Generalized Normal Solutions of the Boltzmann Equation

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Solutions of the linearized Boltzmann equation which may be found from finite moment equations are studied. A general solution to this problem is found and its properties discussed. A more restrictive class of such solutions, called generalized normal solutions, are then uncovered. These constitute a wide class of solutions in any desired number of moments. The governing equations are rendered determinate by exact expressions relating higher moments to the distinguished moments. In certain circumstances the initial data must be altered, and the resulting "ersatz" initial data used in connection with the equations. A case in point is the usual normal solution of the Hilbert–Chapman–Enskog theory. The "ersatz" initial data then renders the latter into an exact asymptotic theory. In addition higher moment systems are discussed in detail. It is shown that the generalized normal solutions are by no means a comprehensive class. Exact solutions of the Boltzmann equation which are also exact solutions of the Euler equations, the Navier–Stokes equations, the Burnett equations, the Thirteen moments equations (among others) are exhibited. Furthermore, these can satisfy the Boltzmann equation in an infinite variety of ways.

I. INTRODUCTION

SINCE this is the last of a series of papers[1,2] on the initial value problem in linearized kinetic theory, we briefly summarize our results and its bearing on some previous work.

In the last published paper II, the chief goal was the solution of the linearized but full Boltzmann equation. This goal was formally achieved with the modal description given there, (II: 2.27).[3] The motives of this paper will be somewhat different. In II the Boltzmann equation was "reduced" to an infinite system of equations in moments of the distribution function,[4] and this system was then solved.[5] In this paper the cardinal concern will be the determination of finite systems of moment equations which yield exact solutions to the infinite system. And to establish criteria under which this may be used in an accurate or approximate manner.

For as long as the Hilbert–Chapman–Enskog theory dominated kinetic theory, the central problem in the field was the determination of a finite moment description,[6] In particular, the Hilbert–Chapman–Enskog method sought to solve the Boltzmann equation in terms of density, temperature, and velocity, $(\rho, \mathbf{u}, T)$. This it did, in general, by describing all the higher moments of the distribution function in terms of $(\rho, \mathbf{u}, T)$.[7,8] It has long been felt that this approach was defective unless clarified. For instance, in connection with an initial value problem, it is evident that the initial values of the higher moments must be of the Hilbert–Chapman–Enskog form. For instance stress and heat conduction (p_{ij}, S_i), if to be related to $(\rho, \mathbf{u}, T)$, must certainly be of this form initially. It has further been felt that the Hilbert–Chapman–Enskog approach did not give, as hoped for, a better description of non-equilibrium phenomena than did the Navier–Stokes equations. Schaaf and Chambre[9] mention this and cite experiments which indicate this.

Furthermore the author[10] showed, for a particular

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[1] L. Sirovich, Phys. Fluids **6**, 10 (1963), henceforth referred to as I.

[2] L. Sirovich, Phys. Fluids **6**, 218 (1963), henceforth referred to as II.

[3] We will use this obvious method of referring to either sections or equations of I and II.

[4] Consistent with current usage we shall use the word "moment" generically to refer to the coefficients in an expansion of the distribution function. In general a coefficient need not even be a sum of moments.

[5] When referring to the Boltzmann equation we shall mean either the true Boltzmann equation or the infinite system in the moments. Strictly speaking, these are not equivalent since the solution to the latter when properly summed to give the distribution function, may not even converge, among other possible difficulties.

[6] A more narrow view is that the Hilbert–Chapman–Enskog procedure reduced kinetic theory to the determination of transport coefficients.

[7] A novel and interesting approach to this problem is given by the Bogoliubov method. There it is assumed that the time dependence of the distribution function is given functionally through $(\rho, T, \mathbf{u})$. For references and an account of this approach see the article by E. G. D. Cohen, "The Boltzmann Equation and its Generalization to Higher Densities" in *Fundamental Problems in Statistical Mechanics* edited by E. G. D. Cohen (North-Holland Publishing Company, Amsterdam, 1962). An equivalent method is given in the next reference.

[8] H. Grad, "Principles of the Kinetic Theory of Gases," in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 12. See section 25 in connection with footnote 7.

[9] S. A. Schaaf and P. L. Chambre, *Flow of Rarefied Gases* (Princeton University Press, Princeton, New Jersey, 1955).

[10] L. Sirovich, "On the Kinetic Theory of Steady Gas Flows," in *Rarefied Gas Dynamics*, edited by L. Talbot (Academic Press Inc., New York, 1961).

boundary value problem, that the Burnett terms only sought to give a finer description of the Navier–Stokes solution. As we shall see, the Hilbert–Chapman–Enskog succeeds only in giving a very high quality description of the hydrodynamic state. It might appropriately be called hyper-hydrodynamics.

A new approach in kinetic theory was taken by Grad,[11] who proposed the use of more than the $(\rho, \mathbf{u}, T)$ moments. This work, published in 1949, explicitly details a theory in the $(\rho, \mathbf{u}, T, p_{ij}, \mathbf{S})$ moments, now known as the Thirteen-moments equations. Shortly after, Wang Chang and Uhlenbeck[12] and Mott-Smith[13] produced works also based on a moments theory and which considered many more than the above thirteen moments. As is well known, the moments methods yields a determinate system by disregarding all but a predetermined finite number of moments. Certain inadequacies of this theory were evident to Grad himself, and he later suggested[14] an amended scheme which he called "interpolation." Both the latter and the earlier moments method are discussed in I. In regard to the initial value problem it was shown that the liabilities presented by a moments method description greatly dimmed its virtues. It was also shown there that the method of interpolation was in the right vein, but, as originally proposed, was incomplete. A more refined method of interpolation as well as a less arbitrary method of choosing moments was then given there. As an example of those considerations, there was exhibited in I a system of equations in nine moments $(\rho, T, \mathbf{u}, \mathbf{S}, Q)$. The latter, a fourth order moment, was considered since its relaxation time is identical to that of $\mathbf{S}$. A major result of the present paper is the delimiting and clarification of the above methods—all of which, strictly speaking, have only a formal status.

Using the terminology of II, the status of the Hilbert–Chapman–Enskog theory may be described, to some degree, in a more general way. If the initial data consists of only the hydrodynamic modes (in the sequel we shall say it belongs to the Hilbert–Chapman–Enskog manifold) then it remains on this manifold for all time. [This was demonstrated in (II: Section II).] Under this condition the solution is here shown to result from a finite determinate system of equations—the conservation equations. The $(p_{ij}, \mathbf{S})$ as well as all higher moments are expressed as convolutions in space of $(\rho, \mathbf{u}, T)$ with functions which we shall call spatial influence functions. The expansion of the influence functions furnishes the Hilbert–Chapman–Enskog theory as it is usually given. More generally suppose the initial data consists only of N-modes (we shall say it belongs to an N-mode manifold) then it belongs to an N-mode manifold for all time. Then it is shown that the solution may be found by solving a determinate N-moment system of equations. Again, all higher moments are expressed in terms of the N distinguished moments by means of space convolutions with spatial influence functions. On expansion of the influence functions, differential operators occur.

