Formal and Asymptotic Solutions in Kinetic Theory

LAWRENCE SIROVICH
Courant Institute of Mathematical Sciences, New York University, New York, New York
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Formal solutions are found to the linearized Boltzmann equation for the initial-value problem. These are decomposed into an infinity of modes, which are orthogonal under a suitable inner product. All but five of these modes exhibit an exponential time decay. These five remaining modes form a generalization of hydrodynamics. Under two different asymptotic assumptions one finds quantitative solutions. If the characteristic wavelength of the initial disturbance is large compared to the mean free path, the solution appears as an expansion in their ratio. If the elapsed time is large compared to the mean free time, the solution may be represented as an expansion in their ratio. As a specific example of the theory, the fundamental solution of the one-dimensional shear-free initial value problem is computed. This appears as an infinity of diffusing modes, a subclass of which also propagate.

I. INTRODUCTION

THIS is the second in a series of papers connected with the initial-value problem in linearized kinetic theory. The first paper[1] was essentially concerned with dispersion relations and solved no specific problems as such. On the other hand, in the present paper we shall develop solutions to the linearized Boltzmann equation. At another time, a general theory of approximate solutions to the Boltzmann equation will be given. This will, include for instance, the Chapman–Enskog theory as a special case. For this reason, no effort is made to connect approximate solutions, e.g. Chapman–Enskog, Moments Theory, Kinetic Models, to the exact results found here. To a large extent many of the natural questions in this context have been answered in reference 1.

In the section that follows we consider the general initial-value problem for the full, but linearized, Boltzmann equation. The solution is found in terms of an infinite series of modes. Although the solution is formal in the sense that one does not obtain precise quantitative results, it is nevertheless a source of information. For instance it is easily shown that all but five of the infinity of modes exhibit a systematic decay in time. These five modes which persist might suitably be considered a generalization of hydrodynamics. By suitably choosing the initial data, one can obtain flows which exhibit only these five modes for all time.

In order to obtain precise quantitative results, an asymptotic analysis of the formal solution is taken up in Sec. 3. Two separate asymptotic situations are studied. In the first instance, smooth initial data is considered. By this is meant that the characteristic wavelength of the initial data is large when compared with the mean free path. This first case is expressed in terms of an expansion in this smoothness ratio. The second case is an expansion in the ratio of mean free time to elapsed time. This is independent of the smoothness, and of course is given by the five hydrodynamic-type modes. Although the latter are dominated by the hydrodynamical moments of density, temperature, and velocity, they contain in general, all moments of the distribution function. Both asymptotic situations lead to the same functional forms. This is really very plausible since a flow naturally smooths itself in time.

As a specific example of the theory, the one-dimensional shear-free fundamental solution is worked out in Sec. 5. The hydrodynamic modes, now three in number, appear as a stationary diffusion, and two oppositely directed diffusing processes centered at the adiabatic speed. The remaining infinity of modes also consists of stationary and propagating diffusion processes. A discussion of extraordinary propagation has already been given in reference 1. It is clear that the hydrodynamic modes which are the sole survivors in time, are not of the same form as the solutions of the Chapman–Enskog theory. The former contains the initial data of the entire distribution function, whereas the latter contains only the hydrodynamic moments of the initial distribution function.

II. FORMAL SOLUTION OF THE LINEARIZED BOLTZMANN EQUATION

Following the notation customarily used,[2] the Boltzmann equation is given by

[1] L. Sirovich, Phys. Fluids **6**, 10 (1963).

[2] H. Grad, *Principles of the Kinetic Theory of Gases*, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, Germany, 1958), Vol. 12.

$$\left(\frac{\partial}{\partial t} + \xi\cdot\nabla\right)f = \frac{1}{m}\int (f'f'_* - ff_*)B(\theta, U)\, d\epsilon\, d\theta\, d\xi_*, \qquad (2.1)$$

with

$$U = |\xi_* - \xi|. \qquad (2.2)$$

Employing the notation of reference 1, linearization yields

$$\left(\frac{\partial}{\partial t} + \xi\cdot\nabla\right)g = L(g) = \int \omega_*[g]B\, d\epsilon\, d\theta\, d\xi_*. \quad (2.3)$$

All quantities in the latter have been made dimensionless with respect to an unspecified time scale τ according to reference 1, Eq. (2.3).

Let us designate the orthonormal eigenfunctions of L by $\{\psi_n\}$ such that

$$L(\psi_n) = \lambda_n\psi_n, \qquad (2.4)$$

and, as is easily shown,[3]

$$\lambda_n \leq 0. \qquad (2.5)$$

The eigenfunctions corresponding to the fivefold zero eigenvalue are

$$(1, \xi_1, \xi_2, \xi_3, \xi^2) = (\psi_0, \psi_1, \psi_2, \psi_3, \psi_4). \qquad (2.6)$$

Assuming completeness we expand g in a series of eigenfunctions

$$g = \sum a_n\psi_n \qquad (2.7)$$

and on substitution we get

$$\left(\frac{\partial}{\partial t} + \xi\cdot\nabla\right)\sum_{n=0}^{\infty} a_n\psi_n = \sum_{n=5}^{\infty} a_n\lambda_n\psi_n. \qquad (2.8)$$

This can be written as an infinite system of partial differential equations

$$\left(\frac{\partial}{\partial t} + (\mathbf{C}^1)\frac{\partial}{\partial x_1} + (\mathbf{C}^2)\frac{\partial}{\partial x_2} + (\mathbf{C}^3)\frac{\partial}{\partial x_3}\right)\mathbf{a} = \mathbf{\Lambda a} \quad (2.9)$$

where we have gone over to matrix notation. In the above

$$C^n_{ij} = \int \omega\bar{\psi}_i\psi_j\xi_n\, d\xi, \qquad (2.10)$$

$$\Lambda_{ij} = \lambda_i\,\delta_{ij} \text{ (no summation convention)}; \qquad (2.11)$$

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \end{bmatrix}, \qquad (2.12)$$

where the bar denotes the complex conjugate. It is immediately clear that each of the matrices are Hermitian and hence that the system is of the symmetric type.[4] The energy associated with such systems[1] in this case turns out to be the negative entropy, and of course, one then sees immediately that the volume entropy is a nondecreasing function of time.

To solve Eq. (2.9) subject to the initial conditions

$$\mathbf{a}(t = 0) = \mathbf{a}^0, \qquad (2.13)$$

we introduce Fourier transforms

$$\mathbf{a} = \frac{1}{(2\pi)^3}\int \exp\,[-i\mathbf{k}\cdot\mathbf{x}]\mathbf{a}(\mathbf{k})\, d\mathbf{k}, \qquad (2.14)$$

where the same letter has been used for the Fourier transformed variable. Introducing this, Eq. (2.9) becomes the system of ordinary differential equations

$$(\partial/\partial t - ik_\mu\mathbf{C}^\mu - \mathbf{\Lambda})\mathbf{a} = 0, \qquad k_\mu\mathbf{C}^\mu = k_1\mathbf{C}^1 + k_2\mathbf{C}^2 + k_3\mathbf{C}^3. \qquad (2.15)$$

The solution to this equation is immediate:

$$\mathbf{a}(\mathbf{k}) = \exp\,[(ik_\mu\mathbf{C}^\mu + \mathbf{\Lambda})t]\mathbf{a}^0(\mathbf{k}), \qquad (2.16)$$

and the solution to Eq. (2.9) is

$$\mathbf{a} = \frac{1}{(2\pi)^3}\int \exp\,[(ik_\mu\mathbf{C}^\mu + \mathbf{\Lambda})t]\,\exp\,[-i\mathbf{k}\cdot\mathbf{x}]\mathbf{a}^0(\mathbf{k})\, d\mathbf{k}. \qquad (2.17)$$

This solution is only formal since the exponential of the matrix sum must be evaluated before quadratures may be performed.

A better understanding of the solution (2.17) comes from the projector[5] decomposition of the exponential matrix, which will be performed in a moment. We first write:

$$\mathbf{G} = ik_\mu\mathbf{C}^\mu + \mathbf{\Lambda} = \mathbf{SDS}^{-1}, \qquad (2.18)$$

where $\mathbf{D}$ is the diagonal matrix of eigenvalues and $\mathbf{S}$ the matrix of column eigenvectors. Strictly speaking, $\mathbf{G}$ is not a Hermitian matrix. This is easily overcome however. We think of (ik) as being a triad of free variables and then $\mathbf{G}$ is a Hermitian form over these free variables. With this dodge, the pertinent theorems of matrix theory are carried over with a slight change of language. The eigenvalues of $\mathbf{D}$ are real functions of (ik) and the eigen-

[3] C. S. Wang Chang, and G. E. Uhlenbeck "On the Propagation of Sound in Monatomic Gases," Engineering Research Institute, University of Michigan (1952). This important report contains the first investigation of the linearized Boltzmann Equation and the attendant eigentheory.

