[2] J. H. Konrad, Ph. D. thesis, California Institute of Technology, Pasadena, CA, 1976.
[3] L. P. Bernal and A. Roshko, J. Fluid Mech. **170**, 499 (1986).
[4] S. J. Lin and G. M. Corcos, J. Fluid Mech. **141**, 139 (1984).
[5] J. C. Lasheras and H. Choi, J. Fluid Mech. **189**, 53 (1988).
[6] K. J. Nygaard and A. Glezer, *Advances in Turbulence 2*, edited by H.-H. Fernholz and H. E. Fiedler (Springer, Berlin, 1989), p. 461.
[7] L.-S. Huang and C.-M. Ho, J. Fluid Mech. **210**, 475 (1990).
[8] R. T. Pierrehumbert and S. E. Widnall, J. Fluid Mech. **114**, 59 (1982).
[9] C. Chandrsuda, R. D. Mehta, A. D. Weir, and P. Bradshaw, J. Fluid Mech. **85**, 693 (1978).
[10] F. K. Browand and T. R. Troutt, J. Fluid Mech. **158**, 489 (1985).
[11] F. K. Browand and S. Prost-Domasky, *New Trends in Nonlinear Dynamics and Patterning Phenomena: The Geometry of Nonequilibrium*, NATO ASI Series 8, edited by P. Coullet and P. Huerre (Plenum, New York, in press).
[12] K. J. Nygaard, MS thesis, University of Arizona, 1987.
[13] H. E. Fiedler, A. Glezer, and I. J. Wygnanski, *Current Trends in Turbulence Research*, Progress in Astronautics and Aeronautics, edited by H. Branover, M. Mond, and Y. Unger (AIAA, Washington, DC, 1988), Vol. 112, p. 30.
[14] N. Didden, Ph.D. thesis, University of Göttingen, 1977.

Empirical and Stokes eigenfunctions and the far-dissipative turbulent spectrum

C. Foias
Department of Mathematics, Indiana University, Bloomington, Indiana 47405

O. Manley
Division of Engineering and Geosciences, U.S. Department of Energy, Office of Basic Energy Sciences, Washington, DC, 20545

L. Sirovich
Center for Fluid Mechanics, Turbulence, and Computation and the Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

(Received 15 September 1989; accepted 6 November 1989)

It is shown that the Stokes eigenfunctions and their corresponding spectra, frequently used in mathematical investigations of the Navier–Stokes equations, provide estimates on the spectrum of the two-point spatial covariance tensor. This, in turn, is used to estimate the far-dissipative turbulent spectrum. An exponential falloff is predicted and evidence given which implies that this is a sharp estimate.

Two widely different eigenfunction approaches have been used in recent years for treating the Navier–Stokes (NS) equations. On the one hand a considerable body of results in the mathematical literature[1–3] is based on the *Stokes eigenfunctions* defined by

$$\nabla\cdot\boldsymbol{\phi} = 0, \qquad \nabla^2\boldsymbol{\phi} = -\lambda\boldsymbol{\phi} + \nabla q, \tag{1}$$

with appropriate side conditions, where q plays the role of pressure.

On the other hand a basis set known as the empirical eigenfunctions, first introduced by Lumley[4] for the study of coherent structures, has proven useful in considering turbulent flows.[5–7] To define these suppose that $\mathbf{u}(\mathbf{x},t)$ satisfies the NS equations

$$\nabla\cdot u = 0, \qquad \frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}, \tag{2}$$

with appropriate boundary conditions. For statistically stationary flows we can define the two-point correlation tensor

$$\mathbf{K}(\mathbf{x},\mathbf{x}') = \langle \delta\mathbf{u}(\mathbf{x}) \otimes \delta\mathbf{u}(\mathbf{x}')\rangle, \tag{3}$$

where $\delta\mathbf{u}$ represents departure from the mean and the brackets indicate a suitable ensemble average. The empirical eigenfunctions are defined by

$$\mathbf{K}\boldsymbol{\psi} = \int \mathbf{K}(\mathbf{x},\mathbf{x}')\boldsymbol{\psi}(\mathbf{x}')d\mathbf{x}' = \mu\boldsymbol{\psi}(\mathbf{x}). \tag{4}$$

As an operator $\mathbf{K}$ is Hermitian, non-negative, and in certain circumstances may be shown to be square integrable. (On physical grounds one may suppose $\mathbf{K}$ to be square integrable.) It then follows from Mercer's theorem[8] that $\{\boldsymbol{\psi}_n\}$ form a complete set.