Reversing the viewpoint, if one seeks a description in terms of N-specific moments, an N-mode description can then be found which results in N determinate equations in the N-moments. In this it is tacitly assumed that the initial data of the higher moments can be approximated (in a sense to be described later) by those relations which are demanded by an N-mode description. In general there are many exact solutions to the Boltzmann equation which depend on only an N-moment description. In the appendix of this paper it is shown how to construct such exact solutions of the Boltzmann equation which, for instance, exactly satisfy the Euler equations, the Navier–Stokes equations, the Thirteen moments equations, and so on. These solutions, as a class, do not contain much structure and are probably not worthy of deep investigation. The moment-mode description, on the contrary, has a great deal of structure and possesses a general theory. The so-called normal solutions found in the Hilbert–Chapman–Enskog theory are the particular members of the class which belong to the Hilbert–Chapman–Enskog manifold. For this reason we shall refer to the entire class of solutions as generalized normal solutions.

In II it was shown that two asymptotic situations were capable of detailed treatment. One was an expansion in smoothness ratio, i.e., the ratio of mean free path to the characteristic length scale of the phenomena. The other, an expansion in mean free time to elapsed time. Both these situations allow detailed calculation in the present paper. Restricting attention to the time asymptotic case, we found in II that only the hydrodynamic modes persist. As time procedes the Hilbert–Chapman–Enskog theory is established to higher and higher orders, which emphasizes our earlier remark that the Hilbert–

[11] H. Grad, Comm. Pure Appl. Math. **2**, 331 (1949).
[12] C. S. Wang Chang and G. E. Uhlenbeck, "On the Propagation of Sound in a Monatomic Gas," (Engineering Research Institute, University of Michigan, Ann Arbor, Michigan, 1952).
[13] H. M. Mott-Smith, "A New Approach to the Kinetic Theory of Gases," Lincoln Laboratories, MIT (1954).
[14] Reference 8, p. 264. See also reference 16.

Chapman–Enskog theory might be called hyperhydrodynamics. It might be characterized as being the closest state to equilibrium without actually being equilibrium.

The question arises as to whether it is possible to deal only with the Hilbert–Chapman–Enskog theory and still obtain from it the asymptotically correct solutions. The answer is in the affirmative if one employs the correct initial data in the Hilbert–Chapman–Enskog development. In this connection it should be mentioned that H. Grad[8] suggested that one should "investigate the initial layer and supplement the asymptotic differential equations (author's note: i.e., the equations obtained in the Hilbert–Chapman–Enskog procedure) with asymptotic initial values computed somehow from the actual initial values." The implied conjecture is therefore proven here. Actually, it can be stated in a different and stronger form. It is shown that the hydrodynamic modes can be dealt with directly by means of a finite system of equations subject to altered initial conditions. Asymptotically the hydrodynamic modes are all that remain of the solution. This, in turn, is governed by the Hilbert–Chapman–Enskog theory with asymptotic initial data. The details of the construction of the "ersatz" initial data are given for the asymptotic case.

Abandoning the five moment description (of the Hilbert–Chapman–Enskog manifold) we find asymptotic equations for any generalized normal solutions. These turn out to be of the same form found in I but the coefficients are somewhat different. It is found, interestingly enough, that the interpolation type scheme has implicit in it a second expansion procedure which, in practice, is not warranted. The asymptotic equations, which are given later, in a sense give the results gotten by summing this second expansion.

Some related topics in the theory of the Boltzmann equation have been recently discussed by McCune, Morse, and Sandri[15] and Grad.[16]

Since this is one of a series of articles it was felt that a repetition of result and analysis would be unnecessary. For this reason frequent reference is made to specific tables and equations in I and II. A knowledge of I and II is needed for an understanding of this paper, and details of this paper depend on details of I and II. The contents are arranged from almost hopeless generality at the outset to specific calculable cases. The section following, poses the problem and gives a general solution. Section III gives a formal method for obtaining the Hilbert–Chapman–Enskog theory and then generalizes it. Following that, in Sec. IV, a more rigorous and tauter analysis is given based on the eigentheory of II. An asymptotic analysis with specific calculations is presented in Sec. V. The appendix gives some examples of exact solutions of the Boltzmann equation which are of a specialized sort. It is presented in a somewhat terse fashion since we are only interested in exhibiting these solutions and not in investigating them.

[15] J. E. McCune, T. F. Morse, and G. Sandri, "On the Relaxation of Gases Toward Continuum Flow," in *Rarefied Gas Dynamics* edited by J. Laurmann (Academic Press Inc., New York, 1963). Nonlinear theory is considered here, but only applied to the Krook equation.

[16] H. Grad, Phys. Fluids **6**, 147 (1963).

II. SEPARATED SYSTEMS OF EQUATIONS

As stated in the introduction the central problem of this study is the reduction of the Boltzmann equation to a determinate finite system of moment equations. This is now carried out in perfect generality.

The reduction of the Boltzmann equation to an infinite system of coupled moment equations is given by (II: 2.9, 11). Using a mixed tensor and vector notation with the summation convention on subscripts this is[17]

$$\left(\delta_{ij}\frac{\partial}{\partial t} + \mathbf{C}_{ij}\cdot\frac{\partial}{\partial \mathbf{x}} - \Lambda_{ij}\right)a_j = 0. \qquad (2.1)$$

Subject to footnote 5, we shall regard (2.1) as equivalent to the Boltzmann equation and its solution as equivalent to finding the distribution function. The investigation will only apply to the initial value problem in circumstances under which we may substitute a Fourier transform or series for the moments. In either case (2.1) becomes the ordinary differential equations

$$\left(\delta_{ij}\frac{\partial}{\partial t} - i\mathbf{k}\cdot\mathbf{C}_{ij} - \Lambda_{ij}\right)a_j = 0. \qquad (2.2)$$

We shall often speak of $-i\mathbf{k}$ and $\partial/\partial\mathbf{x}$ interchangeably. Justification for this will be given when it is deemed necessary.

To achieve a description in terms of the first N moments, say, we introduce the following splitting,

$$\begin{cases} h_i = a_i, \\ \mathbf{H}_{ij} = \mathbf{C}_{ij}, \\ l_{ij} = \Lambda_{ij}, \\ f_i = \sum_{j=N+1}^{\infty} i\mathbf{k}\cdot\mathbf{C}_{ij}a_j: \end{cases} \qquad i, j \leq N \qquad (2.3)$$

[17] See II for definitions and notation.

$$\begin{cases} r_i = a_{i+N}, \\ \mathbf{R}_{ij} = \mathbf{C}_{i+N,j+N}, \\ L_{ij} = \Lambda_{i+N,j+N}, \\ F_i = \sum_{j=0}^{N} i\mathbf{k}\cdot\mathbf{C}_{i+N,j}a_j. \end{cases} \qquad i, j \geq 1 \cdots \quad (2.4)$$

Using vector notation Eq. (2.2) can be written as the two separate systems

$$\left(\frac{\partial}{\partial t} - i\mathbf{k}\cdot\mathbf{H} - 1\right)\mathbf{h} = \mathbf{f}, \qquad (2.5)$$

$$\left(\frac{\partial}{\partial t} - i\mathbf{k}\cdot\mathbf{R} - \mathbf{L}\right)\mathbf{r} = \mathbf{F}. \qquad (2.6)$$

Denoting the initial conditions of $\mathbf{r}$ by $\mathbf{r}_0$, Eq. (2.6) is easily solved to give,

$$\mathbf{r} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{x}}e^{(i\mathbf{k}\cdot\mathbf{R}+\mathbf{L})t}\mathbf{r}_0\, d\mathbf{k}$$

$$+ \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_0^t e^{-i\mathbf{k}\cdot\mathbf{x}}e^{(i\mathbf{k}\cdot\mathbf{R}+\mathbf{L})(t-\tau)}\mathbf{F}(\tau)\, d\tau\, d\mathbf{k}. \quad (2.7)$$