[4] K. O. Friedrichs, Communs. Pure and Appl. Math. 7, 345 (1954).

[5] H. L. Hamburger and M. E. Grimshaw, *Linear Transformations in n-Dimensional Vector Space* (Cambridge University Press, 1951). For results in linear algebra one is referred to this volume.

vectors of **G** are orthogonal functions of (ik). We denote the eigenvectors by $\mathbf{v}^m$, so that

$$G_{ij}v_j^m = d^m v_i^m, \tag{2.19}$$

where we have for the moment gone to tensor notation. The matrix S is then given by

$$S_{ij} = v_i^j, \tag{2.20}$$

and trivially

$$S_{ij}^{-1} = \frac{\bar{v}_j^i}{|v^j|^2}, \tag{2.21}$$

where the bar denotes the complex conjugate. We can write

$$\exp[\mathbf{G}t] = \sum_m \mathbf{P}^m \exp[d_m t], \tag{2.22}$$

with

$$P_{ij}^m = v_i^m \bar{v}_j^m / |v^m|^2. \tag{2.23}$$

It is easily seen that $\mathbf{P}^m$ is Hermitian and that $\mathbf{P}^m \cdot \mathbf{P}^m = \mathbf{P}^m$, and so these are indeed projectors.[5]

On substituting Eq. (2.22) in Eq. (2.17), we decompose our solution into a sum of modes[6]

$$\mathbf{a} = \sum_m \boldsymbol{\alpha}^m, \tag{2.24}$$

$$\alpha_i^m = \frac{1}{(2\pi)^3} \int \exp[-i\mathbf{k}\cdot\mathbf{x} + d^m(\mathbf{k})t] \frac{v_i^m \bar{v}_\mu^m}{|v^m|^2} a_\mu^0 \, d\mathbf{k}. \tag{2.25}$$

As we shall see momentarily, each mode has associated with it a specific exponential decay. If the initial data $a^0(\mathbf{k})$ is decomposed in the eigenvectors of **G**, it is seen from Eq. (2.25) that the Fourier transform of $\boldsymbol{\alpha}_m$ is an eigenvector for all time. Further, if the $\mathbf{a}^0$ lacks any eigenvector, the corresponding mode is absent and never appears. This property is basic in the generalized Chapman–Enskog theory which will appear at another time.

We may put the modes (2.25) in yet another form which is very helpful in actual computations. As a normalizing condition on the eigenvectors we will take

$$v_m^m \equiv 1, \tag{2.26}$$

and therefore the "diagonal" mode member is

$$\alpha_m^m = \frac{1}{(2\pi)^3} \int \exp[-i\mathbf{k}\cdot\mathbf{x} + d^m(\mathbf{k})t] \frac{\bar{v}_\mu^m a_\mu^0}{|v^m|^2} \, d\mathbf{k}. \tag{2.27}$$

In keeping with the notation of using the same letter for both a variable and its transform we may write

$$\alpha_m^m(i\mathbf{k}) = \exp[d^m(\mathbf{k})\, t](\bar{v}_\mu^m a_\mu^0 / |v^m|^2). \tag{2.28}$$

Hence the lth member of the mth mode may be written as

$$\alpha_l^m = \frac{1}{(2\pi)^3} \int \exp[-i\mathbf{k}\cdot\mathbf{x}] \, d\mathbf{k} \, v_l^m * \alpha_m^m, \tag{2.29}$$

where there asterisk denotes the convolution product. The value of this representation will be seen later when asymptotics are considered. Then $v_l^m = v_l^m(i\mathbf{k})$ takes the form of a polynomial. Its transform is then understood in the sense of distributions and is

$$v_l^m(-\nabla_x). \tag{2.30}$$

We shall use this notation as a symbolic form and hence write for Eq. (2.29),

$$\alpha_l^m = v_l^m(-\nabla_x)\alpha_m^m(\mathbf{x}). \tag{2.31}$$

It is clear that by further exploiting this symbolism we can represent α_m^m in terms of a formal operator acting on the integral,

$$I^m = \frac{1}{(2\pi)^3} \int \exp[-i\mathbf{k}\cdot\mathbf{x} + d^m(\mathbf{k})] \, d\mathbf{k} \tag{2.32}$$

Some general properties of $d^m(k_1, k_2, k_3)$ can be found by going back to Eq. (2.9). It is clear that finding the d^m is equivalent to finding exponential solutions of Eq. (2.15) of the form

$$\mathbf{a} = \hat{\mathbf{a}}(k)e^{\sigma t}. \tag{2.33}$$

On substituting we get the "dispersion equation",

$$(-\sigma + ik_\mu \mathbf{C}^\mu + \boldsymbol{\Lambda})\mathbf{a} = 0. \tag{2.34}$$

Multiplying this by $\bar{\mathbf{a}}$ we have

$$\operatorname{Re}\sigma = (\bar{\mathbf{a}}, \boldsymbol{\Lambda}\mathbf{a})/|a|^2. \tag{2.35}$$

The negativity of $\boldsymbol{\Lambda}$ therefore shows that the real part of σ and hence each d^m, is nonpositive. Next it is clear on setting $k \equiv 0$ that the roots of the σ's start as

$$\sigma_i(\mathbf{k} = 0) = \lambda_i. \tag{2.36}$$

It is physically plausible that Re σ is a nonincreasing function of $|\mathbf{k}|$. This has so far proven too difficult to show analytically, except for Maxwell molecules, where some penetration can be made.[1] We shall assume this property on physical grounds. From this we have

$$|\exp[d_m t] - \lambda_m t| \leq 1,$$

and returning to Eq. (2.25), we have

$$|\alpha_i^m| \leq \frac{e^{\lambda_i t}}{(2\pi)^3} \int |v_i^m| \, |\bar{v}_j^m| \, |a_j^0(k)| \, d\mathbf{k}$$
$$< \exp[\lambda_i t]M; M, \text{ a constant}, \tag{2.37}$$

[6] A suitably defined scalar product of two vectors v and w is obtained by taking the customary scalar product of $v(ik)$ and $w(ik)$ considered as vector forms on ik. Under this inner product these modes are normal to one another.

where the absolute value sign is meant in the sense that the absolute value of each element of the matrix and each element of the vector are taken. The convergence of the integral is of course not known because the behavior of $v_\lambda^m \bar{v}_1^m$ is not known as $|\mathbf{k}| \to \infty$. However, we avoid this problem by taking $|a_i^0(\mathbf{k})|$ to be rapidly (enough) decaying as $|\mathbf{k}| \to \infty$.

Equation (2.37) clearly shows that all modes, except those corresponding to $\lambda_i = 0$, vanish exponentially and with a decay time that is not larger than the average time between collisions. Since the $\lambda_i = 0$ eigenvalue is fivefold, there are five separate modes which show no systematic decay in time. We refer to these as the hydrodynamical modes. As a word of caution it should be observed that the hydrodynamical modes do not involve only the hydrodynamical moments of density, temperature, and velocity (ρ, T, $\mathbf{u}$); in general they involve all moments of the distribution function. As will be seen, only under asymptotic conditions do we find that the hydrodynamic modes reduce to a finite number of moments [and in particular (ρ, T, $\mathbf{u}$)]; and make use only of a finite number of moments of the initial distribution function.

III. ASYMPTOTIC ANALYSIS

The solutions of the last section must remain strictly formal for us since no means are available for evaluating the infinite matrix. We must content ourselves with various asymptotic evaluations, which are given in Sec. V. In this section we develop the asymptotic formulas to be used later.

