Class of Exact Solutions in Nonlinear Kinetic Theory

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A class of exact solutions of the one-dimensional steady Krooked equation is obtained. The collision frequency of the latter is left arbitrary. The solutions found depict flows widely removed from equilibrium.

1. INTRODUCTION

FEW exact solutions are available in Kinetic Theory. Certainly the simplest of these is the equilibrium solution itself. In this same category is the spatially dependent equilibrium solution for gas in a potential force field.

The most elementary of the nontrivial solutions are the relaxation solutions. In short, these are obtained by taking the distribution function to be spatially homogeneous. This immediately gives constant hydrodynamical moments and relaxing higher moments. A more general solution of this type was found by Truesdell,[1] who considers a spatially linear velocity field and constant density. He is then able to construct a solution by assuming that all other moments are homogeneous in space but allowing temporal variation. Both of the latter mentioned solutions are given in terms of the moments of the distribution function and depend crucially on having the special type of moments expansion of the collision integral as is obtained for Maxwell molecules.

Another type of exact solution was found by Grad,[2] who sought locally Maxwellian solutions. These were found to be representable by a translation, solid-body rotation, and free expansion. In this case, inasmuch as the collision integral is identically satisfied, the molecular model never enters. Moreover, these solutions do not exhibit dissipation. (They in fact satisfy the Euler equations.)

In this paper a search is made for exact solutions to the single relaxation or Krook equation.[3,4] The solutions found are widely removed from equilibrium in contrast to the presently available exact solutions which are mentioned above. These solutions do not in fact possess higher moments.

[1] C. Truesdell, Ratl. Mech. Anal. **5**, 55 (1956).
[2] H. Grad, Commun. Pure Appl. Math. **2**, 33 (1949); also, see, A. Sommerfeld, *Thermodynamics and Statistical Physics* (Academic Press Inc., New York, 1956).
[3] P. R. Bhatnager, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).
[4] P. Welander, Arkiv Fysik **7**, 44 (1954).

2. FORMULATION OF THE PROBLEM

We consider the one-dimensional form of the single relaxation model equation,

$$\xi_1(\partial/\partial x)f = \nu(x)(f_0 - f), \tag{2.1}$$

where

$$f = f(x, \boldsymbol{\xi}), \quad \text{mass distribution function,}$$

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3), \quad \text{molecular velocity,}$$

$$f_0 = [\rho/(2\pi RT)^{\frac{3}{2}}]e^{-(\boldsymbol{\xi}-\mathbf{U})^2/2RT}, \quad \text{local Maxwellian,}$$

$$\rho(x) = \int f\, d\boldsymbol{\xi}, \tag{2.2}$$

macroscopic mass density,

$$\mathbf{U} = [U(x), 0, 0] = \int \boldsymbol{\xi} f\, d\boldsymbol{\xi}/\rho,$$

macroscopic velocity,

$$T(x) = \int (\boldsymbol{\xi} - \mathbf{U})^2 f\, d\boldsymbol{\xi}/3\rho R, \quad \text{temperature,}$$

and $\nu(x)$ is the collision frequency. (We leave the form of this open for the present.)

Our goal will be to find exact solutions to (2.1). This problem is simplified by introducing the reduced distribution functions

$$F = \int f\, d\xi_2\, d\xi_3, \qquad G = \int (\xi_2^2 + \xi_3^2) f\, d\xi_2\, d\xi_3. \tag{2.3}$$

The model equation (2.1) then reduces to the system

$$\zeta\, \partial F/\partial x = \nu(F_0 - F), \qquad \zeta\, \partial G/\partial x = \nu(2RTF_0 - G), \tag{2.4}$$

where ξ_1 has been replaced by ζ and

$$F_0 = [\rho/(2\pi RT)^{\frac{1}{2}}]e^{-(\zeta-U)^2/2RT}, \tag{2.5}$$

with

$$\rho = \int F\,d\zeta, \qquad \rho U = \int F\zeta\,d\zeta,$$
$$\rho RT = \frac{1}{3}\left[\int (\zeta - U)^2 F\,d\zeta + \int G\,d\zeta\right]. \tag{2.6}$$

The collision frequency is easily removed from the system (2.4) by the transformation

$$dy/dx = \nu(x). \tag{2.7}$$

Integration of (2.7) will be left for later. The system (2.4) now becomes

$$\zeta\,\partial F/\partial y = (F_0 - F), \qquad \zeta\,\partial G/\partial y = (2RTF_0 - G), \tag{2.8}$$

with the moments as before given by (2.6).