$\mathbf{F}$ has been regarded as a known function in the latter, however it must be recalled from (2.4) that it is a linear form on $\mathbf{h}$. Upon substituting $\mathbf{r}$ into $\mathbf{f}$ of (2.5) [$\mathbf{f}$ is a linear form on $\mathbf{r}$] we obtain a finite determinate system of equations in $\mathbf{h}$. Further if we define the Fourier transform of $e^{(i\mathbf{k}\cdot\mathbf{R}+\mathbf{L})}$ to be $\mathbf{M}(\mathbf{x}, t)$, i.e.,

$$\int \mathbf{M}(\mathbf{x}, t)e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} = e^{(i\mathbf{k}\cdot\mathbf{R}+\mathbf{L})t}, \qquad (2.8)$$

then (2.7) may be written as

$$\mathbf{r} = \mathbf{M} * \mathbf{r}_0 + \int_0^t \mathbf{M}(t-\tau) * \mathbf{F}(\tau)\, d\tau. \qquad (2.9)$$

The asterisk denotes the convolution product.

We have therefore succeeded formally in reducing the Boltzmann equation to the solution of a finite determinate system (2.5) [note that the substitutions for $\mathbf{f}$ and $\mathbf{F}$ must be made before (2.9) is substituted into (2.5)]. However the following effects are found in general,

(1) From the first term of (2.7) it is seen that the initial data of the entire distribution function will enter.[18]

(2) Due to the time integral in (2.9), (2.5) will depend on the time history of $\mathbf{h}$.

(3) Due to the convolution product in (2.9), (2.5) will be nonlocal in space.

[18] By methods similar to (II: 2.41) this may be shown to have an exponential time decay. However, it is incorrect to neglect this term on this basis.

III. A GENERALIZED HILBERT-CHAPMAN-ENSKOG THEORY

We now uncover certain requirements on the initial data which, if fulfilled, allow us to relax some of the conditions given at the close of the last section.

We set

$$\mathbf{G} = -(i\mathbf{k}\cdot\mathbf{R} + \mathbf{L}) \qquad (3.1)$$

and consider the transform of (2.7),

$$\mathbf{r} = e^{-\mathbf{G}t}\mathbf{r}_0 + \int_0^t e^{-\mathbf{G}(t-\tau)}\mathbf{F}(\tau)\, d\tau. \qquad (3.2)$$

Parts integrating the last term indefinitely we are led to

$$\mathbf{r} = e^{-\mathbf{G}t}\left[\mathbf{r}_0 - \mathbf{G}^{-1}\frac{1}{1 + \mathbf{G}^{-1}\,\partial/\partial t}\mathbf{F}_0\right]$$

$$+ \mathbf{G}^{-1}\frac{1}{1 + \mathbf{G}^{-1}\,\partial/\partial t}\mathbf{F}. \qquad (3.3)$$

Use has been made of the relation

$$\sum_{n=0}^{\infty}(\mathbf{X})^n = \frac{1}{1-\mathbf{X}}, \qquad (3.4)$$

which is formal in the absence of convergence. An immediate result of (3.3) is that

$$\mathbf{r} = \frac{1}{\mathbf{G} + \partial/\partial t}\mathbf{F} \qquad (3.5)$$

if the relation is true initially.

The time derivatives have no meaning initially, apart from their values as given by space derivatives of the initial data. To find such a representation, we note that from (2.3), (2.4) $\mathbf{F}$ is linear in $\mathbf{h}$ and $\mathbf{f}$ linear in $\mathbf{r}$. More explicitly,

$$\mathbf{f} = \mathbf{b}\cdot\mathbf{r}, \qquad (3.6)$$

$$\mathbf{F} = \mathbf{B}\cdot\mathbf{h}, \qquad (3.7)$$

where neither $\mathbf{b}$ nor $\mathbf{B}$ are necessarily square matrices. Substituting these into Eq. (2.5) gives

$$\frac{\partial\mathbf{h}}{\partial t} = \mathbf{J}\cdot\mathbf{h} + \mathbf{b}\cdot\frac{1}{\mathbf{G} + \partial/\partial t}\mathbf{B}\cdot\mathbf{h}, \qquad (3.8)$$

and from this

$$\frac{\partial}{\partial t}\mathbf{B}\cdot\mathbf{h} = \mathbf{B}\cdot\mathbf{J}\cdot\mathbf{h} + \mathbf{B}\cdot\mathbf{b}\cdot\frac{1}{\mathbf{G} + \partial/\partial t}\mathbf{B}\cdot\mathbf{h}, \qquad (3.9)$$

with

$$\mathbf{J} = i\mathbf{k}\cdot\mathbf{H} + 1. \qquad (3.10)$$

By repeated application of (3.8) on itself with the aid of (3.9), we write finally,

$$\partial \mathbf{h}/\partial t = \mathbf{J}\mathbf{h} + \mathbf{X}\mathbf{h}, \tag{3.11}$$

where $\mathbf{X}$ represents the result of these manipulations.[19] Integrating (3.11) and inverting the transforms we have,

$$\mathbf{h} = \frac{1}{(2\pi)^3} \int e^{[i\mathbf{k}\cdot\mathbf{H}+\mathbf{l}+\mathbf{X}(\mathbf{k})]t} \mathbf{h}_0 e^{-i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}. \tag{3.12}$$

We see therefore that by choosing initial data of the form (3.5) [with $\partial/\partial t$ eliminated by means of (3.8), (3.9)] we have eliminated two of the three conditions given at the close of the last section, and which would exist under general initial data. Now only the initial data of $\mathbf{h}$ enters in the solution for $\mathbf{h}$, (3.12). And secondly the time history effect has been eliminated. As an additional property we note that (3.12) has at most $N + 1$ modes. In the next section we indicate that these are indeed $(N + 1)$-modes of the solution of the Boltzmann equation found in II. This gives a clue to another approach to the problem, i.e. starting with a finite modal condition. This is the approach taken in the next section and it results in a more rigorous and tauter theory. There is one result found there which bears mentioning now. It tells us that before carrying out procedure of this section we should have first eliminated the systematic time decay of $\mathbf{h}$. That is $\bar{\mathbf{h}}e^{lt}$ should be substituted in (2.5) and the exponential eliminated from both (2.5) and (2.6). Although the procedure of this section is correct, it includes implicitly an expansion which is not necessary. A full description of this will be given in Sec. V.

In any case if (2.5) is taken as the conservation equations, $\mathbf{l} = 0$ and no system time decay in $\mathbf{h}$ is present. To get the Hilbert–Chapman–Enskog theory an expansion in small $\mathbf{k}$ is needed. This we know is sufficient for a description of smooth flow and for long elapsed time. Therefore in

$$\frac{1}{\mathbf{G}} \sum_{n=0}^{\infty} \left(-\mathbf{G}^{-1} \frac{\partial}{\partial t} \right)^n \tag{3.13}$$

we write

$$\frac{1}{G} \sim \mathbf{L}^{-1} - \mathbf{L}^{-1}\mathbf{R}_\mu \mathbf{L}^{-1} \frac{\partial}{\partial x_\mu} + \mathbf{L}^{-1}\mathbf{R}_\mu \mathbf{L}^{-1}\mathbf{R}_\nu \mathbf{L}^{-1} \frac{\partial^2}{\partial x_\mu\, \partial x_\nu} \pm \cdots . \tag{3.14}$$

The substitution of $\partial/\partial \mathbf{x}$ for $-i\mathbf{k}$ is justified, for on solving the resulting partial differential equations we obtain the same asymptotic solutions as in the exact case of II. This can be carried to any desired order by carrying a sufficient number of terms in (3.14). As an example on carrying only the first term in (3.13), (3.14) we get the Navier–Stokes equations, and two orders give the Burnett's equations. For the one-dimensional case these equations can be immediately read from table II of II.