We now have need of formal power series representations of v^m and d^m, in the wave vector $\mathbf{k}$. To compute the series expansions, we first write formally

$$\sigma = d^m = \sum_{n=0}^{\infty} (i\mathbf{k})^n \cdot \mathbf{d}^{m,n}, \tag{3.1}$$

$$\mathbf{v}^m = \sum_{n=0}^{\infty} (i\mathbf{k})^n \cdot \mathbf{v}^{m,n}. \tag{3.2}$$

In these, $\mathbf{k}^n$ signifies the tensorial product of $n\mathbf{k}$ vectors, and both $\mathbf{d}^{m,n}$ and $\mathbf{v}^{m,n}$ are nth order tensors. The latter in addition, is a vector. Further, the dot denotes the inner product between tensors. These expansions (3.1) and (3.2) are substituted in Eq. (2.34) and the resulting equation is decomposed into orders of k. The procedure is standard and nothing further need be said.[1] [It should be noted, however, that the first term of Eq. (3.1) is already given by Eq. (2.36).] All further quantities may then be given in series representation by means of the simple formulas in the last section.

Using these expansions, two independent asymptotic situations are considered. One will correspond to an expansion in the ratio of mean free path to characteristic wavelength of the initial disturbance. The other will be an expansion in the ratio of mean free time to elapsed time. For initial data we consider the class of functions $\mathbf{a}$ whose transform satisfy

$$\mathbf{a}(\mathbf{k}) = 0 \quad \text{for} \quad |\mathbf{k}| > R, \tag{3.3}$$

where R is some arbitrary finite number. It will be clear in the analysis that this is far too severe a restriction. Since our goals are more physical than mathematical, we impose condition (3.3) to avoid tedious estimates and we take for granted that our results hold for a wider range of phenomena.

Smooth Phenomena

Rather than being specific, it will suffice to consider a typical integral

$$N = \int_{-\infty}^{\infty} \exp\left[-i\mathbf{k}\cdot\mathbf{x} + d(\mathbf{k})t\right] a\left(\frac{\mathbf{k}}{R}\right) d\mathbf{k}. \tag{3.4}$$

The condition (3.3) has been explicitly exhibited in the latter.

It is convenient in this case to think of the time scale τ as being unity. Next, it is clear that in Eqs. (3.1, 3.2)

$$\mathbf{d}^{m,n}, \mathbf{v}^{m,n} \sim O(1/\lambda^{n-1}), \tag{3.5}$$

where λ is representative of the eigenvalues λ_i. To see this, one has only divide each expression of Eq. (2.34) by the eigenvalue of that equation, and note that one is the solving for σ/λ in terms of $\mathbf{k}/\lambda$. For this reason we write

$$d = \lambda d^{(0)} + i\left(\frac{\mathbf{k}}{R}\right)\cdot d^{(1)} R + \left(\frac{\mathbf{k}}{R}\right)^2 \cdot d^{(2)} \frac{R^2}{\lambda} + \cdots \tag{3.6}$$

to explicitly indicate the parameter dependence. On substituting in Eq. (3.4) and performing the obvious coordinate transformation $\mathbf{n} = \mathbf{k}/R$ we have

$$N = R^3 \int_{|\mathbf{n}|<1} \exp\left[-i\mathbf{n}\cdot\mathbf{x}R\right] \exp\left[t \sum_{n=0}^{N} (i\mathbf{n})^n \cdot \mathbf{d}^{(n)} \left(\frac{R^n}{\lambda^{n-1}}\right)\right] a(\mathbf{n}) \left[1 + tO\left(\frac{R^{n+1}}{\lambda^n} \mathbf{n}^{n+1}\right) + \cdots\right] d\mathbf{n}. \tag{3.7}$$

One may show by standard techniques that

$$tRO\left[\left(\frac{R}{\lambda}\right)^n\right] = \frac{\left|\int \exp\left[-i\mathbf{n}\cdot\mathbf{x}R + t\sum_{n=0}^{N}(i\mathbf{n})^n\cdot\mathbf{d}^{(n)}\left(\frac{R^n}{\lambda^{n-1}}\right)\right]a(\mathbf{n})tRO\left[\left(\frac{R}{\lambda}\right)^n\mathbf{n}^{n+1}\right]d\mathbf{n}\right|}{\left|\int \exp\left[-i\mathbf{n}\cdot\mathbf{x}R + t\sum_{n=0}^{N}(i\mathbf{n})^n\cdot\mathbf{d}^{(n)}\left(\frac{R^n}{\lambda^{n-1}}\right)\right]a(\mathbf{n})\,d\mathbf{n}\right|}. \tag{3.8}$$

If we designate the mean free path by l and the characteristic length scale of the disturbance by L, then

$$R \sim 1/L; \qquad \lambda \sim 1/l. \tag{3.9}$$

Hence Eq. (3.7) leads to an expansion in the smoothness ratio

$$\epsilon = l/L. \tag{3.10}$$

If we think of R as fixed and, for instance, $O(1)$, we see that our expansion, while not uniformly valid in time, is good for, say, $0 \leq t < \lambda^n$. Naturally this is extended by choosing more terms in the exponent of Eq. (3.7). The latter is offset by the practical consideration of evaluating the resulting integrals. As we see shortly, these expansions are valid for all time, through entirely different conconsiderations.

In several instances the approximate integrals may be evaluated; for instance consider the one-dimensional integral

$$\exp[-d^{(0)}t]\int_{-R}^{R}\exp[-ik(x - d^{(1)}t) - d^{(2)}k^2]\cdot a(k)\,dk = \exp[-d^0(t)]\int_{-\infty}^{\infty}\cdots \tag{3.11}$$

which we shall call the Navier–Stokes[7] form. It is natural to consider the corresponding fundamental integral obtained by taking $a(k)$ to be a constant. This gives

$$a\exp[-d^0(t)]\int_{-\infty}^{\infty}\exp[-d^{(2)}k^2t - ik(x - d^{(1)}t)]\,dk = a\left[\frac{\pi}{d^{(2)}t}\right]^{\frac{1}{2}}\exp[-d^{(0)}t]\exp\left[\frac{-(x - d^{(1)}t)^2}{4d^{(2)}t}\right]. \tag{3.12}$$

The corresponding Navier–Stokes form of the fundamental integrals in two and three dimensions can be found without any difficulty and are essentially products of one-dimensional forms. In one dimension, even a cubic term may be retained in the exponent and we shall refer to this as the Burnett[7] form. The evaluation, which is given in Appendix A, is in terms of the Airy function.

It should be noted in Eq. (3.12) that on allowing $d^{(2)} \to 0$, the form approaches a delta "function." This however is not uniform in time.

Long-Time Behavior

Although we again retain condition (3.3), the "smoothness" requirement is now completely relaxed. From Eq. (2.31) and the discussion following it, we know that all modes are exponentially small (in time) compared to the hydrodynamic mode. We consider these as asymptotically negligible and turn to the asymptotic behavior, in time, of the hydrodynamic modes. A typical term is

$$M = \int_{-\infty}^{\infty}\exp[-i\mathbf{k}\cdot\mathbf{x}]\exp[d(\mathbf{k})t]a(\mathbf{k})\,d\mathbf{k}, \tag{3.13}$$

with

$$d(\mathbf{k}) = \sum_{n=1}^{\infty}(i\mathbf{k})^n\cdot\mathbf{d}^{(n)}. \tag{3.14}$$

As before, $\mathbf{k}^n$ denotes the tensorial product and $\mathbf{d}^{(n)}$ an nth order tensor. It is now convenient to think of the normalization τ as being the mean free time.

We decompose Eq. (3.13) as follows:

$$M = M_\epsilon + M_r = \int_{|\mathbf{k}|\leq\epsilon}\exp[-i\mathbf{k}\cdot\mathbf{x} + dt]a\,d\mathbf{k} + \int_{|\mathbf{k}|>\epsilon}\exp[-i\mathbf{k}\cdot\mathbf{x} + dt]a\,d\mathbf{k}. \tag{3.15}$$

From this we have

$$|M_r| \leq \int_{|\mathbf{k}|>\epsilon}\exp[d_rt]\,|a|\,d\mathbf{k}, \tag{3.16}$$

where d_r denotes the real part of d. Here ϵ denotes a small quantity which is still to be specified. Next it is clear from Eqs. (2.35, 36) that d_r vanishes only if $\mathbf{k} = 0$, and so by continuity, d_r is bounded away from zero for $\mathbf{k} \neq 0$. Hence we have

$$\exp[-d_rt] \leq \exp[-C\epsilon^2t] \qquad |k| \geq \epsilon, \tag{3.17}$$

[7] Referring to such integrals as the Navier–Stokes and Burnett forms is, strictly speaking, appropriate to the hydrodynamic modes. It is shown in reference 1 that the expansion (3.1) for the hydrodynamic modes is given correctly to $O(k^2)$ by the Navier–Stokes equations and to $O(k^3)$ by the Burnett equations. We shall use these appellations in general to refer to the number of terms carried in the exponent of an integral of type (3.11).

where C is some positive constant. Therefore from Eq. (3.3) and Eq. (3.16),

$$|M_r| < A \exp[-C\epsilon^2 t], \tag{3.18}$$

where A is some constant.