For our purposes it is convenient to consider $|\zeta|$ instead of ζ. We, therefore, introduce

$$F_\pm = H(\pm\zeta)F, \qquad G_\pm = H(\pm\zeta)G,$$

where H is the Heaviside operator. We now obtain instead of (2.8)

$$\pm\,|\zeta|\,\partial F_\pm/\partial y = (F_0 - F_\pm),$$
$$\pm\,|\zeta|\,\partial G_\pm/\partial y = (2RTF_0 - G_\pm). \tag{2.9}$$

Henceforth, we regard ζ as positive and, so drop the absolute value sign. The moments now are given by

$$\rho = \int_0^\infty (F_+ + F_-)\,d\zeta,$$
$$\rho RT = \frac{1}{3}\int [(\zeta - U)^2(F_+ + F_-) + (G_+ + G_-)]\,d\zeta, \tag{2.10}$$
$$\rho U = \int_0^\infty \zeta(F_+ - F_-)\,d\zeta.$$

3. EXACT SOLUTIONS

One easily shows that the Eqs. (2.9) are invarient under the one parameter group of transformations

$$y = \alpha y', \quad \zeta = \alpha\zeta', \quad F = \alpha^a F', \quad G = \alpha^{a+2}G' \tag{3.1}$$

for arbitrary values of a. We, therefore, investigate the similarity solutions

$$F_\pm = \zeta^a \tilde{F}_\pm(\zeta/y), \qquad G_\pm = \zeta^{a+2}\tilde{G}_\pm(\zeta/y). \tag{3.2}$$

The hydrodynamical moments now become

$$\frac{\rho}{y^{a+1}} = \tilde{\rho} = \int_0^\infty \mu^a[\tilde{F}_+(\mu) + \tilde{F}_-(\mu)]\,d\mu, \tag{3.3}$$

$$\frac{\rho RT}{y^{a+3}} = \tilde{\rho}R\tilde{T} = \frac{1}{3}\int_0^\infty [(\mu - \tilde{U})^{a+2}(\tilde{F}_- + \tilde{F}_-) + \mu^{a+2}(\tilde{G}_+ + \tilde{G}_-)]\,d\mu, \tag{3.4}$$

$$T/y^2 = \tilde{T},$$

$$\frac{\rho U}{y^{a+2}} = \tilde{\rho}\tilde{U} = \int \mu^{a+1}[\tilde{F}_+(\mu) - \tilde{F}_-(\mu)]\,d\mu,$$
$$U/y = \tilde{U}, \tag{3.5}$$

with

$$\mu = \zeta/y. \tag{3.6}$$

Substitution of (3.5) into the continuity equation gives

$$\tilde{\rho}\tilde{U}(\partial/\partial y)y^{a+2} = 0,$$

so that $\tilde{U} = 0$ unless $a = -2$. We shall see that $a \leq -4$, so that only motionless solutions will be considered. Further, in the analysis $y > 0$ will be assumed. The extension to $y < 0$ is straightforward. Substituting (3.2) into (2.9) we find

$$\mp\,(d/d\mu)\tilde{F}_\pm = (\tilde{F}_0/\mu^{2+a}) - (\tilde{F}_\pm/\mu^2), \tag{3.7}$$

$$\mp\,d\tilde{G}_\pm/d\mu = (2R\tilde{T}\tilde{F}_0/\mu^{4+a}) - (\tilde{G}_\pm/\mu^2). \tag{3.8}$$

Integrating we find

$$\tilde{F}_+(\mu) = e^{-1/\mu}\int_\mu^\infty e^{1/\mu'}\,\frac{\tilde{F}_0(\mu')\,d\mu'}{\mu'^{2+a}} + \tilde{F}_+(\infty)e^{-1/\mu}, \tag{3.9}$$

$$\tilde{F}_-(\mu) = e^{1/\mu}\int_0^\mu e^{-1/\mu'}\,\frac{\tilde{F}_0(\mu')\,d\mu'}{\mu'^{2+a}}, \tag{3.10}$$

$$\tilde{G}_+(\mu) = 2R\tilde{T}e^{-1/\mu}\int_\mu^\infty e^{1/\mu'}\,\frac{\tilde{F}(\mu')\,d\mu'}{\mu'^{4+a}} + \tilde{G}_+(\infty)e^{-1/\mu}, \tag{3.11}$$

$$\tilde{G}_-(\mu) = 2R\tilde{T}e^{1/\mu}\int_0^\mu e^{-1/\mu'}\,\frac{\tilde{F}_0(\mu')\,d\mu'}{\mu'^{4+a}}, \tag{3.12}$$

where the limits of integration have been dictated by the equations. In integrating we have allowed $\tilde{F}_+$ and $\tilde{G}_+$ to be nonzero at ∞. Regarding (3.10) and (3.12) we see that $\tilde{F}_-$ and $\tilde{G}_-$ are also nonzero at ∞. Restricting attention to integer values of a, we see from (3.4) that $a \leq -4$. Various other forms for (3.9)–(3.12) are easily obtained by parts integration. It should be noted that the class of solutions (3.9)–(3.12) are algebraically (not exponentially) decreasing at infinity. It is important to note, therefore, that all higher moments do not converge.