If we disregard the earlier remark about eliminating systematic time decay, (3.13), (3.14) give equations in any desired number of moments. In particular the one-dimensional form of the Nine-moments theory suggested in I is gotten by just glancing at table I of II. It is identical to (I: 6.15, 17, 18).

Later we shall see that the form of these equations is correct, only certain coefficients must be altered.

[19] Under certain circumstances the determination of $\mathbf{X}$ can be reduced to find a particular branch of a quadratic in $\mathbf{X}$.

IV. GENERALIZED NORMAL SOLUTIONS

In II we denoted the eigenvalues and eigenvectors of $\mathbf{G}$ by d^m and $\mathbf{v}^m$, respectively, and from this we showed that the solution of (2.2) is[20]

$$a_i = \sum_{m=0}^{\infty} e^{d^m(k)t} \frac{v_i^m(k)\bar{v}_j^m(k)}{|\mathbf{v}^m|^2} a_i^0(k). \tag{4.1}$$

The initial condition $\mathbf{a}^0$ may also be decomposed into a sum of eigenvectors as follows:

$$a_i^0 = \sum_{\mu=0}^{\infty} \beta^\mu v_i^\mu, \tag{4.2}$$

with

$$\beta^\mu = \mathbf{a}^0 \cdot \bar{\mathbf{v}}^\mu / |\mathbf{v}^\mu|^2. \tag{4.3}$$

We now investigate the class of solutions obtained when (4.2) is the finite sum

$$a_j^0 = \sum_{\mu=0}^{N} \alpha_0^\mu v_j^\mu. \tag{4.4}$$

The solution (4.1) is now

$$a_i = \sum_{m=0}^{N} e^{d^m t} \alpha_0^m v_i^m. \tag{4.5}$$

Clearly the α_0^m may be replaced by $(N + 1)$ of the a_i. The most natural replacement is in terms of the diagonal mode members (II: 2.26, 27). The replacement is easily effected with the use of the following notation:

$$\tilde{\mathbf{a}} = \{a_i\} \qquad i = 1, N, \tag{4.6}$$

$$\tilde{\mathbf{V}} = \{v_i^i\} \qquad i, j = 1, N, \tag{4.7}$$

$$\hat{\mathbf{a}} = \{a_{i+N}\} \qquad i = 1, \infty, \tag{4.8}$$

$$\hat{\mathbf{V}} = \{v_{j+N}^i\} \qquad i = 1, N; \quad j = 1, \infty. \tag{4.9}$$

[20] This notation may be somewhat confusing. By $|\mathbf{v}^m|^2$ is meant $\sum_i \bar{v}_i^m v_i^m$ which is not in general real, but rather is a real function of $i\mathbf{k}$.

By means of simple operations we obtain

$$\hat{\mathbf{a}} = \hat{\mathbf{V}}\tilde{\mathbf{V}}^{-1}\tilde{\mathbf{a}}, \tag{4.10}$$

which is a basic relation of this paper. More explicitly, we write

$$a_i = \sum_{j=1}^{N} \omega_i^j(k) a_j(k), \tag{4.11}$$

where the ω_i^j can be computed from (4.10). Naturally $\omega_i^j = \delta_{ij}$ for $i \leq N$. Further, Eq. (4.11) can be regarded as the transform of

$$a_i = \sum_{j=1}^{N} \omega_i^j * a_j, \tag{4.12}$$

with

$$\omega_i^j = \delta_{ij}\delta(x) \qquad i \leq N,$$

We shall refer to the $\omega_i^j(x)$ as spatial influence functions. Returning to the original system of equations we single out the $N + 1$ equations which have time derivatives of the diagonal mode members. (This is not necessary and is only done for definiteness.) In these we replace all higher moments in terms of the distinguished moments by means of (4.12). This renders the system determinate. Upon solving this system, all higher moments can be computed from (4.12). We have, therefore, shown the following:

If the initial data of a distribution function lies on an N-mode manifold, the solution remains in this subspace for all time (see II); further, the distribution function is determined by the solution of a finite system of equations in the same finite number of moments.

For later comparison we point out trivially

$$a(t = 0) = \mathbf{a}^0. \tag{4.13}$$

We note also that, as in Sec. III, only the initial data of the distinguished moments enter, and no time-history effect occurs. This contrasts with the general development of Sec. II. Something which is obscured by the manipulation of Sec. III is that the theory is nonlocal in space. Note also that the influence functions $\omega(x)$ are only dependent on the nature of **G**.

As an example, let us suppose that the initial data lies in the Hilbert–Chapman–Enskog subspace. In this case the natural moments are (ρ, **u**, T) and the natural equations are the conservation equations. These are rendered determinate by the (p_{ij}, **S**) relations[21]

$$\begin{bmatrix} p_{ij} \\ S_i \end{bmatrix} = \begin{bmatrix} \omega_{ij}^{(1)} & \omega_{ij}^{(2)} & \omega_{ijk}^{(3)} \\ \Omega_i^{(1)} & \Omega_i^{(2)} & \Omega_{ik}^{(3)} \end{bmatrix} * \begin{bmatrix} \rho \\ T \\ u_k \end{bmatrix} \tag{4.14}$$

where the asterisk again denotes the convolution product. Specific forms for ω^i and Ω^i will be given in Sec. V.

Let us now consider a related problem. Suppose the initial data is completely arbitrary, but we wish to focus attention on only some particular $N + 1$ modes (let it be the first $N + 1$, for definiteness). From the orthogonality of the eigenvectors $\mathbf{v}^m$ and the linearity of the operator (2.1), we may split the initial data into a sum of eigenvectors, then solve the problem with each of these summands as initial data and finally solve the original problem by summing the partial solutions. However, we have just finished with the case when the initial data is composed of a finite number of modes. Only the initial data of the problem need now be modified. We have therefore shown the following:

Given arbitrary data $\mathbf{a}^0$, *the evolution of* $N + 1$ *particular modes may be found by considering the problem as if only* $N + 1$ *modes existed (i.e., by solving a* $N + 1$ *system of equations) and taking the projection of the initial data on the N-mode manifold as initial conditions,* i.e.,

$$\tilde{\mathbf{a}}^0 = \sum_{\mu=0}^{N} \beta^\mu v^\mu = \mathbf{P}^N \cdot \mathbf{a}^0, \tag{4.15}$$

where β^μ is given by (4.3). We will refer to

$$P^N = \sum_{\mu=0}^{N} \frac{\mathbf{v}^\mu \tilde{\mathbf{v}}^\mu}{|v^\mu|^2} \tag{4.16}$$

as the $(N + 1)$-mode projection operator, it being a sum of the elementary projectors previously defined (II; 2.23). These results have some rather deep consequences, especially in regard to the Hilbert–Chapman–Enskog theory.