Considering M_ϵ we write

$$M_\epsilon = \int_{|k|\le\epsilon} \exp\left[-i\mathbf{k}\cdot\mathbf{x} + t\sum_{n=1}^{N}(i\mathbf{k})^n\cdot\mathbf{d}^{(n)}\right] \cdot[1 + tO(\mathbf{k}^{N+1}) + \cdots]a(\mathbf{k})\,d\mathbf{k}. \tag{3.19}$$

We now take

$$\epsilon = 1/t^\alpha; \qquad \tfrac{1}{2} < \alpha < 1/(N+1). \tag{3.20}$$

It is evident that the expression (3.18) is exponentially small, falling off by some fractional power of t. To show that the second term of Eq. (3.20) is asymptotically small when compared with the first is not difficult. One way is to expand the exponent about $N = 2$. This leads to an integral which may be evaluated and the desired property found by inspection.

The freedom in α as given by Eq. (3.20) is significant and is a source of useful information. We now demonstrate this by examining the one-dimensional form of Eq. (3.19) specifically for $N = 2, 3$.

$$M_\epsilon = \int_{-\epsilon}^{\epsilon} \exp[-ik(x - d^{(1)}t) - d^{(2)}k^2t] \cdot[\beta_0 + \beta_1 k + \beta_2 k^2 + \cdots] \cdot[1 + tO(k^3) + \cdots]\,dk. \tag{3.21}$$

To begin the discussion let us take

$$\epsilon = 1/t^{(\frac{1}{2})^-} \tag{3.22}$$

where $(\frac{1}{2})^-$ is a number slightly smaller than $\frac{1}{2}$. Then, on writing $O(k^3) = id^3k^3 + O(k^4)$, we have, to lowest orders,

$$M_\epsilon \sim \int_{-\infty}^{\infty} \exp[-ik(x - d^{(1)}t) - d^{(2)}k^2t] \cdot[\beta_0 + \beta_1 k + i\beta_0 t d^{(3)}k^3]\,dk, \tag{3.23}$$

where the integration has been extended to infinity since this contributes negligibly. This type of integral has already been evaluated in Eq. (3.12). It is clear from that form that Eq. (3.23) is only valid in x–t space for

$$(x - d^{(1)}t) < \pm(t)^{\frac{1}{2}} \tag{3.24}$$

since outside of this region, Eq. (3.23) and $\exp[-C\epsilon^2 t]$ are of the same order. On the other hand if we take

$$\epsilon = 1/(t)^{(\frac{1}{3})^+} \tag{3.25}$$

where $(\frac{1}{3})^+$ signifies a quantity slightly larger than $\frac{1}{3}$, our results are modified. Next to lowest terms we see that we get

$$M_\epsilon \sim \int_{-\infty}^{\infty} \exp[-ik(x - d^{(1)}t) - d^{(2)}k^2t] \cdot[\beta_0 + i\beta_0 t d^{(3)}k^3]\,dk. \tag{3.26}$$

But on taking $(\frac{1}{3})^+$ closer and closer to $\frac{1}{3}$ we see that all powers of k^3t enter or more precisely we then must consider

$$M_\epsilon \sim \int_{-\infty}^{\infty} \exp[-ik(x - d^{(1)}t) - d^{(2)}k^2t + id^{(3)}k^3t]\beta_0\,dk. \tag{3.27}$$

This is evaluated in Appendix A and is referred to as the Burnett integral. Also since $\exp[-C\epsilon^2 t]$ now decays more rapidly, Eq. (3.27) may be used in the enlarged region

$$(x - d^{(1)}t) < \pm t^{\frac{2}{3}}. \tag{3.28}$$

We can now further delimit our results. This is done by continuing our discussion of the one-dimension integral; however the remarks are applicable to the general case. By employing more terms in the exponent of Eq. (3.27) we can choose α of Eq. (3.20) to be larger and larger. This has the effect of enlarging the domain of validity of the asymptotic expansion. By retaining all terms in the exponent, i.e. by retaining $d(k)$ exactly, our results hold everywhere. This however should not be misunderstood. It is rather the case that the exact evaluation of the integral holds in a limited region of the x–t plane. Let us examine the behavior of the integral

$$\int_{-\infty}^{\infty} \exp[-ikx + d(k)t]\,dk \tag{3.29}$$

as we move out along rays from the origin in the x–t plane. To do this we set $x = st$ and we consider

$$\int_{-\infty}^{\infty} \exp\{t[d(k) - isk]\}\,dk \tag{3.30}$$

as $t \to \infty$. This, however, is in a form which lends itself to the saddle-point technique. This technique produces an asymptotic form of the type $\sim\exp(ht)$. Since we do not have the functional form of $d(k)$ we cannot locate the saddle point (even if we had the closed form expression for $d(k)$, it is highly unlikely that we could locate the saddle). On the other hand

$$\int_{-\infty}^{\infty} |\exp[td(k) - isk]|\,dk \tag{3.31}$$

has a saddle at the origin and hence asymptotically behaves as $O(1/t^{\frac{1}{2}})$. Therefore Re h in $\exp(ht)$ is

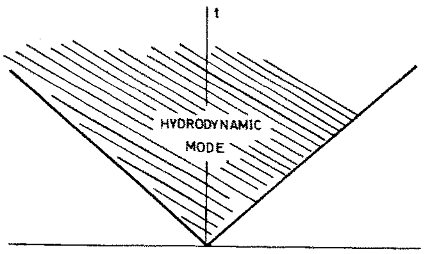

FIG. 1. Range of influence of the hydrodynamic modes.

nonpositive. The point at which we are driving is that along rays, the integral (3.29) is exponentially damped. Further on physical (and mathematical) grounds, this damping increases with α. But in considering the asymptotic limit of $t \to \infty$ we neglect the nonhydrodynamic modes which are $O(\exp[\lambda t])$. Therefore the region in which the $t \to \infty$ limit can be used is $|x/t| < \mathcal{K}$, for some $\mathcal{K}$, as, is indicated in the Fig. 1. If one wants to extend this region higher modes must be included in the analysis, until if one wants to consider $t \to \infty$ but for all x, then all modes must be considered.

Each of the asymptotic cases, as we have just seen, lead to the same types of terms. This is also clear from the fact that each made use only of the expansions for small k. There remains now only the task of connecting the two asymptotic expansions. This can be done only for smooth flows, since nothing can be said of the early evolution of irregular initial data. The connection for smooth initial data is clear since the same functional forms occur. For long times, one has only to neglect the modes with exponential decay, i.e., retain only the hydrodynamic modes.

IV. MAXWELL EIGENFUNCTIONS

We now give a brief description of the eigenfunction theory of $L(g)$. It is due mainly to the pioneering work of Wang Chang and Uhlenbeck.[3]

The orthonormal eigenfunctions of $L(g)$ for Maxwell molecules are given by[8]

$$\Phi_{rlm} = S_{l+\frac{1}{2}}^{(r)}(\tfrac{1}{2}\xi^2)\xi^l P_l^m(\cos\theta)\exp(im\chi)/N_{rlm}^{\frac{1}{2}}, \tag{4.1}$$

with

$$N_{rlm} = \frac{2^{l+1}\Gamma(r+l+\frac{3}{2})}{\pi^{\frac{1}{2}}r!(2l+1)}\frac{(l+|m|)!}{(l-|m|)!}. \tag{4.2}$$

Here θ and χ are the angular variables in spherical velocity space. The functions (4.1) satisfy the eigenrelation,

$$L(\Phi_{rlm}) = \lambda_{rlm}\Phi_{rlm}. \tag{4.3}$$

[8] Here S_l^r are Laguerre polynomials and P_l^m Legendre polynomials. See reference 1 for more details of notation.

Wang Chang and Uhlenbeck[3] have shown that λ_{rlm} are at least $(2l+1)$-fold degenerate—not depending on the subscript m at all. In addition one finds[1]

$$\lambda_{r0} = \lambda_{r-1}, \tag{4.4}$$

and

$$\lambda_{30} = \lambda_{21} = \lambda_{02}. \tag{4.5}$$

These degeneracies are of extreme importance in our later work. Many of the eigenvalues have been evaluated and a table is given in reference 1.