In the event that the convective term of (2) vanishes identically the two systems of eigenfunctions are identical.[9] But in general the two systems are quite different. In fact, the Stokes system may be thought of as arising from (2) for $\mathrm{Re}\downarrow 0$ while the empirical eigenfunctions are associated with the opposite limit, $\mathrm{Re}\uparrow\infty$, i.e., when the flow is turbulent.

In this Brief Communication we relate the two systems; and, as a result of sharp estimates on the spectrum of the Stokes case,[10] we will be able to produce correspondingly sharp results for the spectrum of the correlation operator (3) for which, up until now, only numerical results have been obtained.

A direct application of these results is to the dissipative spectrum of turbulent flows. If η is the Kolmogorov microscale[11] and $\kappa = 1/\eta$ the corresponding wavenumber, then for $k \gg \kappa$ we conclude that the spectrum is given by

$$E(k) = O[\exp(-ck\eta)], \tag{5}$$

 0899-8213/90/030464-04$02.00

where c is a small constant depending weakly on Re. A variety of *ad hoc* arguments have been given which lead to a more rapid decay of the spectrum in this limit. Based on the present investigation we believe these to be inaccurate. Some experimental data are available for comparison. The study by Sreenivasan[12] addresses this issue and leads to a conclusion that agrees with ours. Numerical evidence is also scarce. One exception is a model problem mentioned later which also is in agreement with (5). We believe (5) to be a sharp estimate and now proceed with the formal analysis.

We use $\mathbf{u}$ to denote the flow fluctuation itself so that

$$\langle \mathbf{u} \rangle = \lim_{T \uparrow \infty} \frac{1}{T} \int_0^T \mathbf{u}\, dt = 0. \tag{6}$$

From Mercer's theorem we have the spectral representation

$$\mathbf{K}(\mathbf{x},\mathbf{x}') = \sum_{n=1} \mu_n \boldsymbol{\psi}_n(\mathbf{x}) \otimes \boldsymbol{\psi}_n(\mathbf{x}'), \quad \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant 0, \tag{7}$$

where μ_n may be interpreted as being the mean energy of the flow in the $\boldsymbol{\psi}_n$ direction. As stated earlier, completeness follows if $\mathbf{K}$ is square integrable; $\mathbf{K}$ may be shown to be smooth in two dimensions[13] and we assume this property in three dimensions. If we represent the formal Stokes operator by A then

$$A\,\mathbf{K}A = \sum \mu_n A\boldsymbol{\psi}_n \otimes A\boldsymbol{\psi}_n. \tag{8}$$

(Since the empirical eigenfunctions are incompressible, $\nabla\cdot\boldsymbol{\psi}_n = 0$, it follows that we may take $A = -\nabla^2$.) On taking the trace it follows that

$$\infty > \langle |A\mathbf{u}|^2 \rangle = \mathrm{Tr}(A^2\mathbf{K}) = \mathrm{Tr}(A\,\mathbf{K}A) = \sum_n \mu_n |A\boldsymbol{\psi}_n|^2. \tag{9}$$

Here $|\cdot|$ denotes the usual norm in $L^2(\Omega)^D$.

Next we denote the eigenvalues of the Stokes operator by λ_k,

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \tag{10}$$

and recast (9) into the following form,

$$\begin{aligned} \mathrm{Tr}(A\,\mathbf{K}A) &= (\mu_1 - \mu_2)|A\boldsymbol{\psi}_1|^2 + (\mu_2 - \mu_3)(|A\boldsymbol{\psi}_1|^2 + |A\boldsymbol{\psi}_2|^2) + (\mu_3 - \mu_4)(|A\boldsymbol{\psi}_1|^2 + |A\boldsymbol{\psi}_2|^2 + |A\boldsymbol{\psi}_3|^2) + \cdots \\ &\geqslant (\mu_1 - \mu_2)\lambda_1^2 + (\mu_2 - \mu_3)(\lambda_1^2 + \lambda_2^2) + (\mu_3 - \mu_4)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + \cdots \\ &= \mu_1\lambda_1^2 + \mu_2\lambda_2^2 + \mu_3\lambda_3^2 + \cdots. \end{aligned} \tag{11}$$