We know from II that as the elapsed time becomes large only the hydrodynamic modes persist. Therefore, if we focus attention on the hydrodynamic modes only, we obtain the correct long time behavior of the flow. But as we indicate in the next section, the description of the hydrodynamic mode is equivalent to the Hilbert–Chapman–Enskog theory.[22] We therefore have the following:

For arbitrary initial data, $\mathbf{a}^0$, *the Hilbert–Chapman–Enskog theory gives the correct time asymptotic theory*

[21] The (p_{ij}, **S**) are only known to be coefficients of eigenfunctions for Maxwell molecules. In other cases the summation of coefficients leads to (p_{ij}, **S**), and (4.14) is obtained in that manner.

[22] Only under asymptotic conditions is the Hilbert–Chapman–Enskog theory uncovered. The conservation equations as rendered determinate by the description of the hydrodynamic modes give a more general theory.

if and only if the projection of $\mathbf{a}^0$ *on the Hilbert–Chapman–Enskog manifold is taken as initial data for the problem.*

A conjecture of this nature was made by H. Grad some time ago.[14] Details of such a calculation are given in the following section.

It is also of interest to consider the error involved in considering an $(N + 1)$-mode description without first projecting the initial data onto the $(N + 1)$-mode subspace. Solving a problem in this manner naturally leads only to $(N + 1)$-modes with all higher moments having incorrect initial values. From the linearity of the problem the error term is obtained by solving the same $(N + 1)$-mode problem with initial values given by

$$\Delta\mathbf{a}^0 = \sum_{\mu=N+1}^{\infty} \beta^\mu \mathbf{v}^\mu = \mathbf{a}^0 - \mathbf{P}^N \cdot \mathbf{a}^0, \qquad (4.17)$$

with β^μ given by (4.3). If the solution derived from (4.17) is subtracted from the original incorrect solution, the correct solution results. It is interesting to note that a second incorrect problem must be solved [i.e., taking (4.17) as initial data] in order to rectify the original erroneous calculation.

We have shown that any real or ficticious restriction to N-modes is sufficient to reduce the flow problem to the solution of a finite system of equations. We can then ask if this is in any way a necessary condition for the latter. The appendix provides a negative answer to this question. There it is demonstrated how to construct a wide class of exact solutions to the Boltzmann system (2.1), which are not generalized normal solutions. In particular, it is shown how to construct solutions which exactly satisfy the Euler equations, the Navier–Stokes equations, the Thirteen moments equations, etc., and which can be made to satisfy the Boltzmann equation in infinitely many ways.

We close this section by making some comparisons of this and the preceding section. In all instances which have been considered, the results of the last and this section were equatable (on removing systematic time decay in Sec. 3 identical results were obtained). However, I have not been able to find a genuine demonstration of the equivalence of the two methods. This is also regretable from the mathematical point of view since the method of Sec. III seems to offer a method of computing eigenvectors without first finding the eigenvalues. On examining the two methods we see that the method of Sec. III is more direct and is more easily performed. Although (4.10) was obtained without difficulty, it required all the preparatory machinery that was built up in I and II. Moreover, more calculations are in order before quantitative results can be obtained (see next section). On the other hand, the tauter theory presented in this section offers a firmer foundation for the somewhat cavalier methods of Sec. III. Also the work of this section offers the most promise for an eventual mathematically rigorous theory.

V. ASYMPTOTIC THEORY

The exact determination of the influence functions $\omega(x)$ seems to be a hopeless cause. As in (I) and (II) to gain results we must resort to power series developments in the wave number $\mathbf{k}$. We first carry this out for the basic relation (4.10). Writing[23]

$$\hat{\mathbf{V}} = \sum_{n=1}^{\infty} (i\mathbf{k})^n \cdot \hat{\mathbf{V}}_n, \qquad (5.1)$$

$$\tilde{\mathbf{V}} = \sum_{n=0}^{\infty} (i\mathbf{k})^n \cdot \tilde{\mathbf{V}}_n, \qquad (5.2)$$

where the subscript refers to an nth order tensor and the dot denotes the complete inner product of $\mathbf{V}_n$ with the nth order tensor $(i\mathbf{k})^n$. Then if we write

$$\hat{\mathbf{V}}\tilde{\mathbf{V}}^{-1} = \sum_{n=1}^{\infty} (i\mathbf{k})^n \cdot \mathbf{U}_n, \qquad (5.3)$$

we find

$$\mathbf{U}_1 = \hat{\mathbf{V}}_1\tilde{\mathbf{V}}_0^{-1}, \qquad (5.4)$$

$$\mathbf{U}_2 = \hat{\mathbf{V}}_2\tilde{\mathbf{V}}_0^{-1} - \hat{\mathbf{V}}_1\tilde{\mathbf{V}}_0\tilde{\mathbf{V}}_1\tilde{\mathbf{V}}_0, \qquad (5.5)$$

$$\mathbf{U}_3 = \hat{\mathbf{V}}_1\tilde{\mathbf{V}}_0^{-1}\tilde{\mathbf{V}}_1\tilde{\mathbf{V}}_0^{-1}\tilde{\mathbf{V}}_1\tilde{\mathbf{V}}_0^{-1} - \hat{\mathbf{V}}_1\tilde{\mathbf{V}}_0^{-1}\tilde{\mathbf{V}}_2\tilde{\mathbf{V}}_0^{-1} - \hat{\mathbf{V}}_2\tilde{\mathbf{V}}_0^{-1}\tilde{\mathbf{V}}_1\tilde{\mathbf{V}}_0^{-1}, \qquad (5.6)$$

and so on. A similar expansion holds for the projection operator (4.16), which is also a matrix. These expansions are applicable to two asymptotic situations; an expansion in smoothness ratio and an expansion in mean free time to elapsed time. Both situations were discussed at length in II Sec. III and any further discussion would be repetitious. It suffices to say here that dealing with only a finite number of modes in no way alters the arguments of II.

Evidently the above power series lead to series representations of the influence function ω. To be definite, let us write

$$\omega \sim \sum_{n=1}^{m} \boldsymbol{\omega}_n \cdot (i\mathbf{k})^n. \qquad (5.7)$$

[23] In all cases $\hat{\mathbf{V}}$ does not have a zero order term.

This can be inverted to give

$$\omega(\mathbf{x}) \sim \sum_{n=1}^{m} (-)^n \boldsymbol{\omega}_n \cdot \boldsymbol{\delta}^{(n)}(\mathbf{x}), \tag{5.8}$$

where $\boldsymbol{\delta}^{(n)}$ denotes the nth derivative of the δ-function. When an expression of the type of (5.8) is used in the convolution (4.12) we see that we ultimately wind up with a sum of derivatives. More simply ($i\mathbf{k}$) in (5.7) can be immediately replaced by the gradient $\partial/\partial\mathbf{x}$. This entire procedure, including the replacement wavenumbers by derivatives, is justifiable. From II we know that terminating an expansion yields the correct solution to some particular order. On solving a system in which the derivatives have been replaced by wavenumbers we are immediately led back to what we know are asymptotically correct expressions (this is immediate if the system is solved by transforms). Care must be taken not to use a system to provide results beyond the order of its accuracy. For instance, as we shall see in a moment, the Navier–Stokes equations are accurate asymptotically to the zero order. In gas dynamics it is sometimes found that asymptotic solutions to the Navier–Stokes equations are carried to orders which can only be given by the Burnett equations or higher orders.