Several examples of the eigenfunctions (4.1) and their associated coefficients are given in reference 1, Eq. (2.12). The application of Maxwell eigenfunctions to other molecules is also indicated in reference 1.

V. ASYMPTOTIC SOLUTIONS

As a specific example of the general theory developed, we now work out the asymptotic theory of the one-dimensional shear-free initial-value problem.[9] Two and three dimensions offers nothing new and is only more tedious. Our results will be displayed as modes according to the decomposition of Sec. II. With the exception of the hydrodynamic modes, all modes will be given only to lowest order. The former, because of its special interest, is carried further. Appendix B, contains sufficient data to carry the calculations to even higher orders.

In accordance with the remarks of the last section, we introduce a more specialized notation. First since we consider only one-dimensional motions, the matrix $\mathbf{G}$ becomes

$$ik_1\mathbf{C}^1 + \mathbf{\Lambda} = ik\mathbf{C} + \mathbf{\Lambda} \tag{5.1}$$

As is pointed out in reference 1, we may suppress the index m for one-dimensional shear-free motions. The eigenvector expansion will now be written as

$$v_{ij}^{\mu\nu} = \sum_{n=0}(ik)^n v_{ij}^{\mu\nu,n} \tag{5.2}$$

and the eigenvalues as

$$d^{\mu\nu} = \sum_{n=0} d^{\mu\nu,n}(ik)^n, \tag{5.3}$$

with

$$d^{\mu\nu,0} = \lambda_{\mu\nu}. \tag{5.4}$$

Note that $(\mu\nu)$ refers to the eigenvector (or correspondingly the mode) and ij to the component. A certain ordering of eigenvalues must be assumed, a discussion of this is given in reference 1. In general it is taken, roughly, to correspond to increasing magnitudes of λ_{rl}. Eigenvalues and eigenvectors for the case at hand are given in Appendix B.

[9] The analysis is carried out for Maxwell molecules, but can also be taken as giving the diagonal approximation for other molecular force laws. (See reference 1, Sec. 5).

The types of modes which occur can be split into three categories, according to the degeneracy of the corresponding λ_{rl}. These will be referred to as the nondegenerate, doubly degenerate, and the triply degenerate hydrodynamic modes.[10] Rather than consider any specific initial data, we shall consider the fundamental solution. To do this we replace $\mathbf{a}_0(k)$ by the constant vector

$$\boldsymbol{\beta} = \{\beta_{rl}\}. \tag{5.5}$$

The solution for some particular initial data may then be found in terms of a convolution integral.

Hydrodynamic Modes

The hydrodynamical modes are now three in number, corresponding to

$$\lambda_{00} = \lambda_{10} = \lambda_{01} = 0. \tag{5.6}$$

As we saw in Sec. III, both the long-time and smoothness-ratio expansions follow from an expansion in wavenumber. As stated in Sec. II, we shall express each mode in terms of the diagonal mode member. It is not as yet necessary to give asymptotic forms for the latter. We can first obtain more general relationships from the eigenvector expansion of Appendix B and relation (2.31). This immediately gives the following asymptotic representations of the hydrodynamic modes.

$$\begin{bmatrix} \alpha_{00}^{00} \\ \alpha_{01}^{00} \\ \alpha_{10}^{00} \\ \alpha_{11}^{00} \\ \alpha_{02}^{00} \end{bmatrix} \sim \begin{bmatrix} 1 \\ (\frac{5}{3})^{\frac{1}{2}} + (1/3\lambda_{11} + 2/3\lambda_{02})\,\partial/\partial x \\ -(\frac{2}{3})^{\frac{1}{2}} - [(10)^{\frac{1}{2}}/3\lambda_{11}]\,\partial/\partial x \\ -[(10)^{\frac{1}{2}}/3\lambda_{11}]\,\partial/\partial x \\ [2(5)^{\frac{1}{2}}/3\lambda_{02}]\,\partial/\partial x \end{bmatrix} \alpha_{00}^{00}, \tag{5.7}$$

$$\begin{bmatrix} \alpha_{00}^{01} \\ \alpha_{01}^{01} \\ \alpha_{10}^{01} \\ \alpha_{11}^{01} \\ \alpha_{02}^{01} \end{bmatrix} \sim \begin{bmatrix} -(\frac{3}{5})^{\frac{1}{2}} - \frac{3}{5}(1/3\lambda_{11} + 2/3\lambda_{02})\,\partial/\partial x \\ 1 \\ (\frac{2}{5})^{\frac{1}{2}} - [(\frac{2}{3})^{\frac{1}{2}}(4/5\lambda_{11} - 2/5\lambda_{02})]\,\partial/\partial x \\ [(1/\lambda_{11})(\frac{2}{3})^{\frac{1}{2}}]\,\partial/\partial x \\ [2/(3)^{\frac{1}{2}}\lambda_{02}]\,\partial/\partial x \end{bmatrix} \alpha_{10}^{10}, \tag{5.8a}$$

$$\begin{bmatrix} \alpha_{00}^{10} \\ \alpha_{01}^{10} \\ \alpha_{10}^{10} \\ \alpha_{11}^{10} \\ \alpha_{02}^{10} \end{bmatrix} \sim \begin{bmatrix} (\frac{2}{3})^{\frac{1}{2}} \\ [(1/\lambda_{11})(\frac{2}{3})^{\frac{1}{2}}]\,\partial/\partial x \\ 1 \\ [(1/\lambda_{11})(\frac{5}{3})^{\frac{1}{2}}]\,\partial/\partial x \\ 0 \end{bmatrix} \alpha_{10}^{10}. \tag{5.8b}$$

On referring back to reference 1, Eq. (2.12), we see that to the lowest order for each mode we have

$$\frac{p_{11}}{p_0} = \left(\frac{4}{3\lambda_{02}}\frac{\partial}{\partial x}\right)\frac{u}{(RT_0)^{\frac{1}{2}}}, \tag{5.9}$$

and

$$\frac{S_1}{p_0(RT_0)^{\frac{1}{2}}} = \left(\frac{5}{2\lambda_{11}}\frac{\partial}{\partial x}\right)\frac{T}{T_0}. \tag{5.10}$$

Hence to this order the Navier–Stokes relations hold independently of the approximation to the diagonal mode members. Further to the lowest order, for the (00) and (01) mode, we have the following density–temperature relation:

$$T/T_0 = \tfrac{2}{3}\,\rho/\rho_0. \tag{5.11}$$

Since in linearized theory the "gas low" (identity) is

$$p/p_0 = \rho/\rho_0 + T/T_0, \tag{5.12}$$

Eq. (5.11) expresses that the (00) and (01) are adiabatic to lowest order. On the other hand, inspection of the (10) mode shows it has the isobaric relation,

$$\rho/\rho_0 = -T/T_0. \tag{5.13}$$

This again is to lowest order.

We must still evaluate the diagonal mode members. First we define the following integrals:

$$I^{\mu\nu} = \frac{1}{2\pi}\int \exp\,[-ikx + d^{\mu\nu}t]\,dk. \tag{5.14}$$

Next from the expansions of the eigenvectors in Appendix B, we can write to first order

$$\alpha_{00}^{00} \sim \left\{\left[\frac{3}{10}, \frac{1}{2}\left(\frac{3}{5}\right)^{\frac{1}{2}}, -\left(\frac{6}{10}\right)^{\frac{1}{2}}\right]\begin{bmatrix}\beta_{00}\\ \beta_{01}\\ \beta_{10}\end{bmatrix} - \left[\frac{1}{10}\left(\frac{3}{5}\right)^{\frac{1}{2}}\left(\frac{3}{\lambda_{11}} + \frac{2}{\lambda_{02}}\right), \frac{1}{5\lambda_{11}}, \frac{1}{5}\left(\frac{3}{5}\right)^{\frac{1}{2}} \cdot\left(\frac{1}{\lambda_{11}} - \frac{1}{\lambda_{02}}\right), \frac{1}{\lambda_{11}(10)^{\frac{1}{2}}}, \frac{1}{\lambda_{02}(5)^{\frac{1}{2}}}\right] \cdot \begin{bmatrix}\beta_{00}\\ \beta_{01}\\ \beta_{10}\\ \beta_{11}\\ \beta_{02}\end{bmatrix}\frac{\partial}{\partial x}\right\} I^{00}, \tag{5.15}$$

