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Dispersion Relations in Rarefied Gas Dynamics

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The one-dimensional initial-value problem of a monatomic single component gas is considered. Using the linearized Boltzmann equation the dispersion relation is studied. In addition to the usual gas-dynamic sound waves, one finds an infinity of decaying propagating waves. The phenomenon exhibits itself as a sequence of epochs, the last state of which is hydrodynamic. With reference to the same problem, macroscopic equations such as Euler, Navier–Stokes, Burnett, moments equations, etc., are considered. In addition, the recently considered "kinetic models" of Gross *et al.* are applied to the problem. These various formulations are critically analyzed and compared with each other and with the Boltzmann analysis. Lastly, several modifications are offered which remedy some of the shortcomings which appear in the approximate theories.

I. INTRODUCTION

In the period between 1952 and 1956 Wang Chang and Uhlenbeck[1–4] and Mott-Smith[5] produced a series of highly original and important studies in kinetic theory. These dealt with extremely simple geometries and with situations where linearization could be used with confidence. Through their formulations of problems and high level of analysis, these reports (none are published) have provided the inspiration for much of the recent theoretical work in kinetic theory. The present study is in no small way indebted to these researchers, as will be evidenced by the frequent reference to their work.

Despite the simplicity of the problems in these studies, perhaps the simplest problem has been overlooked. This is the initial-value problem in an unbounded domain. Closely connected to this is the problem of sound propagation considered by Wang Chang and Uhlenbeck.[1] It was their intention to describe sound propagation for all ratios of the mean-free-path to sound wavelength. In a certain sense they were unsuccessful since they only achieved their results in power-series representations of this ratio. For the most part, we shall be content with such representations, but will be interested in more than just sound propagation. We shall, in fact, uncover many other phenomena, including propagations other than ordinary sound.

Our initial-value problem is, in itself, relatively sterile from the physical point of view, since neither laboratory nor reality comply with its requirements. On the other hand, due to its simplicity it allows greater penetration than any other nontrivial problem in kinetic theory. It therefore provides a suitable setting in which to examine existing approximate formulations in gas dynamics. We include in this latter designation the Chapman–Enskog theory,[6] moments equations,[7] and kinetic models.[8,9] Furthermore, new approximate formulations will be suggested.[10]

In the present paper the initial-value problem is studied in a narrow sense. It will be based strictly on dispersion relations. Such an analysis is decidedly restrictive. For one thing it does not consider the effect of initial data which, for instance, is of crucial importance in delimiting the Chapman–Enskog theory. This will be taken up at another time. In a following paper[11] the initial-value problem is carried further.

II. LINEARIZED BOLTZMANN EQUATION

In the interest of brevity we give only a short review of the one-dimensional unsteady linearized Boltzmann equation for Maxwell molecules,

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[1] C. S. Wang Chang and G. E. Uhlenbeck, "On the Propagation of Sound in Monatomic Gases," Engineering Research Institute, University of Michigan (1952).
[2] C. S. Wang Chang and G. E. Uhlenbeck, "The Heat Transfer between Two Parallel Plates," Engineering Research Institute, University of Michigan (1953).
[3] C. S. Wang and G.E. Uhlenbeck, "The Couette Flow between Two Parallel Plates," Engineering Research Institute, University of Michigan (1954).
[4] C. S. Wang Chang and G. E. Uhlenbeck, "On the Behavior of a Gas near a Wall," Engineering Research Institute, University of Michigan (1956).
[5] H. M. Mott-Smith, "A New Approach in the Kinetic Theory of Gases," Lincoln Laboratory, MIT (1954).
[6] S. I. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1952).
[7] H. Grad, "Principles of the Kinetic Theory of Gases," in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 12.
[8] E. P. Gross and E. A. Jackson, Phys. Fluids **2**, 432 (1959).
[9] P. F. Bhatnager, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).
[10] Using the initial-value problem to generate approximate equations which are to be used in more general situations, is a widespread practice. This is explicit or implicit in the Chapman–Enskog theory and in the current gas-dynamic and kinetic derivations from the Liouville equation in statistical mechanics.
[11] L. Sirovich, Phys. Fluids (to be published).

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x}\right) g = L(g) = \frac{1}{m} \int f_*^0 [g] B(\theta)\, d\epsilon\, d\theta\, d\boldsymbol{\xi}_*,$$

$$[g] = g' + g'_* - g - g_*, \tag{2.1}$$

$$f^0 = \frac{\rho_0}{(2\pi R T_0)^{\frac{3}{2}}} \exp\left(\frac{-\xi^2}{2RT_0}\right) = \frac{\rho_0}{(RT_0)^{\frac{3}{2}}}\, \omega.$$

$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ denotes the molecular velocity and $\boldsymbol{\xi}_*$ the velocity of the struck particle. We define g by

$$f = f^0(1 + g), \tag{2.2}$$

where f is the distribution function. We make the linearized Boltzmann equation dimensionless with respect to an unspecified frequency $\nu = 1/\tau$ as follows:

$$\nu t = t', \frac{x\nu}{(RT_0)^{\frac{1}{2}}} = x', \frac{\boldsymbol{\xi}}{(RT_0)^{\frac{1}{2}}} = \boldsymbol{\xi}', B' = \frac{\rho_0 B \nu}{m}. \tag{2.3}$$

Rather than introduce a cumbersome notation for the dimensionless quantities, we consider (2.3) as having been carried out and then remove the primes. The dimensionless equation then is

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x}\right) g = L(g) = \int \omega_* [g] B\, d\epsilon\, d\theta\, d\boldsymbol{\xi}_* \tag{2.4}$$

The eigentheory of the operator L was discovered by Wang Chang and Uhlenbeck[1] (see also references 5, 12). For shear-free one-dimensional motions, the eigenfunctions of L are[5]

$$\psi_{rl} = \frac{S^r_{l+\frac{1}{2}}(\frac{1}{2}\xi^2)\xi^l P_l(\cos\theta)}{\left[\dfrac{2^{l+1}\Gamma(r+l+\frac{3}{2})}{\pi^{\frac{1}{2}} r!\,(2l+1)}\right]^{\frac{1}{2}}}, \tag{2.5}$$

where S^r_l, P_l denote the Laguerre and Legendre polynomials. In the form given we have

$$\int \omega \psi_{rl} \psi_{r'l'}\, d\xi = \delta_{rr'}\, \delta_{ll'} \tag{2.6}$$

and[5]

$$\begin{aligned}\xi_1 \psi_{rl} = (l+1)\Bigg\{&\psi_{r,l+1}\left[\frac{(r+l+\frac{3}{2})2}{(2l+1)(2l+3)}\right]^{\frac{1}{2}} \\ &- \psi_{r-1,l+1}\left[\frac{2r}{(2l+3)(2l+1)}\right]^{\frac{1}{2}}\Bigg\} \\ &+ l\Bigg\{\psi_{r,l-1}\left[\frac{2(r+l+\frac{1}{2})}{(2l+1)(2l-1)}\right]^{\frac{1}{2}} \\ &- \psi_{r+1,l-1}\left[\frac{2(r+1)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}}\Bigg\}.\end{aligned} \tag{2.7}$$

The eigenvalues of L are given by[1]

$$\begin{aligned}\lambda_{rl} = 2\pi \int d\theta\, B(\theta)[&\cos^{2r+l} \tfrac{1}{2}\theta P_l(\cos \tfrac{1}{2}\theta) \\ &+ \sin^{2r+l} \tfrac{1}{2}\theta P_l(\sin \tfrac{1}{2}\theta) - (1 + \delta_{r0}\, \delta_{l0})],\end{aligned} \tag{2.8}$$

[12] L. Waldmann, "Transporterscheinungen in Gasen von mittlerem Druck," in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 12.