The eigentheory of the linearized Boltzmann collision integral L [see (II: 2, 3)] is only known for Maxwell molecules. For this reason we restrict all further study to the diagonal approximation [see (I; Sec. V)] of the collision integral.[24] The operator L, for Maxwell molecules, has a discrete spectrum. And by construction the diagonal approximation gives a perturbed operator for other molecules which also gives rise to a discrete spectrum. The general theory developed earlier and in II assumes at the outset the existence of a discrete spectrum, and therefore the cases mentioned fall within the pale of the theory. On the other hand it has recently been shown[25] that under certain restrictions a continuous spectrum occurs. It would then seem that the mathematical simplicity of the Maxwell molecules is possibly offset by an inadequate physical description. This has not been shown yet. Moreover for simple gases there is not one known phenomenon predicted by arbitrary force laws which is not predicted by Maxwell molecules.[26]

We now consider several examples of the results of this paper. As in I and II we shall for simplicity only consider one dimensional shear-free flow. We can then find $\hat{\mathbf{V}}$ and $\tilde{\mathbf{V}}$ for the first two orders in Tables I through VI of II.

HILBERT–CHAPMAN–ENSKOG THEORY

We state at the outset that no rigorous identification will be established between the well-known Hilbert–Chapman–Enskog theory, the theory of Sec. III, and that which is developed below. In all instances tried, the results produced were identical.

For the Hilbert–Chapman–Enskog theory the distinguishing moments are $(\rho, \mathbf{u}, T)$, or using the notation of II, (b_{00}, b_{01}, b_{10}). We therefore only need tables I through III of II in order to compute U_n. Also, since the conservation equations are the natural equations, we will exhibit only (b_{02}, b_{11}) [which are proportional to p_{11} and S_1 respectively]. By inspection and elementary operations, we get to lowest order [i.e. corresponding to U_1 of (5.4)],

$$\begin{bmatrix} b_{11} \\ b_{02} \end{bmatrix}_1 = ik \begin{bmatrix} (10)^{\frac{1}{2}}/3\lambda_{11} & -\frac{1}{\lambda_{11}}\left(\frac{2}{3}\right)^{\frac{1}{2}} & -\frac{1}{\lambda_{11}}\left(\frac{5}{3}\right)^{\frac{1}{2}} \\ -\frac{2(5)^{\frac{1}{2}}}{3\lambda_{02}} & -\frac{2(3)^{\frac{1}{2}}}{\lambda_{02}3} & 0 \end{bmatrix}$$
$$\cdot \begin{bmatrix} \frac{3}{10} & \frac{1}{2}\left(\frac{3}{5}\right)^{\frac{1}{2}} & -\frac{1}{5}\left(\frac{3}{2}\right)^{\frac{1}{2}} \\ -\frac{1}{2}\left(\frac{3}{5}\right)^{\frac{1}{2}} & \frac{1}{2} & \frac{1}{(10)^{\frac{1}{2}}} \\ \frac{2}{5}\left(\frac{3}{2}\right)^{\frac{1}{2}} & 0 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} b_{00} \\ b_{01} \\ b_{10} \end{bmatrix} \tag{5.9}$$

The second matrix is $\tilde{\mathbf{V}}_0^{-1}$ in this case and the first is the appropriate truncation of $\hat{\mathbf{V}}_1$ to give the column vector on the left. Substituting for the b_{rl} their definitions (I: 2.12), and inverting we get

$$[S_1]_1 = \frac{5}{2\lambda_{11}} \frac{\partial T}{\partial x} \tag{5.10}$$

$$[p_{11}]_1 = \frac{4}{3\lambda_{02}} \frac{\partial u}{\partial x} \tag{5.11}$$

which are of course the Navier–Stokes relations, customarily gotten in the first step of the Hilbert–Chapman–Enskog method. The subscript 1 in (5.9–11) denotes the first order theory.

Second order theory can also be obtained from the calculations of II. This gives[27]

[24] Note that this is exact for Maxwell molecules.

[25] See reference 16.

[26] For gas mixtures, Maxwell molecules fail to predict thermal diffusion. See S. I. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1952).

[27] The (02), $j = 2$ entry of (II: Table I) should have a coefficient of -1. The (02), $j = 2$ entry of (II: Table III) should be 0.

$$\begin{bmatrix} b_{11} \\ \\ b_{02} \end{bmatrix}_2 = (ik)^2 \left\{ -\begin{bmatrix} \left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{4}{3\lambda_{02}\lambda_{11}} & \frac{(10)^{\frac{1}{2}}}{15\lambda_{11}} \left(\frac{2}{\lambda_{02}} - \frac{1}{\lambda_{11}}\right) & 0 \\ \frac{2}{\lambda_{02}3^{\frac{1}{2}}} \left(\frac{1}{\lambda_{11}} - \frac{1}{\lambda_{02}}\right) & \frac{(25)^{\frac{1}{2}}}{3\lambda_{02}} \left(\frac{2}{5\lambda_{11}} - \frac{1}{\lambda_{02}}\right) & \frac{2\sqrt{2}}{3\lambda_{11}\lambda_{02}} \end{bmatrix} \begin{bmatrix} \frac{3}{10} & \frac{1}{2}\left(\frac{3}{5}\right)^{\frac{1}{2}} & -\frac{1}{5}\left(\frac{3}{2}\right)^{\frac{1}{2}} \\ -\frac{1}{2}\left(\frac{3}{5}\right)^{\frac{1}{2}} & \frac{1}{2} & \frac{1}{(10)^{\frac{1}{2}}} \\ \frac{2}{5}\left(\frac{3}{2}\right)^{\frac{1}{2}} & 0 & \frac{3}{5} \end{bmatrix} \right.$$

$$\left. - \begin{bmatrix} -\left(\frac{2}{3}\right)^{\frac{1}{2}} \left(\frac{1}{5\lambda_{02}\lambda_{11}} + \frac{1}{10\lambda_{11}^2}\right) & \left(\frac{2}{5}\right)^{\frac{1}{2}} \left(\frac{1}{3\lambda_{11}\lambda_{02}} - \frac{3}{2\lambda_{11}^2}\right) & \frac{1}{15}\left(\frac{2}{\lambda_{02}\lambda_{11}} + \frac{1}{\lambda_{11}^2}\right) \\ \frac{2}{5(3)^{\frac{1}{2}}} \left(\frac{1}{\lambda_{02}^2} + \frac{5}{2\lambda_{02}\lambda_{11}}\right) & \frac{2}{3(5)^{\frac{1}{2}}} \left(\frac{1}{2\lambda_{02}\lambda_{11}} + \frac{1}{\lambda_{02}^2}\right) & \frac{\sqrt{2}}{3}\left(-\frac{2}{5\lambda_{02}^2} + \frac{1}{\lambda_{11}\lambda_{02}}\right) \end{bmatrix} \right\} \begin{bmatrix} b_{00} \\ b_{01} \\ b_{10} \end{bmatrix}. \quad (5.12)$$

The first term of the right hand side is the product $\hat{\mathbf{V}}_2\tilde{\mathbf{V}}_0^{-1}$ and the second is the result of $\hat{\mathbf{V}}_1\tilde{\mathbf{V}}_0^{-1}\tilde{\mathbf{V}}_1\tilde{\mathbf{V}}_0^{-1}$. Carrying out the matrical operations, substituting the usual expressions for b_{rs}, and inverting we get,

$$\begin{bmatrix} S_1 \\ \\ p_{11} \end{bmatrix}_2 = \frac{\partial^2}{\partial x^2} \begin{bmatrix} 0 & \left(\frac{4}{3\lambda_{11}\lambda_{02}} - \frac{5}{3\lambda_{11}^2}\right) & 0 \\ -\frac{4}{3\lambda_{02}^2} & 0 & \frac{2}{3}\left(-\frac{2}{\lambda_{02}^2} + \frac{2}{\lambda_{11}\lambda_{02}}\right) \end{bmatrix} \begin{bmatrix} \rho \\ u \\ T \end{bmatrix}. \quad (5.13)$$

These are the usual terms furnished in addition to (5.12) in the Burnett approximation. Higher orders can also be obtained in this manner. However the preparation and tedious calculations soon make such a course prohibitive. For higher orders it is far more expedient and direct to use the method of Sec. III.