The inequality follows from the property that for an orthonormal system $\{\boldsymbol{\psi}_m\}^n$ of n elements the sum

$$\sum_{m=1}^{n} |A\boldsymbol{\psi}_m|^2$$

achieves its minimum on the eigenvectors of the first n eigenvalues of A, ordered in a nondecreasing sequence.[14]

As is well known,

$$\lambda_n = O(n^{2/D}), \tag{12}$$

in D-dimensional space $\Re^D$. In all cases $\mathrm{Tr}(A\,\mathbf{K}A)$ is bounded so that $\mu_n \downarrow 0$. In particular, on taking (9) into account we conclude from (11) and (12) that

$$\mu_n = O(n^{-4/D}) \quad \text{in } \Re^D.$$

Since the range of $\mathbf{K}$ is in C^∞, A can be applied to it repeatedly leading to the conclusion that, asymptotically, μ_n decreases faster than any inverse power of n. In three dimensions the index beyond which this rapid decay occurs can be estimated by $n = O(\mathrm{Re}^{9/4})$, which is Landau's[15,16] estimate for the number of modes needed before reaching the dissipative range. This is conservative since the number of empirical eigenfunctions is optimal.[17]

This heuristic result is further supported by the recent proof of the Gevrey class regularity for solutions of the Navier–Stokes equations.[10] They show that a solution $\mathbf{u}$ satisfies

$$|A \exp[c(\eta^2 A)^{1/2}]\mathbf{u}| < b, \tag{13}$$

where the constant $c > 0$ is $O(1)$ and b is a constant (in the case of two dimensions it is also independent of the solution).

From (13) it follows that

$$\begin{aligned} b^2 &\geqslant \sup\langle |A \exp[c(\eta^2 A)^{1/2}]|^2 \rangle \\ &\geqslant \mathrm{Tr}\{A^2 \exp[2c(\eta^2 A)^{1/2}]K\} \\ &\geqslant \sum \mu_n \lambda_n^2 \exp[2c(\eta^2\lambda_n)^{1/2}], \end{aligned} \tag{14}$$

where the same manipulations as were used in (11) have been applied. From this it follows that

$$\mu_n = o[\lambda_n^{-2} \exp(-2c\eta\lambda_n^{1/2})], \tag{15}$$

and hence falls off more rapidly than any inverse power of n.

We can now clarify the need for relatively few modes, $\{\boldsymbol{\psi}_n\}$, to represent $\mathbf{K}$ in the following way. Namely, we can estimate the fraction of time that $\mathbf{u}$ spends outside the slab

$$S_n = \left\{\mathbf{u}: \sum_{j=n}^{\infty} (\mathbf{u},\boldsymbol{\psi}_j)^2 \leqslant \epsilon\right\}, \tag{16}$$

with

$$\epsilon = \mu_n^{1/4}\left(\sum_{j=n}^{\infty} \lambda_j^{-2} \exp[-2c(\eta^2\lambda_j)^{1/2}]\right)^{1/4}.$$

Note that the slab is in the infinite-dimensional function space spanned by $\{\boldsymbol{\psi}_n\}$. Toward that end consider the function

$$\chi_n(\mathbf{u}) = \begin{cases} 0, & \text{if } \mathbf{u} \in S_n, \\ 1, & \text{otherwise.} \end{cases}$$

We have

$$\chi_n^* = \lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_n[\mathbf{u}(t)]dt = \int \chi_n(\mathbf{u})d\mu(\mathbf{u}), \tag{17}$$

where μ is an adequate statistical solution of the Navier–Stokes equations, in the w^* closure of the time averages of the Dirac measures $\delta_{\mathbf{u}(t)}$ on $L^2(\Omega)^D$. (In two dimensions this is precisely the invariant measure.) The relations (17) follow from the ergodic property (in a weak sense) of the statistical solutions of the Navier–Stokes equations, and moreover,[18]

$$\langle \mathbf{u} \rangle = \int \mathbf{u}\, d\mu(\mathbf{u}), \quad \mathbf{K} = \int \mathbf{u} \otimes \mathbf{u}\, d\mu(\mathbf{u}). \tag{18}$$

There, as here, the statistical solution depends on the solution $\mathbf{u}$ and the limits in (16) and should be taken in a generalized sense.