TABLE I. Normalized eigenvalues. (The values of the table were obtained from reference 5.)

r \ l	0	1	2	3	4	5
0	0	0	3/2	9/4	2.808	3.274
1	0	1	7/4	2.354	2.864	3.318
2	1	3/2	2.014	2.500	2.952	
3	3/2	1.8420	2.238	2.646	3.064	
4	1.8420	2.106	2.428	2.780		
5	2.106	2.320	2.598			
6	2.320					

from which we see

$$\lambda_{r0} = \lambda_{r-1,1}. \tag{2.9}$$

Values of $\lambda_{rl}/\lambda_{11}$ taken from references 1 and 5 are given in Table I. The $\lambda_{00} = \lambda_{01} = \lambda_{10} = 0$ degeneracy comes from the conservation laws (these will be referred to as the hydrodynamic roots). We take $\nu = |\lambda_{11}|$ in (2.4), this being the smallest eigenvalue.[13]

Expanding the perturbed distribution function in the ψ_{rl},

$$g = \sum_{r,l} b_{rl} \psi_{rl}; \qquad b_{rl} = \int \omega g \psi_{rl}\, d\boldsymbol{\xi}, \tag{2.10}$$

we can immediately "reduce" the linearized Boltzmann equation to the infinite system of coupled equations,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \lambda_{\mu\nu}\right) b_{\mu\nu} + \frac{\partial}{\partial x}\Bigg(\nu\Bigg\{&b_{\mu,\nu-1}\left[\frac{(\mu+\nu+\frac{1}{2})2}{(2\nu-1)(2\nu+1)}\right]^{\frac{1}{2}} \\ &- b_{\mu+1,\nu-1}\left[\frac{2\mu+2}{(2\nu-1)(2\nu+1)}\right]^{\frac{1}{2}}\Bigg\} \\ + (\nu+1)\Bigg\{&b_{\mu,\nu+1}\left[\frac{(\mu+\nu+\frac{3}{2})2}{(2\nu+3)(2\nu+1)}\right]^{\frac{1}{2}} \\ &- b_{\mu-1,\nu+1}\left[\frac{2\mu}{(2\nu+3)(2\nu+1)}\right]^{\frac{1}{2}}\Bigg\}\Bigg) = 0,\end{aligned} \tag{2.11}$$

Several examples of the coefficients and their associated eigenfunctions are

$$\begin{aligned}\psi_{00} &= 1, & b_{00} &= \rho, \\ \psi_{01} &= \xi_1, & b_{01} &= u, \\ \psi_{10} &= (\tfrac{3}{2})^{\frac{1}{2}}(1 - \tfrac{1}{3}\xi^2), & b_{10} &= -(\tfrac{3}{2})^{\frac{1}{2}} T, \\ \psi_{02} &= \tfrac{1}{2}(3)^{\frac{1}{2}}(\xi_1^2 - \tfrac{1}{3}\xi^2), & b_{02} &= \tfrac{1}{2}(3)^{\frac{1}{2}} p_{11}, \\ \psi_{11} &= (\tfrac{5}{2})^{\frac{1}{2}}(1 - \tfrac{1}{5}\xi^2)\xi_1, & b_{11} &= -(\tfrac{2}{5})^{\frac{1}{2}} S_1.\end{aligned} \tag{2.12}$$

The quantities of the right column are, respectively, the dimensionless perturbed density, velocity, temperature, stress, and heat conduction. The dimensionless pressure p will be eliminated by the "gas law"

$$p = \rho + T. \tag{2.13}$$

[13] C. Truesdell, J. Rat. Mech. Anal. 5, 55 (1956).

TABLE II. Eigenvalue ordering.

$b_{00}=\rho$	$b_{01}=u_1$	$b_{10}=(\frac{3}{2})^{\frac{1}{2}}T$	b_{11}	b_{20}	b_{02}	b_{21}	b_{30}	b_{12}	b_{31}	b_{40}	b_{22}	b_{41}	b_{50}	b_{32}	b_{03}	
σ	$-ik$															
$-ik$	σ	$(2/3)^{\frac{1}{2}}ik$			$-2/(3)^{\frac{1}{2}}ik$											
	$(2/3)^{\frac{1}{2}}ik$	σ	$-(5/3)^{\frac{1}{2}}ik$													1
		$-(5/3)^{\frac{1}{2}}ik$	$\sigma-\lambda_{11}$	$2/(3)^{\frac{1}{2}}ik$	$(8/15)^{\frac{1}{2}}ik$			$-(28/15)^{\frac{1}{2}}ik$								
			$2/(3)^{\frac{1}{2}}ik$	$\sigma-\lambda_{20}$		$-(7/3)^{\frac{1}{2}}ik$										2
	$-2/(3)^{\frac{1}{2}}ik$		$(8/15)^{\frac{1}{2}}ik$		$\sigma-\lambda_{02}$										$-3/(5)^{\frac{1}{2}}ik$	2′
				$-(7/3)^{\frac{1}{2}}ik$		$\sigma-\lambda_{21}$	$(2)^{\frac{1}{2}}ik$	$4/(15)^{\frac{1}{2}}ik$			$-6/(15)^{\frac{1}{2}}ik$					
						$(2)^{\frac{1}{2}}ik$	$\sigma-\lambda_{30}$		$-(3)^{\frac{1}{2}}ik$							3
			$-(28/15)^{\frac{1}{2}}ik$			$4/(15)^{\frac{1}{2}}ik$		$\sigma-\lambda_{12}$							$(18/35)^{\frac{1}{2}}ik$	
							$-(3)^{\frac{1}{2}}ik$		$\sigma-\lambda_{31}$							
		1		2	2′		3									

As examples of the equations of (2.11), we have

$$(\partial/\partial t)\rho + (\partial/\partial x)u = 0,$$
$$(\partial/\partial t)u + (\partial/\partial x)p_{11} + (\partial/\partial x)(\rho + T) = 0, \quad (2.14)$$
$$(\partial/\partial t)T + \tfrac{2}{3}(\partial/\partial x)S_1 + \tfrac{2}{3}(\partial/\partial x)u = 0,$$

which are, of course, the conservation equations. In writing down any finite collection of equations the ordering of the b's becomes important (a complete solution is, of course, independent of ordering). Generally, in questions of ordering, the degree of the ψ_{rl} is most often taken as the determining factor. For some purposes, ordering the b's according to the magnitudes of λ is certainly as important.