[10] There is in addition a triply degenerate mode corresponding to Eq. (4.5). This in reality, however, behaves separately as nondegenerate and doubly degenerate modes.

$$\alpha_{01}^{01} \sim \left\{ \left[-\frac{1}{2}\left(\frac{3}{5}\right)^{\frac{1}{2}}, \frac{1}{2}, \frac{1}{(10)^{\frac{1}{2}}} \right] \begin{bmatrix} \beta_{00} \\ \beta_{01} \\ \beta_{10} \end{bmatrix} - \left[\frac{1}{5\lambda_{11}}, -\frac{1}{(15)^{\frac{1}{2}}}\left(\frac{1}{2\lambda_{11}} - \frac{1}{\lambda_{02}}\right), \frac{1}{5\lambda_{11}}\left(\frac{3}{2}\right)^{\frac{1}{2}}, -\frac{1}{\lambda_{11}(6)^{\frac{1}{2}}}, -\frac{1}{\lambda_{02}(3)^{\frac{1}{2}}} \right] \begin{bmatrix} \beta_{00} \\ \beta_{01} \\ \beta_{10} \\ \beta_{11} \\ \beta_{02} \end{bmatrix} \frac{\partial}{\partial x} \right\} I^{01}, \tag{5.16}$$

$$\alpha_{10}^{10} \sim \left\{ \left[\frac{(6)^{\frac{1}{2}}}{5}, \frac{3}{5} \right] \begin{bmatrix} \beta_{00} \\ \beta_{10} \end{bmatrix} - \left[-\frac{(6)^{\frac{1}{2}}}{5\lambda_{11}}, \frac{1}{\lambda_{11}}\left(\frac{3}{5}\right)^{\frac{1}{2}} \right] \begin{bmatrix} \beta_{01} \\ \beta_{11} \end{bmatrix} \frac{\partial}{\partial x} \right\} I^{10}. \tag{5.17}$$

From the work of Sec. III we know that our considerations apply to two different asymptotic limits. On one hand is the theory in the limit of large collision frequency (or equivalently smooth phenomena), and on the other hand is the theory in the limit of large times. The former analysis was restricted to finite time, but this was lifted by the latter development. The time asymptotic analysis makes no demands on the smoothness of the initial data.

The basic difference in these two approaches has an immediate effect on the interpretation of the results given above and in particular on the fact that $I^{\mu\nu}$ was not given any particular approximation. First let us regard the case of smooth initial data, and consider the type of integral which occurs:

$$I \sim \int_{-\infty}^{\infty} \exp\,[d(k)t - ikx][1 + O(k)]\,dk. \tag{5.18}$$

From the analysis of Sec. III we know that the term $O(k)$ for any finite time, is of a lower order than the contribution of an infinite remainder in the expansion of $d(k)$. However any term in the expansion of $d(k)$ is of a lower order than $O(k)$ for sufficiently large t. Therefore, including more and more terms in the approximation of $d(k)$ extends the validity in time until, on keeping all the terms, it holds for all time.

Improving the approximation of $d(k)$ in the long-time theory has quite another effect. There is now no requirement on the smoothness of the initial data. (As already remarked, a flow naturally smooths itself in time). In Sec. III, we found that the inclusion of more terms in the approximation of $d(k)$ extended the region, in x space, of the long-time asymptotics; that for x sufficiently large, any term in the expansion of $d(k)$ was of lower order than $O(k)$ in Eq. (5.18). By finally choosing the exact expression for $d(k)$ the integral is valid over all x as the lowest-order contribution. But the asymptotic theory has already neglected exponentially decaying terms. Therefore as shown in Sec. III, the integral is only valid in x–t region indicated in Fig. 1. This distinction between the value of the integral (5.18) and its role in an asymptotic theory is extremely important.

To reconcile the two asymptotic developments of the hydrodynamic modes we note that the hydrodynamical mode has the same form in both expansions. From the analysis of the long-time behavior, we know that the finite-time requirement in the smooth phenomena theory may be lifted, and that those results are valid for all time. It is however the case, that as time becomes large, these results can be used only in some finite x interval. This interval is delimited by Fig. 1, and its extent in that region is determined by the level of the approximation.

An interesting comparison between the two asymptotic theories comes from entropy considerations. Rather than entropy we must of course consider the H function mentioned in Sec. II. From that section we see that for large times the H function is given only by the contribution from the hydrodynamic mode, but the smooth-flow expansion has contributions from all the other modes. They of course agree for large times since the contributions from the higher modes vanish.

We now turn to specific approximations for the integrals $I^{\mu\nu}$. By retaining $O(k^2)$ in the exponent, the Navier–Stokes approximation is obtained and we denote it by $I^{\mu\nu}_{NS}$. For the hydrodynamic mode we have

$$I^{00}_{NS} = \left(\frac{3\lambda_{11}\lambda_{02}}{4\pi t\,|\lambda_{02} + 2\lambda_{11}|}\right)^{\frac{1}{2}} \cdot \exp\left[\frac{(x - (\frac{5}{3})^{\frac{1}{2}}t)^2 3\lambda_{11}\lambda_{02}}{4(\lambda_{02} + 2\lambda_{11})t}\right], \tag{5.19}$$

$$I^{01}_{NS} = \left(\frac{3\lambda_{11}\lambda_{02}}{4\pi t\,|\lambda_{02} + 2\lambda_{11}|}\right)^{\frac{1}{2}} \cdot \exp\left[\frac{(x + (\frac{5}{3})^{\frac{1}{2}}t)^2 3\lambda_{11}\lambda_{02}}{4(\lambda_{02} + 2\lambda_{11})t}\right], \tag{5.20}$$

$$I^{01}_{NS} = \left(\frac{|\lambda_{11}|}{4\pi t}\right)^{\frac{1}{2}} \exp\left(\frac{x^2\lambda_{11}}{4t}\right). \tag{5.21}$$

Each of these forms characterizes a diffusion process, but in the first two [Eqs. (5.19), (5.20)], this is centered about a propagating wave. The speed of the wave is $(\frac{5}{3})^{\frac{1}{2}}$ which is the adiabatic speed in our normalization. Only the zeroth order expression may be validly used with the Navier–Stokes integrals.[11] This is true since the correction term $O(k^3t)$ is of the same order as $O(k)$ which leads to the second-order theory (see Sec. III). Further the first two modes [Eqs. (5.15), (5.16)] apply in a region given by

$$[x - (\tfrac{5}{3})^{\frac{1}{2}}t] < \pm(t)^{\frac{1}{2}}, \qquad (5.22)$$

and the third mode [Eq. (5.21)], on a region given by

$$x < \pm(t)^{\frac{1}{2}}. \qquad (5.23)$$

Outside of these regions in x–t space, the solution is ~ 0, to this order. These regions of dependence are sketched in Fig. 2.

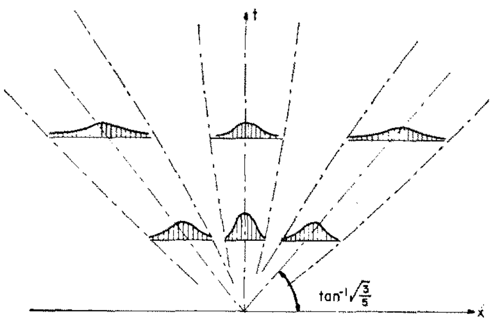

FIG. 2. Evolution of the hydrodynamical modes.

If terms of $O(k^3)$ are retained in the approximation to $d(k)$ in the evaluation, we get the Burnett integral and write it as

$$I_B^{\mu\nu}.$$

The evaluation and discussion for this integral is given in Appendix A. On using this integral we may now use the relations (5.7)–(5.9) and (5.15)–(5.17) to describe the hydrodynamic mode to the first order. Also the range of application in x–t space is widened and for instance the I_B^{00} integral may be used in

$$[x - (\tfrac{5}{3})^{\frac{1}{2}}t] < \pm(t)^{\frac{1}{3}} \qquad (5.24)$$

The range of application of I_{NS}^{00} and I_B^{00} are sketched

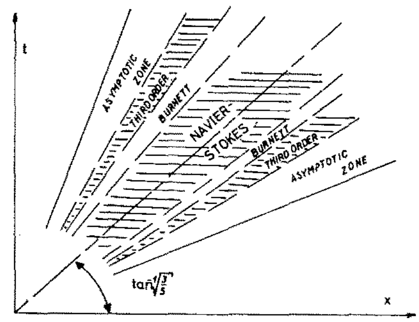

FIG. 3. Domains of accuracy in the Chapman–Enskog integrals.

in Fig. 3. Also the effect of retaining $O(k^4)$ in the exponent, or what is designated in the sketch as third-order theory. By choosing more terms in the approximation of the eigenvalue $d(k)$, the range of application is widened. However, it will not pass out of the wedge region. Past this region, the integral is of an exponential decay comparable to already neglected quantities (the high modes).