The determination of $\tilde{F}_+(\infty)$ and $\tilde{G}_+(\infty)$ now follow from (3.3) and (3.4). Considering the expression for ρ,

$$\tilde{\rho} = \int_0^\infty \mu^a \left[e^{-1/\mu} \int_\mu^\infty e^{1/\mu'} \frac{\tilde{F}_0(\mu')\, d\mu'}{\mu'^{2+a}} + e^{1/\mu} \int_0^\mu e^{-1/\mu'} \frac{\tilde{F}_0(\mu')\, d\mu'}{\mu'^{2+a}} \right] d\mu + \tilde{F}_+(\infty) \int_0^\infty \mu^a e^{-1/\mu}\, d\mu,$$

we write

$$\tilde{\rho} = \tilde{\rho}_1 + \tilde{\rho}_2 + \tilde{\rho}_3,$$

where

$$\tilde{\rho}_1 = \int_0^\infty \mu^a e^{-1/\mu} \int_\mu^\infty e^{-1/\mu'} \frac{\tilde{F}_0(\mu')}{\mu'^{2+a}}\, d\mu'\, d\mu,$$

$$\tilde{\rho}_3 = \tilde{F}_+(\infty) \int_0^\mu \mu^a e^{-1/\mu}\, d\mu.$$

Interchanging orders of integration,

$$\tilde{\rho}_1 = \int_0^\infty \int_0^{\mu'} \mu^a e^{-1/\mu}\, d\mu\, \frac{\tilde{F}_0(\mu') e^{1/\mu'}}{\mu'^{2+a}}\, d\mu'.$$

Writing

$$a = -n$$

one easily shows

$$\int_0^\mu \mu'^{-n} e^{1/\mu'}\, d\mu' = e^{-1/\mu} \sum_{i=0}^{n-2} \left(\frac{1}{\mu}\right)^{n-2-i} \frac{(n-2)!}{(n-2-i)!}.$$

Therefore,

$$\tilde{\rho}_1 = \sum_{i=0}^{n-2} \frac{(n-2)!}{(n-2-i)!} \int_0^\infty F_0(\mu) \mu^i\, d\mu.$$

In the same way we find

$$\tilde{\rho}_2 = \sum_{i=0}^{n-2} \frac{(n-2)!\,(-)^i}{(n-2-i)!} \cdot \int_0^\infty F_0(\mu)\mu^i\, d\mu + (-)^{n-1}(n-2)! \cdot \int_0^\infty F_0(\mu)\mu^{n-2} e^{-1/\mu}\, d\mu,$$

$$\tilde{\rho}_3 = (n-2)!\, \tilde{F}_+(\infty).$$

Combining terms we find after some manipulation

$$0 = \sum_{k=1}^{[\frac{1}{2}(n-2)]} \frac{(2k)!\,(R\tilde{T})^k}{(n-2-2k)!\,k!\,2^k} - (-)^n (R\tilde{T})^{\frac{1}{2}(n-2)} I_{n-2}[1/(R\tilde{T})^{\frac{1}{2}}] + \frac{\tilde{F}_+(\infty)}{\tilde{\rho}}, \qquad (3.13)$$

where

$$I_n(x) = \int_0^\infty e^{-\frac{1}{2}\zeta^2 - (x/\zeta)} \zeta^n \frac{d\zeta}{(2\pi)^{\frac{1}{2}}} \qquad (3.14)$$

and [ ] denotes the integral part operator. The expression (3.14) has been considered in the literature[5] and is partially tabulated. Equation (3.13) determines $\tilde{F}_+(\infty)$ for given values of $\tilde{\rho}$ and $\tilde{T}$.