III. DISPERSION RELATIONS

To develop dispersion relations we consider plane wave solutions to (2.11), i.e., we assume

$$b_{rl} = \hat{b}_{rl}e^{\sigma t - ikx} \quad (3.1)$$

for all dependent variables, where the $\hat{b}_{rl}$ are constants. This corresponds to a Laplace transform in time and Fourier transform in space. Substituting this into (2.11) we get the linear homogeneous system

$$(\sigma - \lambda_{\mu\nu})\hat{b}_{\mu\nu} - ik\Bigg(\nu\Bigg\{\hat{b}_{\mu,\nu-1}\left[\frac{(\mu+\nu+\frac{1}{2})2}{(2\nu-1)(2\nu+1)}\right]^{\frac{1}{2}} - \hat{b}_{\mu+1,\nu-1}\left[\frac{2\mu+2}{(2\nu-1)(2\nu+1)}\right]^{\frac{1}{2}}\Bigg\} - (\nu+1)\Bigg\{\hat{b}_{\mu-1,\nu+1}\left[\frac{2\mu}{(2\nu+3)(2\nu+1)}\right]^{\frac{1}{2}} - \hat{b}_{\mu,\nu+1}\left[\frac{(\mu+\nu+\frac{3}{2})2}{(2\nu+3)(2\nu+1)}\right]^{\frac{1}{2}}\Bigg\}\Bigg) = 0. \quad (3.2)$$

The determinant of the linear homogeneous system (3.2) will be referred to as $D(\sigma, k)$. This, of course, defines the dispersion law. The fact that $D(\sigma, k)$ is infinite introduces some difficulties, e.g., the existence roots. To avoid such questions we can consider (3.2) as being arbitrarily large but finite. As will be seen, the results are unchanged by going to the limit. A truncated matrix of the system (3.2) is given for the eigenvalue ordering in Table II and for the polynomial ordering in Table III. We now determine the roots $\sigma(k)$.[14]

On setting $k = 0$ in

$$D(\sigma, k) = 0, \quad (3.3)$$

we see immediately that

$$\sigma = \lambda_{rl} \quad (3.4)$$

are roots. Note that $\sigma = 0$ and $\sigma = \lambda_{02}(=\lambda_{30} = \lambda_{21})$ are triple roots and $\sigma = \lambda_{r0}(=\lambda_{r-1,1})$ are double roots. We shall develop the roots in a power series in k in the neighborhood of each of these $k = 0$ points. Each $k = 0$ point defines the origin of one or more of the branches of (3.3). It can be shown[15] that D is a function of k^2, and, therefore, in the neighborhood of a simple zero the root σ may be developed in a series of ascending powers of k^2. However, in the neighborhood of a double or triple point a Puiseux series must be used. In our case this can be shown to be a power series expansion in k.

We write for each root σ

$$\sigma = \sigma_0 + \sigma_1 ik + \sigma_2 k^2 + \cdots, \quad (3.5)$$

[14] For a boundary-value problem, one is essentially interested in $k = k(\sigma)$. This is considered in reference 1.

[15] L. Sirovich, Courant Institute of Mathematical Sciences, New York University, Rept. MF 17 (1961).

TABLE III. Polynomial ordering.

$b_{00}=\rho$	$b_{01}=u_1$	$b_{10}=(\frac{3}{2})^{\frac{1}{2}}T$	b_{02}	b_{11}	b_{03}	b_{20}	b_{12}	b_{04}	b_{21}	b_{13}	
σ	$-ik$										
$-ik$	σ	$(2/3)^{\frac{1}{2}}ik$	$-2/(3)^{\frac{1}{2}}ik$								
	$(2/3)^{\frac{1}{2}}ik$	σ		$-(5/3)^{\frac{1}{2}}ik$							1
	$-2/(3)^{\frac{1}{2}}ik$		$\sigma-\lambda_{02}$	$(8/15)^{\frac{1}{2}}ik$	$-3/(5)^{\frac{1}{2}}ik$						
		$-(5/3)^{\frac{1}{2}}ik$	$(8/15)^{\frac{1}{2}}ik$	$\sigma-\lambda_{11}$		$2/(3)^{\frac{1}{2}}ik$	$-(28/15)^{\frac{1}{2}}ik$				2
			$-3/(5)^{\frac{1}{2}}ik$		$\sigma-\lambda_{03}$		$(18/35)^{\frac{1}{2}}ik$	$-4/(7)^{\frac{1}{2}}ik$			
				$2/(3)^{\frac{1}{2}}ik$		$\sigma-\lambda_{20}$			$-(7/3)^{\frac{1}{2}}ik$		
				$-(28/15)^{\frac{1}{2}}ik$	$(18/35)^{\frac{1}{2}}ik$		$\sigma-\lambda_{12}$		$(16/15)^{\frac{1}{2}}ik$	$-(81/35)^{\frac{1}{2}}ik$	3
					$-4/(7)^{\frac{1}{2}}ik$			$\sigma-\lambda_{04}$			
		1		2			3				

and for $D(\sigma, k)$ the expansion

$$D = D_0 + kD_0^{(1)} + (k^2/2)\, D_0^{(2)} + (k^3/3!)\, D_0^{(3)} + \cdots . \qquad (3.6)$$

The enclosed superscript denotes differentiation and the zero subscript the evaluation at $k = 0$. Relation (3.5) is placed in (3.6) and this results in the evaluation of the σ_i when (3.6) is set equal to zero. The latter states that

$$0 = D_0 = D_0^{(1)} = D_0^{(2)} = \cdots = D_0^{(n)} = \cdots . \qquad (3.7)$$

Because of the special nature of D, several techniques can be found which simplify the calculations.[15]

Hydrodynamical Branches

In this case it is more convenient to examine Table III, the polynomial ordering, rather than the eigenvalue ordering of Table II which will be used in all other cases. (Of course, the results are indifferent to whatever ordering we use. A particular ordering is used only for the convenience it affords.)