With either (5.12), or (5.12) plus (5.13), the conservation equations become determinate, and with proper initial data describe the hydrodynamic modes in the two already mentioned asymptotic regimes. The determination of the asymptotic initial data proceeds as the other calculations and it is not necessary to go into it. Denoting the asymptotic initial conditions by a superbar, these are to the first two orders given by,

$$\begin{bmatrix} \bar{\rho}^0 \\ \bar{u}_0 \\ \bar{T}_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{\lambda_{02}}\frac{\partial}{\partial x} \\ 0 & 0 & 1 & \frac{2}{3\lambda_{11}}\frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \rho^0 \\ u^0 \\ T^0 \\ S_1^0 \\ p_{11}^0 \end{bmatrix}. \quad (5.14)$$

The coefficient matrix is the transform of the projection operator (4.16) to the first two orders. From our study of fundamental solutions in II [see especially the discussion following (II: 5.21)] we know that restrictions exist on the solution of the conservation equations in any Hilbert–Chapman–Enskog approximation. Thus in solving the conservation equations with (5.12) Navier–Stokes, only the lowest order (zeroth order) remains unchanged in a finer approximation. The first order is correctly given by the Burnett equations. It also follows that the "ersatz" initial data (5.14) must fit the order of the equations. Thus with the Navier–Stokes equation just the zeroth order of (5.14) is to be used, i.e.,

$$(\bar{\rho}^0, \bar{u}^0, \bar{T}^0) = (\rho^0, u^0, T^0). \quad (5.15)$$

On the other hand all of (5.14) should be used with the Burnett equations.

With the use of "ersatz" initial data the hydrodynamic mode is obtained, at least, in two asymptotic senses. For long time, as stated in Sec. IV, it furnishes us with asymptotically correct (ρ, u, T) to any desired order. By using the Burnett equations we find it through the first two orders. On the other hand, for the smooth ratio asymptotics, it is incorrect to assume that (ρ, u, T) will come from the asymptotic hydrodynamic mode. Certainly it will do so for a long time since this is just the prior case. But from II we do know that to lowest order the asymptotic hydrodynamic mode furnishes the correct (ρ, u, T). Hence:

For smooth phenomena [in the sense of (II)] the Navier–Stokes equations will, with (5.15), *give the correct lowest order (zeroth order in smoothness) solution for* $(\rho, \mathbf{u}, T)$ *for all time.*

On the other hand, we also know from II that first order (in smoothness) contributions to $(\rho, \mathbf{u}, T)$ come from the (11) and (02) modes, i.e., from the heat conduction and stress modes. Only asymptotically in time are these contributions washed out of the picture.

We mention in passing that the fundamental solutions of the Navier–Stokes and Burnett equations are furnished to us by the calculations in II.[28]

As a second example of the theory we next consider the case associated with (I: 6.15).

NINE MOMENTS THEORY

The distinguished moments now are (ρ, T, u, S_1, Q) or in the b_{rl} notation $(b_{00}, b_{10}, b_{01}, b_{11}, b_{20})$. To terminate the natural equations [see (I: 6.15)] we need to find (b_{02}, b_{21}, b_{12}). From tables I through V of II, we get to lowest approximation (corresponding to the truncation of U_1)

$$\begin{bmatrix} b_{02} \\ b_{21} \\ b_{12} \end{bmatrix} = ik \begin{bmatrix} \frac{-2(5)^{\frac{1}{2}}}{3\lambda_{02}} & \frac{-2}{\lambda_{02}3^{\frac{1}{2}}} & 0 & \frac{2}{\lambda_{02}-\lambda_{20}}\left(\frac{2}{15}\right)^{\frac{1}{2}} & \frac{2}{\lambda_{02}-\lambda_{20}}\left(\frac{2}{15}\right)^{\frac{1}{2}} \\ 0 & 0 & 0 & \frac{1}{\lambda_{21}-\lambda_{20}}\left(\frac{7}{3}\right)^{\frac{1}{2}} & -\frac{1}{\lambda_{21}-\lambda_{20}}\left(\frac{7}{3}\right)^{\frac{1}{2}} \\ 0 & 0 & 0 & \frac{2}{(\lambda_{20}-\lambda_{21})}\left(\frac{7}{15}\right)^{\frac{1}{2}} & \frac{2}{(\lambda_{20}-\lambda_{12})}\left(\frac{7}{15}\right)^{\frac{1}{2}} \end{bmatrix}$$

$$\cdot \begin{bmatrix} \frac{3}{10} & \frac{1}{2}\left(\frac{3}{5}\right)^{\frac{1}{2}} & -\frac{1}{5}\left(\frac{3}{2}\right)^{\frac{1}{2}} & 0 & 0 \\ -\frac{1}{2}\left(\frac{3}{5}\right)^{\frac{1}{2}} & \frac{1}{2} & \frac{1}{(10)^{\frac{1}{2}}} & 0 & 0 \\ \frac{2}{5}\left(\frac{3}{2}\right)^{\frac{1}{2}} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} b_{00} \\ b_{01} \\ b_{10} \\ b_{11} \\ b_{20} \end{bmatrix}. \tag{5.16}$$

The first matrix is the truncated $\hat{V}_1$ and the second $\hat{V}_0^{-1}$. Carrying out the matrical operations, inverting the transform, and going to more familiar notation, we obtain

$$\begin{bmatrix} p_{11} \\ b_{21} \\ b_{12} \end{bmatrix} = \begin{bmatrix} \frac{4}{3\lambda_{02}} & \frac{8}{15(\lambda_{02}-\lambda_{20})} & 0 \\ 0 & 0 & \frac{1}{\lambda_{21}-\lambda_{20}}\left(\frac{7}{10}\right)^{\frac{1}{2}} \\ 0 & \frac{2}{5(\lambda_{20}-\lambda_{12})}\left(\frac{14}{3}\right)^{\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial S_1}{\partial x} \\ \frac{\partial Q}{\partial x} \end{bmatrix}. \tag{5.17}$$

We now compare this to (I: 6.15–18).[29] We see immediately that to obtain the latter from (5.17) we expand the denominators of (5.17) and only retain the first term. This bears out the previous remark that the method of Sec. III, the Grad interpolation technique, and its refinement in I all have implicit in it a second expansion. Successive terms of this expansion are picked up in successive interpolations. Regarding table I of I we see that such an expansion is unwarranted in practice. The forms in (5.17) have in them the exact summation of these

[28] The initial value fundamental solution of the linearized Navier–Stokes equations in one-dimension for Prandtl number of $\frac{3}{4}$ has been calculated by T. Y. Wu, J. Math. Phys. **35**, 13 (1956).
The method of (II) allows the rigorous calculation of the Navier–Stokes fundamental solution for all Prandtl numbers and in three dimensions. In addition, the Burnett solution can also be obtained.