In a similar manner (3.4) yields

$$0 = \sum_{k=1}^{[\frac{1}{2}(n-4)]} \frac{(2k+1)!\,(R\tilde{T})^k}{(n-4-2k)!\,k!\,2^k} - (-)^n (R\tilde{T})^{\frac{1}{2}(n-4)} I_{n-2}[1/(R\tilde{T})^{\frac{1}{2}}] + 2 \sum_{k=1}^{[\frac{1}{2}(n-4)]} \frac{(2k)!\,(R\tilde{T})^k}{(n-4-2k)!\,2^k k!} - 2(-)^n (R\tilde{T})^{\frac{1}{2}(n-4)} I_{n-4}[1/(R\tilde{T})^{\frac{1}{2}}] + \frac{\tilde{F}_+(\infty) + \tilde{G}_+(\infty)}{\tilde{\rho} R\tilde{T}}, \qquad (3.15)$$

which with (3.13) determines $\tilde{G}_+(\infty)$. {The summations in (3.15) are defined to be zero for $[\frac{1}{2}(n-4)] < 1$.}

As an example, we consider the case $n = 4$. From (3.13) we obtain

$$-1 + I_2[1/(R\tilde{T})^{\frac{1}{2}}] = \tilde{F}_+(\infty)/\tilde{\rho}R\tilde{T}, \qquad (3.16)$$

and from (3.15)

$$1 + 2I_0[1/(R\tilde{T})^{\frac{1}{2}}] = \tilde{G}_+(\infty)/\tilde{\rho}R\tilde{T}. \qquad (3.17)$$

Inspection of (3.14) reveals that $I_2 \leq \frac{1}{2}$ so that $\tilde{F}_+(\infty)$ is negative. However, by choosing $\tilde{\rho}R\tilde{T}$ sufficiently small, $\tilde{F}_+(\infty)$ may be made as small as we please. $\tilde{G}_+(\infty)$ on the other hand is definitely positive. In general one must take $\tilde{\rho}R\tilde{T}$ small to make $\tilde{F}_+(\infty)$ small in magnitude.

4. EXTENSIONS

The solutions obtained in the last section apply directly to the case of constant collision frequency. In this case we take $y = \nu x$ as the integral of (2.7) to complete the solution. A constant collision frequency is of course unrealistic, and it now remains to consider a general form for the collision frequency.

A simple calculation relates the viscosity $\bar{\mu}$ to collision frequency as follows[6]:

$$\bar{\mu} = \rho RT/\nu. \qquad (4.1)$$

As is well-known from Kinetic Theory,[7]

$$\bar{\mu} \sim T^\beta,$$

[5] C. T. Zahn, Phys. Rev. **52**, 67 (1937); C. Laporte, *ibid.* **52**, 72 (1937); M. Abramowitz, J. Math. & Phys. **32**, 188 (1953).

[6] As is well-known both the viscosity and the heat conductivity cannot be simultaneously chosen for the single relaxation model. Our choice of viscosity is only made for the sake of definiteness.

[7] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, New York, 1952).

where $\frac{1}{2} \leq \beta \leq 1$ (ranging from rigid spheres to Maxwell molecules, respectively). We, therefore take[8]

$$\nu = \lambda\rho T^{\alpha}, \qquad 0 \leq \alpha \leq \tfrac{1}{2}. \tag{4.2}$$

Substitution into (2.7) yields

$$dy/dx = \lambda\tilde{\rho}\tilde{T}^{\alpha}y^{-n+1+2\alpha},$$

where we have made use of (3.3) and (3.4). Since $n \geq 4$ we integrate from the origin and find

$$y = [(n - 2\alpha)\lambda\tilde{\rho}\tilde{T}^{\alpha}x]^{1/(n-2\alpha)}. \tag{4.3}$$

Finally, the hydrodynamical moments are given by

$$\begin{aligned} T &= \tilde{T}[(n - 2\alpha)\lambda\tilde{\rho}\tilde{T}^{\alpha}x]^{1/(n-2\alpha)}, \\ \rho &= \tilde{\rho}[(n - 2\alpha)\lambda\tilde{\rho}\tilde{T}x]^{(-n+1)/(n-2\alpha)}. \end{aligned} \tag{4.4}$$

[8] More general choices of ν can easily be taken without additional difficulty. In fact, since the distribution functions are well removed from equilibrium, having faithful transport coefficient expressions seems hardly necessary.

The reduced distribution functions are given by (3.2) with (3.9)–(3.12), and this completes the problem.

5. CONCLUSIONS

A wide class of exact solutions of the single relaxation model has been found. In all, three parameters $\tilde{T}$, $\tilde{\rho}$, and $n \geq 4$ may be chosen arbitrarily. The hydrodynamical moments ρ and T in each instance are such that the density falls off to zero and the temperature becomes unbounded at infinity. An especially noteworthy feature of these solutions is that all sufficiently high moments diverge. The solutions exhibited, therefore, are widely removed from equilibrium.

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