The hydrodynamical branches are found by taking

$$\sigma = i\alpha k + \beta k^2 + i\gamma k + \delta k^4 + i\epsilon k^5 + \cdots \qquad (3.8)$$

in $D(\sigma, k)$, i.e., in Table III. One finds that α is obtained from the coefficient of k^3 in the expansion of the first 3×3 determinant. We write this as

$$\text{coef}\,(k^3)\begin{vmatrix} \sigma & -ik & 0 \\ -ik & \sigma & (\frac{2}{3})^{\frac{1}{2}}ik \\ 0 & (\frac{2}{3})^{\frac{1}{2}}ik & \sigma \end{vmatrix} = 0. \qquad (3.9)$$

β is given by

$$\text{coef}\,(k^4)\begin{vmatrix} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & (\frac{2}{3})^{\frac{1}{2}}ik & \frac{-2}{(3)^{\frac{1}{2}}}\,ik & 0 \\ 0 & (\frac{2}{3})^{\frac{1}{2}}ik & \sigma & 0 & -(\frac{5}{3})^{\frac{1}{2}}ik \\ 0 & \frac{-2}{(3)^{\frac{1}{2}}}\,ik & 0 & -\lambda_{02} & 0 \\ 0 & 0 & -(\frac{5}{3})^{\frac{1}{2}}ik & 0 & \lambda_{11} \end{vmatrix} = 0, \qquad (3.10)$$

and γ by

$$\text{coef}\,(k^5)\begin{vmatrix} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & (\frac{2}{3})^{\frac{1}{2}}ik & \frac{-2}{(3)^{\frac{1}{2}}}\,ik & 0 \\ 0 & (\frac{2}{3})^{\frac{1}{2}}ik & \sigma & 0 & -(\frac{5}{3})^{\frac{1}{2}}ik \\ 0 & \frac{-2}{(3)^{\frac{1}{2}}}\,ik & 0 & \sigma-\lambda_{02} & (\frac{8}{15})^{\frac{1}{2}}ik \\ 0 & 0 & -(\frac{5}{3})^{\frac{1}{2}}ik & (\frac{8}{15})^{\frac{1}{2}}ik & \sigma-\lambda_{11} \end{vmatrix}. \qquad (3.11)$$

The result of these calculations are the three hydrodynamic branches

$$\sigma = k^2/\lambda_{11} + O(k^4), \tag{3.12}$$

$$\sigma = \pm i\left(\frac{5}{3}\right)^{\frac{1}{2}}\left[k + k^3\left(\frac{-2}{5\lambda_{02}\lambda_{11}} + \frac{1}{10\lambda_{11}^2} + \frac{8}{15\lambda_{02}^2}\right)\right] + k^2\left(\frac{1}{3\lambda_{11}} + \frac{2}{3\lambda_{02}}\right) + O(k^4). \tag{3.13}$$

The branch given by (3.12) is purely decaying and is connected with heat conduction. Relation (3.13) furnishes two propagating and decaying modes. To $O(k)$, (3.13) is the adiabatic speed and the $O(k^3)$ term gives the increase in the phase velocity. One easily sees that (3.9) is the Laplace–Fourier transform of the Euler equations, (3.10) the transform of the Navier–Stokes equations,[16] and (3.11) the transform of the thirteen moments equations. Therefore, from the point of view of kinetic theory, the three roots (3.12), (3.13) are given correctly to $O(k)$ by the Euler equations and to (k^3) by the thirteen moments equations.

With respect to the latter, one sees that two other roots appear. This is discussed in Sec. VI. Actually, instead of σ appearing in the last two rows of (3.11), one may insert their expansions. On doing this one finds that (3.11) is equivalent to the transform of the Burnett equations of the Chapman–Enskog theory.[17]

Referring to Table III and the determinant given by $i, j < 3$, the coefficient of k^6 leads to δ, and the coefficient of k^7 to ϵ of the expansion for σ in (3.15). The determinant is identical to the one which would be gotten from a "26 moments" theory. In this way we may determine the dispersion relation of the hydrodynamical branches to any desired order. Although we found the transforms of the equations of the Chapman–Enskog procedure in the above analysis, it should not in any way be construed as a verification of the Chapman–Enskog procedure itself. This is clear from the fact that the inversions of the relations do not, without proper initial conditions, give Chapman–Enskog relations for stress and heat conduction. This important point will be taken up at another time. Next we note that the expansion of σ is in inverse powers of λ, i.e., with terms of the type (k/λ_{rl}). As we take larger determinants we gain larger $|\lambda_{rl}|$ and we can presumably describe higher-frequency phenomena. Unfortunately, the calculations become more tedious with larger determinants.

[16] The viscosity and heat conductivity coefficients are proportional respectively to the negative reciprocals of λ_{02} and λ_{11}.

[17] The reduction of the thirteen moments equations to the Burnett equation was noted by S. A. Schaaf and P. L. Chambre [*Flow of Rarefied Gases* (Princeton University Press, Princeton, New Jersey, 1955)].

Nondegenerate Branches

We may express the branch of a nondegenerate root as

$$\sigma = \lambda_{\mu\nu} + \beta k^2 + \delta k^4 + \cdots. \tag{3.14}$$

One finds[15]

$$\beta_{\mu\nu} = -[(\nu+1)^2\left[\frac{2(\mu+\nu+\frac{3}{2})}{(2\nu+1)(2\nu+3)}\frac{1}{\lambda_{\mu\nu}-\lambda_{\mu,\nu+1}} + \frac{2\mu}{(2\nu+3)(2\nu+1)}\frac{1}{\lambda_{\mu\nu}-\lambda_{\mu-1,\nu+1}}\right] + \nu^2\left[\frac{2(\mu+\nu+\frac{1}{2})}{(2\nu+1)(2\nu-1)}\frac{1}{\lambda_{\mu\nu}-\lambda_{\mu,\nu-1}} + \frac{2\mu+2}{(2\nu-1)(2\nu+1)}\frac{1}{\lambda_{\mu\nu}-\lambda_{\mu+1,\nu-1}}\right]. \tag{3.15}$$

If one inserts into (3.15) those values which are given in Table I, it is found that $\beta_{\mu\nu}$ is negative. It is plausible to presume that the coefficient of k^2 in the nondegenerate branches are all negative.

Doubly Degenerate Branches

From (2.9) we know that the doubly degenerate eigenvalues are given by $\lambda_{r0} = \lambda_{r-1,1}$. Referring to (3.8) one finds for α

$$\alpha_{r0} = \pm(2r/3)^{\frac{1}{2}} \tag{3.16}$$

and for β_{r0}

$$\beta_{r0} = -\left[\frac{2r+3}{6(\lambda_{r,0}-\lambda_{r,1})} + \frac{2r+1}{6(\lambda_{r,0}-\lambda_{r-1,0})} + \frac{2(2r+3)}{15(\lambda_{r0}-\lambda_{r-1,2})} + \frac{2(2r-2)}{15(\lambda_{r0}-\lambda_{r-2,2})}\right]. \tag{3.17}$$

The β's which can be computed from Table I are negative. It is again plausible to presume this negativity for all β given by (3.17).