[29] In (I: 6.17) replace $-$ by $+$, and in (I: 6.18) the second expression should contain λ_{21} instead of λ_{12}.

expansions. We see that the result of the differences of eigenvalues in (5.17) is to enhance the effect of the "interpolated" terms.

It is fairly easy to see where the interpolation argument breaks down.[30] In ignoring time derivative terms, interpolation implies that exponentially decaying terms are negligible. But on checking it is found that certain other exponentials are being kept, which is inconsistent. Interpolation may easily be made consistent by taking into account the relative rates of decay. In the method of Sec. III, this simply means that the systematic decay of (2.5) must be removed. The method of accomplishing this was given in Sec. III.

The initial conditions for the system (I: 6.16) with (5.17) are diagonal, i.e.,

$$(\bar{\rho}^0, \bar{u}^0, \bar{T}^0, \bar{S}_1^0, \bar{Q}^0) = (\rho^0, u^0, T^0, S^0, Q^5). \quad (5.18)$$

In general, the lowest order "ersatz" initial is always diagonal.

The asymptotic equations resulting from (I: 6.16) in conjunction with (5.17), and in general all such asymptotic equations need some word of explanation. Certainly for the asymptotic expansion in smoothness ratio (this is for time, see II), the equations have a simple explanation. They give a description of modes which for *a priori* reasons one has chosen to investigate. On the other hand, the time asymptotic theory becomes somewhat obscure, for, aside from the hydrodynamic modes, all modes have exponential decay. We must therefore view equations such as (I: 6.16) as giving the time asymptotic theory when for other reasons the hydrodynamic modes are negligible. Indeed, the nine moments theory contains the most persistent of the systematically decaying modes.

We have refrained from giving any but one-dimensional examples in the above. To do otherwise would have greatly increased the length of the paper. Also, the Maxwell eigenfunctions do not lend themselves as well to higher dimensions as they do to the one-dimensional case. It is hoped that in the near future a nonlinear theory will appear. The approach will necesarily be quite different and mathematically less satisfying, but the generalized normal solutions will fall out as a special case, and in three space dimensions.

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APPENDIX. SOME ELEMENTARY EXACT SOLUTIONS

As in Sec. V, we consider the one dimensional linearized Boltzmann equation in the diagonal approximation. In passing, we remark that the following treatment can be carried out for three dimensions and for nonlinear cases. At the present time we are only interested in exhibiting certain exact solutions and not investigating them. The simple one-dimensional case will suffice for this purpose.

We wish to find elementary exact solutions to the Boltzmann equation. Rather than deal with space derivatives we consider the transformed equation since it in some respects allows us more latitude. Equation (2.2) in the diagonal approximation can be written as (see I: 3.2)

$$\begin{aligned} 0 = \left(\frac{\partial}{\partial t} - \lambda_{\mu\nu}\right) b_{\mu\nu} \\ - ik\Big(\nu\Big\{\hat{b}_{\mu,\nu-1}\left[\frac{(\mu+\nu+\tfrac{1}{2})2}{(2\nu-1)(2\nu+1)}\right]^{\frac{1}{2}} \\ - b_{\mu+1,\nu-1}\cdot\left[\frac{2\mu+2}{(2\nu-1)(2\nu+1)}\right]^{\frac{1}{2}}\Big\} \\ - (\nu+1)\Big\{b_{\mu-1,\nu+1}\left[\frac{2\mu}{(2\nu+3)(2\nu+1)}\right]^{\frac{1}{2}} \\ - b_{\mu,\nu+1}\cdot\left[\frac{(\mu+\nu+3/2)2}{(2\nu+3)(2\nu+1)}\right]^{\frac{1}{2}}\Big\}\Big). \end{aligned} \quad (A1)$$

We first look for solutions which are of hydrodynamic type, i.e. for which the solution, in an essential way, depends only on (ρ, u, T), or in the notation of (A1) on (b_{00}, b_{01}, b_{10}). To do this we fix the following moments as indicated,

$$\begin{bmatrix} b_{02} \\ b_{in} \end{bmatrix} = \begin{bmatrix} a_{00}(ik) & a_{01}(ik) & a_{10}(ik) \\ a_{00}^n(ik) & a_{01}^n(ik) & a_{10}^n(ik) \end{bmatrix} \begin{bmatrix} b_{00} \\ b_{01} \\ b_{10} \end{bmatrix}, \quad n \geq 1. \quad (A2)$$

The $a(ik)$, within reason, are perfectly arbitrary. If an $a(ik)$ is a polynomial it indicates a differential

[30] For a description of the interpolation technique see the discussion on pages 273–274 of reference 8. Also the discussion following (I: 6.16).

operator, in general only a convolution product is indicated. Since b_{02} and b_{11} are proportional to the stress and heat conduction, (A2) renders the conservation equations determinate. On solving the now determinate conservation equations for $(\rho, \mathbf{u}, T)$ we next see that all $b_{\mu\nu}$ are determined from them without further equation solving. To see this one substitutes (A2) in (A1) and makes use of the fact that, from the conservation equations, using (A2), the time derivatives of (b_{00}, b_{10}, b_{01}) are linear functions of (b_{00}, b_{10}, b_{01}). It is then immediate that all b_{0n} are determined. Next, assuming all $b_{n\nu}$ are known in terms of (b_{00}, b_{10}, b_{01}), it is clear that (A1) gives all $b_{n+1,\nu}$.

As an example let the a's all be zero in (A2). We are therefore considering certain solutions of the Boltzmann equation which satisfy the inviscid equations exactly. Setting $\mu = 0$ in (A1), we have,

$$\left(\frac{\partial}{\partial t} - \lambda_{0\nu}\right)b_{0\nu} - ik\left(b_{0,\nu-1}\left[\frac{(\nu + \frac{1}{2})2}{(2\nu - 1)(2\nu + 1)}\right]^{\frac{1}{2}} + (\nu + 1)b_{0,\nu+1}\left[\frac{(\nu + 3/2)2}{(2\nu + 3)(2\nu + 1)}\right]^{\frac{1}{2}}\right). \quad \text{(A3)}$$

Then since $b_{02} = 0$ we see that all b_{0n} are determined by (A3). Next, setting $\mu = 1$ in (A1), we have

$$b_{2,\nu-1}\left[\frac{2}{(2\nu - 1)}\right]^{\frac{1}{2}} = -(\nu + 1)b_{0,\nu+1}\left[\frac{1}{2\nu + 3}\right]^{\frac{1}{2}}, \quad \text{(A4)}$$

which determines all b_{2n}. A glance at (A1) shows us that we may get all $b_{\mu\nu}$ in this manner. It is clear that by properly choosing the $a(ik)$, we may have $(\rho, \mathbf{u}, T)$ determined by the Navier–Stokes equations, the Burnett equations, or for that matter any step of the Hilbert–Chapman–Enskog procedure. Next, because of the arbitrariness in choosing the $a^n(ik)$, we see that there are infinitely many distribution functions which have the same $(\rho, \mathbf{u}, T)$.

It is furthermore evident that we may choose more than $(\rho, \mathbf{u}, T)$ as distinguished variables and by the same methods find various exact solutions in such variables. The remarks at the close of the last paragraph again apply.