Clearly χ_n^* is the fraction of the time that $\mathbf{u}(t)$ is on the outside Ω_n of the slab S_n in the function space $L^2(\Omega)^D$.

Now we have

$$\chi_n^* \epsilon = \mu(\Omega_n)\epsilon \leqslant \int \sum_{j=n}^{\infty} (\mathbf{u},\psi_j)^2\, d\mu(\mathbf{u})$$

$$= \sum_{j=n}^{\infty} (\mathbf{K}\psi_j,\psi_j) = \sum_{j=n}^{\infty} \mu_j$$

and thus

$$\chi_n^* \leqslant \frac{1}{\epsilon} \left(\sum_{j=n}^{\infty} \mu_j \lambda_j^{-2} \exp[-2c(\eta^2\lambda_j)^{1/2}] \right)^{1/2}$$

$$\times \left(\sum_{j=n}^{\infty} \mu_j \lambda_j^2 \exp[2c(\eta^2\lambda_j)^{1/2}] \right)^{1/2}$$

$$\leqslant \mu_n^{1/4} \left(\sum_{j=n}^{\infty} \lambda_j^{-2} \exp[-2c(\eta^2\lambda_j)^{1/2}] \right)^{1/4} b. \tag{19}$$

Therefore if either the average energy μ_n or $\Sigma_{j=n}^{\infty} \lambda_j^{-2} \exp[-2c(\eta^2\lambda_j)^{1/4}]$ is very small we can assume that $\mathbf{u}$ actually lives on the linear manifold

$\mathbf{K}\psi_1 + \cdots + \mathbf{K}\psi_{n-1}$.

The deliberations presented above can now be applied to estimating the energy spectrum for scales that are small compared with the Kolmogorov microscale, η. To accomplish this we first observe that since μ_n is the average energy in the nth mode and therefore represents the energy spectrum as a function of the index n. [In dimensional units, μ_n is $\ell^5 t^{-2}$, and thus has the same units as the three-dimensional spectral energy density $E(k)$.] A second observation of importance is that as the index tends toward infinity, the empirical eigenfunction become sinusoidal, as do the Stokes eigenfunctions, at least locally. This follows from the fact that, generally, as $n \uparrow \infty$, WKB theory applies and hence leads to local trigonometric forms for the eigenfunctions.[19,20] Furthermore, in the present instance, given the damping effect of the dissipation on the fine structure of the flow, for $n \uparrow \infty$ the convective term in the NS equations diminishes in importance. Thus by an earlier remark, the Stokes and empirical eigenfunctions approach one another. The former, except for boundary layers, clearly tend to sinusoids.

It therefore follows that for $n \uparrow \infty$ we can transform from the index n to the wavenumber

$$k = \sqrt{k_1^2 + k_2^2 + \cdots},$$

and instead of (12) we have

$$\lambda_n \sim k^2, \tag{20}$$

which holds in any number of dimensions. Then, writing $E(k)$ instead of $\mu_{n(k)}$, we obtain from (15)

$$E(k) = o[\exp(-c\eta k)] \tag{21}$$

in any number of dimensions. Of course in two dimensions η is to be interpreted as the Kraichnan[21] cutoff. In writing (21) we are ignoring unimportant algebraic factors of k.

Many attempts have been made to determine the dissipative spectrum. For example, Corrsin[22] and Pao[23] predict an $O[\exp(-ck^{4/3})]$ falloff while a number of authors predict $O[\exp(-ck^2)]$ (Townsend[24]; Novikov[25]; Saffman[26]; see Tennekes and Lumley[11] and Monin and Yaglom[27] for further discussion). These derivations are heuristic and may be questioned. A noteworthy exception is Kraichnan's DIA equations[28] which lead to $O[k^3 \exp(-ck)]$, which though less sharp, is not far from (21) in form. Of course (21) does not contradict these results. The issue hinges on the degree of sharpness present in (21).