Triply Degenerate Eigenvalues $(\lambda_{02} = \lambda_{21} = \lambda_{30})$

The results of this calculation are

$$\sigma = \lambda_{30} - k^2\left[\frac{4}{3\lambda_{02}} + \frac{8}{15(\lambda_{02}-\lambda_{11})} - \frac{9}{5(\lambda_{02}-\lambda_{03})}\right] + O(k^4),$$

$$\sigma = \lambda_{30} \pm (2)^{\frac{1}{2}} ik - k^2 \cdot\left[\frac{8}{15(\lambda_{02}-\lambda_{12})} + \frac{18}{15(\lambda_{02}-\lambda_{22})} + \frac{7}{6(\lambda_{02}-\lambda_{20})} + \frac{3}{2(\lambda_{02}-\lambda_{31})}\right]. \tag{3.18}$$

Again one may see, on using the values of Table I, that each of the modes is dissipative. No general relation is given in this case since no general triple degeneracy is to be found. In a sense, this last case is not a triple degeneracy at all, at least not in the same way as the hydrodynamical branches. One can easily see that the branch corresponding to λ_{02} may be computed as if it were a nondegenerate case. Further, the branches corresponding to $\lambda_{30} = \lambda_{21}$ may be computed as if they were a double degeneracy.

IV. SOME REMARKS ON SOLUTIONS OF THE BOLTZMANN EQUATIONS

In Fig. 1 we have plotted the roots of the dispersion relation in the complex σ plane. k occurs as a parameter along each branch, and the dependence of each σ on k may be found by returning to the appropriate calculation of the last section. The dispersion relation is given in a power series, and, having only a few terms of the series, the representation will be valid for only small k. We are therefore restricted to problems with sufficiently smooth initial data or to an observer far removed from the initial disturbance in space or time.[11]

From Fig. 1 we can obtain a qualitative picture of the solution to the initial-value problem. We see that three branches emanate from the origin. The purely real branch leads to a strictly decaying mode, whereas the two complex branches have propagation in addition to decay. These two conjugate branches correspond to ordinary sound propagation. The presence of other complex branches indicate additional propagating modes. Also, we see the presence of additional purely decaying modes. These latter modes are distinguished from the hydrodynamic modes in that they begin with a negative real part. This results in an immediate exponential decay in time for all such modes.

The solution of the Boltzmann equation includes all modes and may be thought of as a succession of epochs, or temporal boundary layers. Each epoch is determined by a "folding time" λ_{rl}^{-1}. After each of these times a cluster of modes becomes, roughly speaking, e^{-1} of its initial value. This does not at all mean that it is negligible when compared to the modes immediately to the right of it in Fig. 1. Inspection of Table I shows that ratios of eigenvalues are not large enough to warrant such a statement. However, any particular mode is asymptotically negligible compared to its neighbor to the right in Fig. 1. A cluster of modes is washed out of the picture more quickly than its neighboring

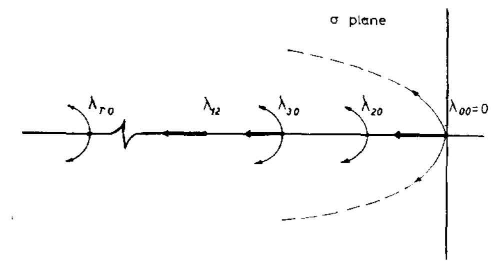


FIG. 1. Dispersion relation of the Boltzmann equation.

cluster to the right and less quickly than the one to the left. The hydrodynamic mode is the last of these epochs, and all other modes are asymptotically small compared to it.

V. NON-MAXWELL MOLECULES

A knowledge of the Maxwell eigenfunctions has made the collision integral amenable to analysis. For general molecular force laws the eigentheory of the collision operator is unknown and only an approximate treatment may be given. The appearance of nonsound propagations comes directly from the twofold degeneracy (2.9). There does not seem to be any analytical basis for supposing that this degeneracy persists for non-Maxwell molecules. And hence the existence of nonsound propagations are cast into doubt. In this short section we are chiefly interested in this question.

For general molecular force laws one may show[15]

$$L(g) = \sum_{r\mu\nu} b_{r\mu}\Lambda_{r\mu\nu}\psi_{\nu\mu} \tag{5.1}$$

with[18]

$$\Lambda_{r\mu\nu} = \int \omega\omega_*[\psi_{r\mu}]\psi_{\nu\mu}B(\theta, |\xi_* - \xi|)\, d\epsilon\, d\theta\, d\xi_*\, d\xi. \tag{5.2}$$

The diagonal approximation is obtained by taking

$$\Lambda_{r\mu\nu} = 0, \quad \text{for} \quad r \neq \mu. \tag{5.3}$$

This is exact for Maxwell molecules and leads to the first approximation of the heat conductivity and viscosity coefficients in the Chapman–Enskog procedure.[6] Mott-Smith[5] presents a strong case in favor of this approximation by showing that off-diagonal terms rapidly vanish, and Wang Chang and Uhlenbeck[1] give a systematic procedure for improving it. It is clear that the results of Sec. III are immediately applicable to the diagonal approximation. It is only in this sense that we consider more general molecular modes.

[18] $\Lambda_{r\mu\nu}$ is the normalized form of the usually defined "bracket integrals"; see references 5 and 6.

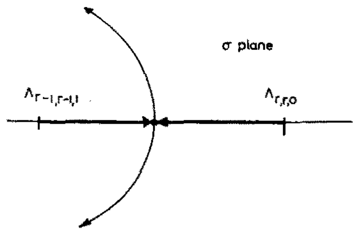


Fig. 2. Bifurcation leading to propagation.

Mott-Smith[5] has evaluated $\Lambda_{r\mu\nu}$ for general molecular models in terms of compound cross sections. From his results one may show

$$\Lambda_{220} = \Lambda_{111}. \tag{5.4}$$

This degeneracy exists independently of the molecular model, and corresponds to $\lambda_{20} = \lambda_{11}$ for Maxwell eigenvalues. Hence the first cluster (propagating) to the left of the origin in the figure exists independently of the model (in the diagonal approximation). On the other hand, one finds

$$\Lambda_{rr0} \neq \Lambda_{r-1,r-1,1} \qquad (r > 2) \tag{5.5}$$

except, of course, for Maxwell molecules. Since the degeneracy is lost we may use (3.23) to examine the branches defined by (5.5). One finds that, although we have the inequality (5.5), the values of Λ_{rr0} and $\Lambda_{r-1,r-10}$ are close to one another. Their difference, which is small and occurs in the denominator in (3.15), governs the behavior of the two branches under question. Inspection of (3.15) shows that the branch starting at the left moves to the right along the real line and vice versa. The two branches meet in a point for some value of k and bifurcation takes place, resulting in propagation. This is sketched in Fig. 2.

Improvement of the diagonal approximation should not change this situation. From the essentially diagonal character of $\Lambda_{r\mu\nu}$ we can expect the eigenvalues to differ only slightly from $\Lambda_{\mu\mu\nu}$. We can then expect the occurrence described in the last paragraph to persist and still have propagation.

The phenomenon just described certainly seems unphysical. The decrease in decay with increasing wavenumber is, in itself, suspicious. Nevertheless, the case for propagation is strong. A possible reconciliation is that the degeneracy given by (2.9) is independent of molecular model. This conjecture has been difficult to prove or disprove.

