitation to spend the year at the Institute and for the gracious hospitality received during that year.

* This work was supported by the U.S. Office of Naval Research under Contract 562(39).

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[20] See Eq. (35) in Sec. 4. K just represents a constant.

[21] Henceforth, $\delta > 0$ represents a small quantity, whose definition may change from equation to equation.

[22] The presence of infinite limits is of no importance, since we have already demonstrated that the contribution exterior to the neighborhood of the origin is asymptotically small. We can, in fact, consider ρ as having compact support at this point.

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[29] Reference 26, pp. 403 and 408.

[30] Reference 26, p. 324.

[31] Reference 26, p. 331.

[32] Reference 26, p. 326.

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