Experimental confirmation is complicated by the difficulty in collecting reliable data. Since in the asymptotic limit of k large, the signal is small and noise plays an increasingly important role. The most recent discussion of experimental data in this limit was given by Sreenivasan.[12] He finds that $E(k)$ is best fit by an exponential. Rewriting his fit in our notation he finds

$$E(k) = O[\exp(-12.7\eta k)], \quad 0.1 \leqslant k\eta \leqslant 0.5,$$

$$= O[\exp(-8.8\eta k)], \quad 0.5 \leqslant k\eta \leqslant 1.5. \tag{22}$$

Computation is also of limited help in fixing the far-dissipative range. Memory and performance limitations force almost all computations to stop at roughly the Kolmogorov scale. The resulting spectrum is therefore unreliable for $k\eta = O(1)$ and beyond. An exception to this are calculations for the Ginzburg–Landau (GL) equation[29] for which the calculations were taken well into the dissipative range. We mention this case since the analysis of the GL equation is virtually the same as that given earlier, and the result for the dissipative spectrum is (15). Under an appropriate definition for η, (21) also applies. The numerical results found in Ref. 29 (see also Ref. 30) are consistent with the exponential fall suggested by (21). In fact, as explained there, one should expect behavior of the sort shown in (22).

In view of the experimental data of Sreenivasan[12] and of the GL computations[29] we suggest that (21) is in fact a reasonably sharp description of the far-dissipative term.

ACKNOWLEDGMENTS

The authors are grateful to Peter Constantin and Bruce Knight for their helpful comments.

The research reported here was supported by the U.S. Department of Energy under Contract No. DE.FGO2-86ER25020 and by DARPA/URI under Grant No. N00014-86-K0754.

[1]P. Constantin, C. Foias, O. P. Manley, and R. Temam, J. Fluid Mech. **150**, 427 (1985).

[2]P. Constantin and C. Foias, *Navier–Stokes Equations, Chicago Lectures in Mathematics* (University of Chicago Press, Chicago, 1988).

[3]R. Temam, *Infinite Dimensional Dynamic Systems in Mechanics and Physics* (Springer, Berlin, 1988).

[4]J. L. Lumley, *Atmospheric Turbulence and Radio Wave Propagation*, edit-

ed by A. M. Yaglom and V. I. Tatarski (Nauka, Moscow, 1967), pp. 166–178.
[5]P. Moin and R. D. Moser, J. Fluid Mech. **200**, 471 (1989).
[6]L. Sirovich, H. Tarman, and M. Maxey, *Proceedings on the Sixth Symposium of Turbulent Shear Flows*, edited by B. E. Launder (Springer, Berlin, 1988).
[7]N. Aubry, P. Holmes, J. L. Lumley, and E. Stone, J. Fluid Mech. **192**, 115 (1988).
[8]F. Reisz and B. Sz. Nagy, *Functional Analysis* (Ungar, New York, 1955).
[9]L. Sirovich, *Low and High Dimensional Dynamics of Chaotic Flow*, Proceedings of the 1989 Newport Conference on Turbulence (Springer, New York, 1990).
[10]C. Foias and R. Temam, J. Funct. Anal. **87**, 359 (1989).
[11]H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT Press, Cambridge, 1972).
[12]K. R. Sreenivasan, J. Fluid Mech. **151**, 81 (1985).
[13]C. Foias and G. Prodi, Rend. Semin. Univ. Padova **39**, 1 (1967).
[14]R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1962), Vol. I.
[15]L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics, Fluid Mechanics* (Pergamon, New York, 1987), Vol. 6.
[16]C. Foias, O. Manley, and R. Temam, Phys. Lett. A **122**, 140 (1987).
[17]L. Sirovich, Q. Appl. Math. **45**, 561 (1987).
[18]C. Foias, Rend. Semin. Univ. Padova **49**, 9 (1973).
[19]L. Sirovich and B. W. Knight, Proc. Natl. Acad. Sci. USA **82**, 8275 (1985).
[20]B. W. Knight and L. Sirovich, Proc. Natl. Acad. Sci. USA **83**, 527 (1986).
[21]R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
[22]S. Corrsin, Phys. Fluids **7**, 1156 (1964).
[23]Y. H. Pao, Phys. Fluids **8**, 1063 (1965).
[24]A. A. Townsend, Proc. R. Soc. London Ser. A **208**, 534 (1951).
[25]E. A. Novikov, Dokl. Akad. Nauk SSR **139**, 331 (1961).
[26]P. G. Saffmann, J. Fluid Mech. **16**, 545 (1963).
[27]A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975), Vol. II.
[28]R. H. Kraichnan, J. Fluid Mech. **5**, 497 (1959).
[29]L. Sirovich, B. W. Knight, and J. D. Rodriguez, to appear in Physica D.
[30]L. Sirovich, Physica D **37**, 126 (1989).