VI. KINETIC AND MACROSCOPIC MODELS

We now go to the analysis of approximate formulations and the comparison of these with the Boltzmann results. Rather than start with any particular approximate formulation, we shall find it more convenient to extract these from the Boltzmann equation itself. Our study of the Boltzmann equation has limited us to the study of relatively small wavenumbers. On the other hand, in examining the approximate theories we will consider the full range of wavenumbers. In all cases these calculations are possible and often serious shortcomings are revealed at the high wavenumbers.

The most straightforward approximation is gotten by using a truncated system. For instance, referring to Table III, if we truncate off $i, j < 1$,[19] we get the Euler equations (2.14). The roots of the corresponding dispersion relation are

$$\sigma = \pm(\tfrac{5}{3})^{\frac{1}{2}} ik, 0, \tag{6.1}$$

which are plotted in Fig. 3.

The truncations of i, $j < 2$ in Table III leads to the thirteen moments equations of Grad.[7] As mentioned in connection with (3.11), two roots exist in addition to the hydrodynamic roots (3.12, 13). For small wavenumber these are

$$\sigma = \lambda_{11} - \frac{k^2}{\lambda_{11}}\left[\frac{5}{3} + \frac{8\lambda_{11}}{15(\lambda_{11} - \lambda_{02})}\right] + O(k^4), \tag{6.2}$$

$$\sigma = \lambda_{02} - \frac{k^2}{\lambda_{02}}\left[\frac{4}{3} + \frac{8\lambda_{02}}{15(\lambda_{02} - \lambda_{11})}\right] + O(k^4). \tag{6.3}$$

On substituting the values of Table I we find that both these branches move to the right, and at

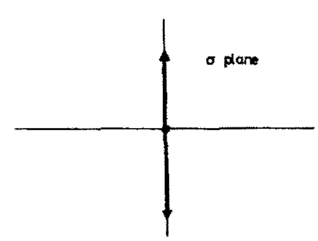
$$k \sim \pm 0.35, \qquad \sigma \sim 0.6\lambda_{02}, \tag{6.4}$$

the two branches come together in a double point and leave the axis in the way shown in the circled region of Fig. 4. (The same type effect occurred in connection with Fig. 2 as discussed in Sec. V.) To complete the picture the asymptotics for large k must be found. A simple calculation shows the five roots to be

$$\sigma = (\tfrac{5}{9})\lambda_{02} + O(1/k^2), \tag{6.5}$$

$$\begin{aligned} \sigma &= \pm ik(4.54)^{\frac{1}{2}} + 0.33\lambda_{02} + O(1/k), \\ \sigma &= \pm ik(0.66)^{\frac{1}{2}} + 0.22\lambda_{02} + O(1/k). \end{aligned} \tag{6.6}$$

Fig. 3. Euler dispersion relation.



[19] The i, j refer to the integers located along, the side and bottom of Tables II and III. In this connection we associate σ, $-ik$ and dispersion relations with $\partial/\partial t$, $\partial/\partial x$ and partial differential equations respectively.

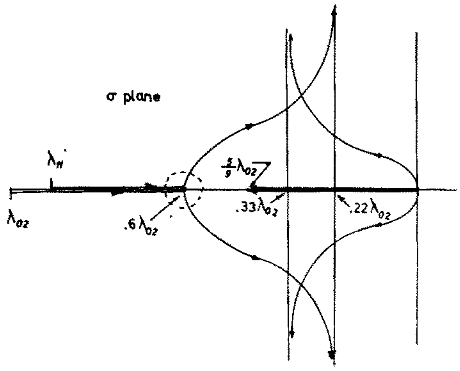


FIG. 4. Thirteen moments dispersion relation.

The coefficients of ik in (6.6) should be recognized as the thirteen moments characteristic speed. On carrying the expansion of (6.6) several more orders we find that the first expression is to be identified with the hydrodynamic branch and the second with the extra sound. This completes the picture sketched in Fig. 4.

We can make the following remarks in connection with the thirteen moments equations. For smooth phenomena, $k \ll 1$, they accurately describe the hydrodynamic branches but not so for the branches near λ_{11} and λ_{02}. Comparison with Fig. 1 shows that the dispersion relation is completely distorted in that region, the most evident inconsistency being that decay decreases with wavenumber. For large-wavenumber phenomena the inconsistency is more striking. The indication is that all branches of the dispersion relation Fig. 1 move toward negative infinity with increasing wavenumber. The thirteen moments system has a finite absorption cutoff. Further aggravating this is the fact that there is a branch for which the absorption decreases as k becomes unbounded, see Fig. 4. Similar shortcomings appear in general for truncated systems.

One may easily show that any truncation, no matter how large, gives a symmetric hyperbolic system of equations.[15] This has several consequences which are of importance to us. One immediate consequence is that as $k \to \infty$ we are led to finite speeds of propagation (some of which may vanish). We represent a truncated system by

$$[I(\partial/\partial t) + A(\partial/\partial x) - \Lambda]b = 0, \qquad (6.7)$$

where I is the identity matrix, A the coefficient matrix [as gotten from (2.11)] and Λ the matrix made up of the eigenvalues of the Boltzmann integral operator.

Equivalent to dispersion theory is finding eigenvalues σ, such that

$$(ikA + \Lambda)b_\sigma = \sigma b_\sigma, \qquad (6.8)$$

where b_σ is the corresponding eigenvector. On multiplying (6.4) by the conjugate of b_σ, b_σ^*, we easily obtain

$$\sigma_r = (\Lambda b_\sigma, b_\sigma^*)/(b_\sigma, b_\sigma^*), \qquad (6.9)$$

where σ_r denotes the real part of σ. Since Λ is nonpositive, we have

$$\sigma_r \leq 0. \qquad (6.10)$$

Denoting the eigenvalue of largest magnitude by λ^*, we have

$$\lambda^* \leq \sigma_r \leq 0. \qquad (6.11)$$

Hence any truncated system has limited decay, and, as the size of the truncation increases, λ^* increases in magnitude. It should be borne in mind that σ_r is a function of k, as is the eigenvector b_σ. The work of the previous sections indicates that σ_r becomes more negative with increasing k. A statement which is certainly physically reasonable. Further, by increasing the size of the system it is plausible to assume that $\sigma_r(k)$ will become more negative. This is clearly implied by Eq. (6.9). We conclude from this line of reasoning that a truncated system, in limiting the decay of high-wavenumber phenomena, inadequately describes this part of the spectrum. Also in support of this statement are the calculations of Sec. III. We showed there that any coefficient in the series representation of σ is not correctly given until a large enough matrix is taken.

Another type of approximate theory is furnished by the Navier–Stokes equations. As pointed out, this may be extracted from the thirteen moments equation.[7,17] It gives the hydrodynamic roots [(3.12, 13)] to $O(k^2)$.