Nondegenerate Modes

From Appendix B, the lowest order is

$$\alpha_{\mu\nu}^{\mu\nu} \sim \frac{\beta_{\mu\nu} \exp\,[\lambda_{\mu,\nu}t - x^2/4d^{\mu\nu,2}t]}{(4\pi d^{\mu\nu,2}t)^{\frac{1}{2}}}. \qquad (5.25)$$

Aside from the diagonal mode member, all other entries vanish to the lowest order. We see that in addition to the systematic decay in time, we have diffusion. A sketch of this behavior is given in Fig. 4.

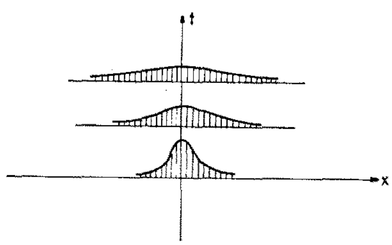

FIG. 4. Evolution of a typical non-degenerate mode.

Doubly-Degenerate Mode

It will be remembered that in this case $\lambda_{r0} = \lambda_{r-1,1}$. With the results of Appendix B, we have for the two modes associated with this degeneracy,

$$\alpha_{r-1,1}^{r-1,1} \sim \tfrac{1}{2}(\beta_{r-1,1} - \beta_{r,0})\,\frac{\exp\,[-(x + d^{r0,1}t)^2/4d^{r0,2}t] + \lambda_{r0}t}{\{4\pi d^{r0,2}t\}^{\frac{1}{2}}}, \qquad \alpha_{r0}^{r0} \sim -\alpha_{r-1,1}^{r-1,1}, \qquad (5.26)$$

$$\alpha_{r0}^{r0} \sim \tfrac{1}{2}(\beta_{r-1,1} + \beta_{r,0})\,\frac{\exp\,[-(x - d^{r0,1}t)^2/4d^{r0,2}t] + \lambda_{r0}t}{\{4\pi d^{r0,2}t\}^{\frac{1}{2}}}, \qquad \alpha_{r-1,1}^{r-1,1} \sim \alpha_{r0}^{r0}. \qquad (5.27)$$

Aside from the systematic decay, we now have a diffusing wave traveling to the right and to the left, each having the same speed. This is sketched in Fig. 5.

[11] By this is meant undifferentiated terms. It is easily seen that the notion of order in each of the expansions can be transferred to the appearance of derivatives. Note that both Eqs. (5.7–5.9) and Eqs. (5.15–5.17) fall into orders according to appearance of derivatives.

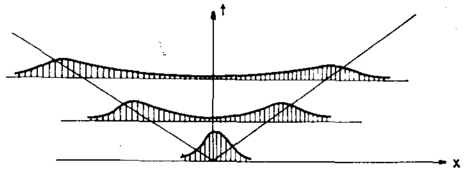

FIG. 5. Evolution of a typical doubly degenerate mode.

The expression for $d^{r0,1}$, $d^{r0,2}$ in terms of the λ's are to be found in Appendix B. In only taking the lowest orders, we have obscured an important property of these modes. This is the fact that a single mode will in general, contribute to an infinity of moments. This can be seen by examining the eigenvectors in Appendix B. This mushrooming of a mode was clearly exhibited for the hydrodynamic modes.

In closing we can briefly note that the solution to the Boltzmann equation is far more complex than given by the Chapman–Enskog theory. The latter can give no more than the hydrodynamic modes and hence can at best be a time asymptotic theory. Further, the Chapman–Enskog theory takes into consideration only the initial data of the hydrodynamical moments. This is clearly insufficient for a general theory as is seen from Eqs. (5.15–5.17). Nevertheless the Chapman–Enskog theory has a valid asymptotic sense, as will be demonstrated at another time.

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APPENDIX A. THE BURNETT INTEGRAL

We consider

$$\int_{-\infty}^{\infty} \exp\left[-iak - bk^2 + ick^3\right] dk \qquad \text{(A.1)}$$

with a, b, c, real.

Defining

$$u = (a + b^2/3c); \qquad v = 2b^3/27c^2 + ab/3c,$$

we can transform Eq. (A.1) into

$$B = \frac{1}{(3c)^{\frac{1}{3}}} e^{v} \mathrm{Ai}_0(u/(3\,|c|)^{\frac{1}{3}}) \qquad \text{(A.2)}$$

where Ai_0 is an Airy integral. For tables and properties of this function, one is referred to the literature.[12]

APPENDIX B

We give here (see Tables I–VI) a résumé of the results necessary for the calculations made in Sec. V. The eigenvalue determinations are taken from reference 1, and the eigenvector vector determinations are given without reference or calculations. Both computations follow from the procedure given in Sec. III, and are in general straightforward, though somewhat tedious. Some of the tedium is removed by the computational aids found in reference 1. The determination of the hydrodynamical modes are carried to a higher order because of its natural importance. A description of the notation is found in Sec. V.

Hydrodynamic eigendeterminations

TABLE I. $b_{\mu\nu}^{00,j}$, (00) values.

$$d^{00} = (\tfrac{5}{3})^{\frac{1}{2}} ik + (1/3\lambda_{11} + 2/3\lambda_{02})k^2 + (\tfrac{5}{3})^{\frac{1}{2}}(-2/5\lambda_{02}\lambda_{11} + 1/10\lambda_{11}^2 + 8/15\lambda_{02}^2)ik^3 + \cdots$$

$\mu\nu$	$j = 0$	$j = 1$	$j = 2$	
00	1	0	0	0
01	$(\frac{5}{3})^{\frac{1}{2}}$	$-(1/3\lambda_{11} + 2/3\lambda_{02})$	$(\frac{5}{3})^{\frac{1}{2}}(-2/5\lambda_{02}\lambda_{11} + 1/10\lambda_{11}^2 + 8/15\lambda_{02}^2)$	
10	$-(\frac{2}{3})^{\frac{1}{2}}$	$(10)^{\frac{1}{2}}/3\lambda_{11}$	$(\frac{2}{3})^{\frac{1}{2}}(1/3\lambda_{11}^2 + 4/\lambda_{02}^2 - 14/3\lambda_{02}\lambda_{11})$	
11	0	$(10)^{\frac{1}{2}}/3\lambda_{11}$	$(\frac{2}{3})^{\frac{1}{2}}4/3\lambda_{02}\lambda_{11}$	
20	0	0	$(10/3)^{\frac{1}{2}}2/3\lambda_{20}\lambda_{11}$	
02	0	$-2(5)^{\frac{1}{2}}/3\lambda_{02}$	$[2/\lambda_{02}(3)^{\frac{1}{2}}](1/\lambda_{11} - 1/\lambda_{02})$	
12	0	0	$(14/3)^{\frac{1}{2}}2/3\lambda_{11}\lambda_{12}$	
03	0	0	$-2/\lambda_{02}\lambda_{03}$	
	0	0	0	

[12] See for instance, J. C. P. Miller, *The Airy Integral* (Cambridge University Press 1958), Part B.