A simple calculation shows that for large k the three roots are

$$\sigma = \frac{5\lambda_{11}}{5\lambda_{02} + 4\lambda_{11}} + O\left(\frac{1}{k^2}\right), \qquad (6.12)$$

$$\sigma = 4k^2/3\lambda_{02} + O(1), \qquad (6.13)$$

$$\sigma = 5k^2/3\lambda_{11} + O(1). \qquad (6.14)$$

This completes the sketch given in Fig. 5.

This leads us to an interesting situation. From calculations made later in this section and from what has been said we may guess that the hydrodynamic mode in Fig. 1 is qualitatively given by the dotted line. It may return to the real line, but in any case the decay decreases with wavenumber. The Navier–Stokes equations qualitatively give

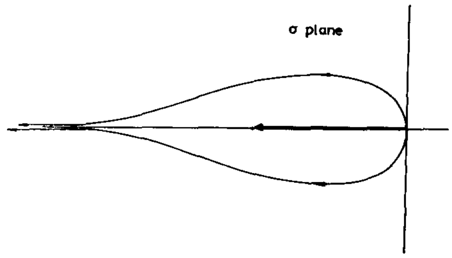


FIG. 5. Navier–Stokes dispersion relation.

this behavior. They may or may not be correct in predicting the loss of propagation—but they do lead to increasing decay with wavenumber. The thirteen moments theory, to the contrary, has bounded decay. They may or may not be correct in predicting propagation at high wavenumbers. But this is really immaterial, since the attenuation at high wavenumbers becomes so great that the presence of a wave no longer seems a sensible notion. Therefore the thirteen moments equations are incorrect at high wavenumber and by the previous discussion they incorrectly give the first propagating mode to the left of the origin in Fig. 2. The only instance in which they are correct is in their description of the hydrodynamic mode for small wavenumber, which is the same as Navier–Stokes to $O(k^2)$. Alternately both the Navier–Stokes and thirteen moments equations have the same solution asymptotically in time.[20] From this discussion we can conclude that it seems fruitless to consider the thirteen moments equations in an initial value problem, since the simpler Navier–Stokes equations are superior in many respects.

We can now develop a system of equations which describes the hydrodynamic and $\lambda_{20} = \lambda_{11}$ mode in Fig. 1. It is obvious that the relative placing of the eigenvalues must be given consideration. From relaxation considerations it is clear that in a homogeneous problem, the moment b_{20} persists for a longer time than does the stress b_{02} (since it is represented by a smaller eigenvalue). Looking at Table II, the eigenvalue ordering, it is tempting to consider the truncated system $i, j < 2$. For small k this would give the first two clusters of branches in Fig. 1. The immediate shortcoming to this system is that it gives insufficient attention to the stress. This would have the consequence of giving the hydrodynamic branches incorrectly to $O(k^2)$. Alternatively the Navier–Stokes could not be extracted from this system. The difficulties of bounded absorption also reappear.

The five exact equations corresponding to the truncated system just mentioned are immediately gotten from (2.11),

$$\begin{aligned}
&(\partial/\partial t)\rho + \partial u/\partial x = 0, \\
&\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\rho + T) + \frac{\partial p_{11}}{\partial x} = 0, \\
&\frac{\partial T}{\partial t} + \frac{2}{3}\frac{\partial u}{\partial x} + \frac{2}{3}\frac{\partial S_1}{\partial x} = 0, \\
&\frac{\partial S_1}{\partial t} - \lambda_{11} S_1 + \frac{5}{2}\frac{\partial T}{\partial x} + \frac{\partial Q}{\partial x} \\
&\qquad + \frac{\partial p_{11}}{\partial x} - \left(\frac{14}{3}\right)^{\frac{1}{2}} \frac{\partial}{\partial x} b_{12} = 0, \\
&\frac{\partial Q}{\partial t} - \lambda_{20} Q + \frac{4}{3}\frac{\partial}{\partial x} S_1 - \frac{(70)^{\frac{1}{2}}}{3}\frac{\partial}{\partial x} b_{21} = 0,
\end{aligned} \tag{6.15}$$

where

$$Q = (10/3)^{\frac{1}{2}} b_{20}, \tag{6.16}$$

and b_{12}, b_{21}, b_{20} are defined by (2.10). In order to close off the system and introduce a Navier–Stokes type of dissipative character into it we can resort to the interpolation scheme of Grad.[7] In short, this consists of dropping b_{21}, b_{12} from (6.15) and neglecting the time derivative of stress in the stress equation (the idea being that it would introduce a negligible transient). The latter yields

$$p_{11} = (4/3\lambda_{02})(\partial u/\partial x) - (8/15\lambda_{02})(\partial S/\partial x). \tag{6.17}$$

A finer analysis, however, shows that this is asymptotically incorrect, and that one must terminate b_{12}, b_{21}, in the same way as b_{02}.[21] This is easily carried out and gives

$$\begin{aligned}
b_{12} &= (-2/5\lambda_{12})(14/3)^{\frac{1}{2}}(\partial S/\partial x), \\
b_{21} &= (1/\lambda_{12})(7/10)^{\frac{1}{2}}(\partial Q/\partial x).
\end{aligned} \tag{6.18}$$

Equations (6.15) coupled with (6.17, 18) constitute a generalization of the Navier–Stokes equation to include the first decaying mode of Fig. 1. In a succeeding paper this will appear as a special case of a generalization of the Chapman–Enskog procedure to include higher modes.

Implicit in each of the truncation and associated interpolation methods is a particular form of the distribution function. Equivalent to the method

[20] D. K. Ai, "Small Perturbations in the Unsteady Flow of a Rarefied Gas Based on Grad's Thirteen-Moments Approximation," in *Rarefied Gas Dynamics* (Academic Press Inc., New York, 1961). Actually since (3.12), (13) are correct to $O(k^4)$, the thirteen moments equations are asymptotically accurate to one higher order.

[21] This finer analysis will be given in a later paper. In it will be given a more rigorous derivation of expressions (6.17, 18).

of truncation is imposing a distribution function of the finite form

$$g = \sum_{r l}^{N} b_r \psi_r, \qquad (6.19)$$

where some particular ordering is taken. When this is substituted into the linearized Boltzmann equation, one finds that the first[22] $(N + 1)$ generated equations form a determined system for the coefficients $b_r(r = 0, 1, \cdots, N)$. This is identical to the truncated system. After solving this system of equations for the b's they are substituted into (6.19) above, and the corresponding distribution function is known. An interpolation scheme only changes the method of finding the b_r.

The criticisms of the moment systems now apply to the corresponding distribution functions, e.g., they can only be presumed to be asymptotically valid. One further remark in support of this is the following: in demanding that a distribution be of the form (6.19) we eliminate consideration of free flow or near-free flow. The insertion of such a distribution function in a free-flow operator immediately leads to the generation of the higher moments which have been excluded from consideration. We now consider another type of approximation which remedies some of these shortcomings.

In a well-known paper, Gross and Jackson[8] give a general technique for approximating the Boltzmann equation. Their equations are gotten by replacing all λ's in Table III past a certain point by a single constant. One of the virtues of this technique is the simplicity of the resulting forms. For instance, on taking all λ's equal to λ_{02}, one gets

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} - \lambda_{02}\right) g = \lambda_{02}[\rho + u\xi_1 + \tfrac{3}{2}T(\tfrac{1}{3}\xi^2 - 1)], \qquad (6.20)$$

which is the linearized "Krooked" equation.[9] The dispersion relation of (6.20) was studied in reference 9. A more detailed analysis is to be found in reference 23.

To gain some understanding of the accuracy of the related dispersion relations, we briefly consider the dispersion relation of (6.20).

We recall that (6.20) was obtained from Table III by replacing all λ's past a certain point by a single constant. Let us consider the hydrodynamic branch of (6.20). Referring to Table III, we see that $i, j < 1$

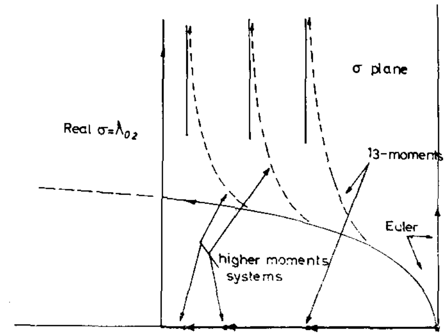


FIG. 6. Single relaxation time model dispersion relation.

remains unchanged and the three branches in the hydrodynamic cluster are given correctly to $O(k)$. To the next order, however, we see that λ_{11} is replaced by λ_{02} and the expansions for σ are somewhat modified. In fact, all higher orders are modified in this way. However, the complexion of the matrix of Table III has not been changed, and the series given by the "Krooked" model is of the correct form but with incorrect constants. The effect of successively higher truncations on the hydrodynamic branches are sketched by the dotted lines moving to the left in Fig. 6. The diffusion mode moves to the left along the real axis, as is shown. The exact theory of the single relaxation model is given by the heavy lines terminating in $\sigma_r = \lambda_{02}$.[23] From the remarks made earlier, we know that no truncated system can have a greater absorption than that given by λ_{02}. For this reason the line $\sigma_r = \lambda_{02}$ plays the role it does in Fig. 6.[24] Nothing has been said of the infinity of branches which have their origin at $\sigma = \lambda_{02}$. For any finite truncation these extraordinary branches appear in the strip $\lambda_{02} < \sigma_r < 0$. As the order of the truncation increases these are shoved over to line $\sigma_r = \lambda_{02}$ and, in fact, the latter line belongs to the spectrum. For a finer discussion the reader is referred to reference 23.

A similar discussion can be given for more detailed models. However, one soon reaches the same difficulties as were encountered with the thirteen moments equations, and the Gross–Jackson technique must be slightly modified to avoid them. Care must be taken in choosing the distinguished moments. For instance, in almost any conceivable problem in fluid mechanics the hydrodynamic mode is all one is interested in. As we saw in Sec. III, the polynomial ordering is the most natural for

[22] Actually more than $(N + 1)$ equations may be generated, and one has the embarrassing situation of more equations than unknowns. This can be avoided by ignoring equations without time derivatives.

[23] L. Sirovich and J. Thurber, Courant Institute of Mathematical Sciences, New York University, Rept. NYO-9757 (1961).

[24] The dashed extension to the left of $\sigma_r = \lambda_{02}$ represents the analytic continuation which appears.

determining the hydrodynamic branches. An entirely different ordering (neither eigenvalue nor polynomial) would be used if one wanted to examine another branch. There is no systematic way in which to order the eigenfunctions. This can be more clearly seen if we allow ourselves a large number of moments in our description. Suppose, for example, we choose to use the polynomial ordering. We would then have the hydrodynamical branch given quite accurately, but we would also have the incorrect evolution of a great number of moments. Many of the moments which occur between those involved in the hydrodynamical calculation will not even appear. On the other hand, strict use of the eigenvalue ordering leads to a correct dispersion for small k, but does not necessarily give any branch to a great accuracy. A compromise between the two methods might be advantageous. One chooses which branches to describe with some degree of accuracy and then collects the moments accordingly. This, of course, corresponds to neither the eigenvalue nor polynomial ordering.

To investigate the nature of the solutions of model equations, as gotten by the Gross–Jackson technique, it suffices to consider (6.20). Letting $\lambda_{02} = \nu$, the solution to (6.19) is

$$\begin{aligned} g(x, t, \xi) = e^{-\nu t} \int_0^t e^{\nu s}[\rho(x + \xi_1\{s - t\}, s) \\ + \xi_1 u(x + \xi_1\{s - t\}, s) \\ + T(x + \xi_1\{s - t\}, s)(\tfrac{1}{2}\xi^2 - \tfrac{3}{2}] \, ds \\ + g(x - \xi_1 t, \xi\, 0)e^{-\nu t}. \end{aligned} \tag{6.21}$$

On taking the appropriate moments, this reduces to

$$\begin{aligned} \begin{bmatrix} \rho(x, t) \\ u(x, t) \\ T(x, t) \end{bmatrix} = e^{-\nu t} \int \begin{bmatrix} 1 \\ \xi_1 \\ (\tfrac{1}{3}\xi^2 - 1) \end{bmatrix} g(x - \xi t, \xi_1 0)\omega \, d\xi \\ + e^{-\nu t} \int_0^t e^{\nu s} f^0 \begin{bmatrix} 1 & \xi_1 & (\tfrac{1}{2}\xi^2 - \tfrac{3}{2}) \\ \xi_1 & \xi_1^2 & \xi_1(\tfrac{1}{2}\xi^2 - \tfrac{3}{2}) \\ (\tfrac{1}{3}\xi^2 - 1) & \xi_1(\tfrac{1}{3}\xi^2 - 1)^2 & \tfrac{2}{3}(\tfrac{1}{2}\xi^2 - \tfrac{3}{2})^2 \end{bmatrix} \begin{bmatrix} \rho(s) \\ u(s) \\ T(s) \end{bmatrix} ds \, d\boldsymbol{\xi}, \end{aligned} \tag{6.22}$$

where the argument of the elements in the last vector is $(x + \xi_1\{s - t\}, s)$. For instance, $\rho(s) = \rho(x + \xi_1\{s - t\}, s)$.

On solution of this system, the flow (ρ, u, T) is substituted into (6.21) giving the perturbed distribution function. We see that the distribution function is given in terms of the initial distribution and the evolution of the flow (ρ, u, T). In general, any model gives the distribution function in terms of the initial distribution and the evolution of a finite number of moments. In a certain sense this is a generalization of the moments method. There, as we saw earlier (6.19), the distribution is given in terms of a finite number of moments and one solves differential equations for the moments. However, full use of the initial data is not made there and, as our considerations indicate, the distribution function is only asymptotically valid. On the other hand, the model solution describes the flow as we approach free flow. In fact, on setting the ν's to zero in (6.21) we get the free flow solution.

ACKNOWLEDGMENTS

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