TABLE II. $b_{\mu\nu}^{01,j}$, (01) values.

$$d^{01} = -ik(\tfrac{5}{3})^{\frac{1}{2}} + (1/3\lambda_{11} + 2/3\lambda_{02})k^2 - (-2/5\lambda_{02}\lambda_{11} + 1/10\lambda_{11}^2 + 8/15\lambda_{02}^2)ik^3 + \cdots$$

$\mu\nu$	$j = 0$	$j = 1$	$j = 2$	
00	$-(\frac{3}{5})^{\frac{1}{2}}$	$\frac{3}{5}(1/3\lambda_{11} + 2/3\lambda_{02})$	$(\frac{3}{5})^{\frac{1}{2}}(1/6\lambda_{11}^2 - 2/15\lambda_{02}\lambda_{11} + 4/5\lambda_{02}^2)$	
01	1	0	0	0
10	$(\frac{2}{5})^{\frac{1}{2}}$	$(\frac{2}{3})^{\frac{1}{2}}(4/5\lambda_{11} - 2/5\lambda_{02})$	$[1/(10)^{\frac{1}{2}}](1/2\lambda_{11}^2 + 2/\lambda_{11}\lambda_{02} - 4/\lambda_{02}^2)$	
11	0	$(-1/\lambda_{11})(\frac{2}{3})^{\frac{1}{2}}$	$[(10)^{\frac{1}{2}}/15\lambda_{11}](2/\lambda_{02} - 1/\lambda_{11})$	
20	0	0	$2(2)^{\frac{1}{2}}/3\lambda_{11}\lambda_{02}$	
02	0	$-2/\lambda_{02}(3)^{\frac{1}{2}}$	$[2(5)^{\frac{1}{2}}/3\lambda_{02}](2/5\lambda_{11} - 1/\lambda_{02})$	
12	0	0	$-2/3\lambda_{11}\lambda_{12}(14/5)^{\frac{1}{2}}$	
03	0	0	$-(\frac{3}{5})^{\frac{1}{2}}2/\lambda_{02}\lambda_{02}$	

TABLE III. $b_{\mu\nu}^{10,j}$, (10) values.

$$d^{10} = k^2/\lambda_{11} + O(k^4) + \cdots$$

$\mu\nu$	$j = 0$	$j = 1$	$j = 2$	
00	$(\frac{2}{3})^{\frac{1}{2}}$	0	$-(\frac{2}{3})^{\frac{1}{2}}(4/3\lambda_{11}\lambda_{02} + 1/\lambda_{11})$	
01	0	$-(\frac{2}{3})^{\frac{1}{2}}1/\lambda_{11}$	0	
10	1	0	0	0
11	0	$-1/\lambda_{11}(\frac{5}{3})^{\frac{1}{2}}$	0	
20	0	0	$2(5)^{\frac{1}{2}}/3\lambda_{11}\lambda_{20}$	
02	0	0	$+2(2)^{\frac{1}{2}}/3\lambda_{11}\lambda_{02}$	
12	0	0	$-2(7)^{\frac{1}{2}}/3\lambda_{11}\lambda_{12}$	
03	0	0	0	

Propagating Mode $\lambda_{r0} = \lambda_{r-1,1}$

TABLE IV. $b_{\mu\nu}^{r-1,1;j}$, $(r - 1, 1)$ values.

$$d^{r-1,1} = \lambda_{r0} + \left(\frac{2r}{3}\right)^{\frac{1}{2}} ik - \left[\frac{2r+3}{6(\lambda_{r0} - \lambda_{r1})} + \frac{2r+1}{6(\lambda_{r0} - \lambda_{r-1,0})} + \frac{2(2r+3)}{15(\lambda_{r0} - \lambda_{r-1,2})} + \frac{2(2r-2)}{15(\lambda_{r0} - \lambda_{r-2,2})}\right]k^2 + \cdots$$

$\mu\nu$	$j = 0$	$j = 1$	$j = 2$
$r-1, 1$	1	0	0
$r, 0$	-1	$\left(\frac{3}{2r}\right)^{\frac{1}{2}}\left[\frac{2r+1}{3(\lambda_{r0} - \lambda_{r-1,0})} - \frac{2(2r+3)}{15(\lambda_{r0} - \lambda_{r-1,2})} - \frac{2(2r-2)}{15(\lambda_{r0} - \lambda_{r-2,2})} + \frac{2r+3}{6(\lambda_{r0} - \lambda_{r1})}\right]$	
$r, 1$	0	$\frac{1}{\lambda_{r1} - \lambda_{r0}}\left(\frac{2r+3}{3}\right)^{\frac{1}{2}}$	
$r-1, 0$	0	$\frac{1}{\lambda_{r0} - \lambda_{r-1,0}}\left(\frac{2r+1}{3}\right)^{\frac{1}{2}}$	
$r-1, 2$	0	$\frac{1}{\lambda_{r0} - \lambda_{r-1,2}}\left(\frac{2r+3}{15}\right)^{\frac{1}{2}}$	
$r-2, 2$	0	$\frac{2}{\lambda_{r-2,1} - \lambda_{r0}}\left(\frac{2r-2}{15}\right)^{\frac{1}{2}}$	

TABLE V. $b_{\mu\nu}^{r;0;i}$, $(r, 0)$ values.

$$d^{r0} = \lambda_{r0} + \left(\frac{2r}{3}\right)^{\frac{1}{2}} ik - \left[\frac{2r+3}{6(\lambda_{r0} - \lambda_{r1})} + \frac{2r+1}{6(\lambda_{r0} - \lambda r_{-1,0})} + \frac{2(2r+3)}{15(\lambda_{r0} - \lambda_{r-1,2})} + \frac{2(2r-1)}{15(\lambda_{r0} - \lambda_{r-2,2})}\right]k^2 + \cdots$$

$\mu\nu$	$j = 0$	$j = 1$	$j = 2$
$r-1, 1$	1	$-\left(\frac{3}{2r}\right)^{\frac{1}{2}}\left[\frac{2r+3}{2(\lambda_{r0} - \lambda_{r1})} + \frac{2r+1}{6(\lambda_{r0} - \lambda_{r-1,0})} + \frac{2(2r+3)}{15(\lambda_{r0} - \lambda_{r-1,2})} + \frac{2(2r-1)}{15(\lambda_{r0} - \lambda_{r-2,2})}\right]$	
$r, 0$	1	0	0
$r, 1$	0	$\frac{1}{\lambda_{r0} - \lambda_{r1}}\left(\frac{2r+3}{3}\right)^{\frac{1}{2}}$	
$r-1, 0$	0	$\frac{1}{\lambda_{r0} - \lambda_{r-1,0}}\left(\frac{2r+1}{3}\right)^{\frac{1}{2}}$	
$r-1, 2$	0	$\frac{2}{\lambda_{r0} - \lambda_{r-1,2}}\left(\frac{2r+3}{15}\right)^{\frac{1}{2}}$	
$r-2, 2$	0	$\frac{2}{\lambda_{r-2,2} - \lambda_{r0}}\left(\frac{2r-2}{15}\right)^{\frac{1}{2}}$	

Nonpropagating Modes

TABLE VI. $b_{rl}^{\mu\nu,1}$.

$$d^{\mu\nu} = \lambda_{\mu\nu} - k^2\left\{(\nu+1)^2\left[\frac{2\mu+2\nu+3}{(2\nu+1)(2\nu+3)}\frac{1}{\lambda_{\mu\nu} - \lambda_{\mu,\nu+1}} + \frac{2\mu}{(2\nu+3)(2\nu+1)}\frac{1}{\lambda_{\mu\nu} - \lambda_{\mu-1,\nu+1}}\right]\right.$$
$$\left. + \nu^2\left[\frac{2\mu+2\nu+1}{(2\nu+1)(2\nu-1)}\frac{1}{\lambda_{\mu\nu} - \lambda_{\mu,\nu-1}} + \frac{2\mu+2}{(2\nu-1)(2\nu+1)}\frac{1}{\lambda_{\mu\nu} - \lambda_{\mu+1,\nu-1}}\right]\right\} + \cdots$$

$r_1 l$	$j = 0$	$j = 1$	
μ, ν	1	0	0
$\mu, \nu - 1$	0	$\frac{-\nu}{\lambda_{\mu,\nu-1} - \lambda_{\mu\nu}}\left(\frac{(\mu+\nu+\frac{1}{2})2}{(2\nu-1)(2\nu+1)}\right)^{\frac{1}{2}}$	
$\mu+1, \nu+1$	0	$\frac{\nu}{\lambda_{\mu+1,\nu+1} - \lambda_{\mu\nu}}\left(\frac{2\mu+2}{(2\nu+1)(2\nu-1)}\right)^{\frac{1}{2}}$	
$\mu, \nu+1$	0	$\frac{-(\nu+1)}{\lambda_{\mu,\nu+1} - \lambda_{\mu\nu}}\left(\frac{2\mu+2\nu+3}{(2\nu+3)(2\nu+1)}\right)^{\frac{1}{2}}$	
$\mu-1, \nu+1$	0	$\frac{\nu+1}{\lambda_{\mu-1,\nu+1} - \lambda_{\mu\nu}}\left(\frac{2\mu}{(2\nu+1)(2\nu+3)}\right)^{\frac{1}{2}}$	
	0	0	