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Boundary-Value Problems in Compressible Magnetohydrodynamics

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Linearized steady two-dimensional compressible magnetohydrodynamics is considered. No restriction on the gas law or on the field orientations is made. Thin-body flow problems are solved for all flow field regimes (e.g. doubly hyperbolic, hyperliptic). The method of solution is based on replacing material boundaries by surfaces of discontinuity. The discontinuous forms of the magnetohydrodynamic equations are derived in full generality. It is then shown that the solution to any problem may be represented in terms of the fundamental solution. The latter is obtained in closed form for all regimes. The final solution is then reduced to a single integral equation which may be solved in all cases.

I. INTRODUCTION

THIS investigation is concerned with the study of steady nondissipative, two-dimensional magnetohydrodynamic flow past bodies. This problem has been considered previously by many authors.[1] However, the present method has the advantage of systematically dealing with all cases and thereby furnishing solutions to many situations not previously considered. The fluid is compressible, no specific gas law is employed, and the applied magnetic field can assume any orientation with respect to the free stream direction.

A linearized theory is presented in which small disturbances from a constant free-stream velocity and applied magnetic field are considered. The solutions to particular boundary value problems are obtained through the use of fundamental solutions. Therefore, in the next section we show how the fundamental solutions are superposed to give the flow field for the flow past bodies. This is achieved by defining the flow field everywhere and regarding the body surface as a distribution of sources. In Appendix A, we carry out the derivation of the magnetohydrodynamic equations under such conditions. The derivation is given for unsteady flows in n dimensions with moving boundaries and for an arbitrary dissipative gas. This generality, although greater than what is required for the present case, is included since it will be needed in subsequent work, and it is expedient to include it here for future reference.

In Sec. III the fundamental solutions are obtained, and in Sec. IV we use them to solve several boundary value problems.

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[1] A. Jeffrey and T. Taniuti, *Non-Linear Wave Propagation* (Academic Press, Inc., New York, 1964).

II. THE USE OF FUNDAMENTAL SOLUTIONS IN MAGNETOHYDRODYNAMICS

The equations of magnetohydrodynamics are well known.[1] If ρ', e', p', $\mathbf{u}'$ represent the mass density, internal energy, fluid pressure, and velocity; $\mathbf{E}'$, $\mathbf{J}$, $\mathbf{B}'$ represent the electric field, total current density, and magnetic induction; then the equations of conservation of mass, momentum, and energy are

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot \rho' \mathbf{u}' = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho'\mathbf{u}') + \nabla \cdot (\rho'\mathbf{u}'\mathbf{u}') + \nabla p' - (\mathbf{J} \times \mathbf{B}') = \mathfrak{S}, \tag{2.2}$$

$$\frac{\partial}{\partial t}\rho'\left(e' + \frac{u'^2}{2}\right) + \nabla \cdot \left[\rho'\mathbf{u}'\left(e' + \frac{u'^2}{2}\right)\right] + \nabla \cdot (p'\mathbf{u}') - \mathbf{J} \cdot \mathbf{E}' = 0, \tag{2.3}$$

$$\nabla \cdot \mathbf{B}' = 0, \tag{2.4}$$

$$\mathbf{J} - \frac{1}{\mu} \nabla \times \mathbf{B}' = \mathfrak{J}, \tag{2.5}$$

$$\mathbf{E}' + \mathbf{u}' \times \mathbf{B}' = 0, \tag{2.6}$$

$$\nabla \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial t} = 0. \tag{2.7}$$

The quantities on the right-hand sides account for the presence of material boundaries within the fluid. More general source terms are derived in Appendix A. For the case of a closed impermeable body immersed in the fluid, it is shown there that

$$\mathfrak{S} = p'\mathbf{n}\ \delta(S), \tag{2.8}$$

where $S = 0$ denotes the equation of the body surface (the one-dimensional delta function $\delta(S)$ is defined in Appendix A) and $\mathbf{n}$ is the outward drawn normal. It is also shown that for a nonconducting body $\mathfrak{J}$ in (2.5) is given by

$$\mathfrak{J} = \frac{1}{\mu} [\mathbf{B}_t'] \times \mathbf{n}\ \delta(S), \tag{2.9}$$

where $[\mathbf{B}_t'] \times \mathbf{n} = [\mathbf{B}'] \times \mathbf{n}$ represents the jump in the tangential component of the magnetic field across the surface $S = 0$. The requirement that the tangential component of the Maxwell stress tensor must be continuous across $S = 0$ can be shown to imply $B_n'[\mathbf{B}_t'] = 0$, where B_n' is the component of the magnetic-field normal to $S = 0$. (B_n' is, of course, continuous across $S = 0$.) Consequently, if $B_n' \neq 0$, $[\mathbf{B}_t'] = 0$.

Consider an infinitely long cylindrical body with generators parallel to the y axis of a set of rectangular coordinates. We assume, without loss of generality, that the free-stream velocity lies in the x–y plane, and take the applied magnetic field to lie in the x–z plane. Assuming steady flow, the flow field will depend only on x and z. We now distinguish two cases.

If the upstream magnetic field is parallel to the flow velocity, one can show that the stream lines and magnetic lines of force remain parallel. This means that the field is parallel to the body at the outer surface and $B_n'=0$. $[\mathbf{B}_t']$ is just $\mathbf{B}'$ since the magnetic field inside the body vanishes.

Next, consider the case where the projections of the velocity and magnetic fields lines onto the x–z plane are not aligned upstream. It is easy to show that they are then nowhere aligned in the x–z plane, so $B_n' \neq 0$. Hence $[\mathbf{B}_t'] = 0$ and $\mathfrak{J}$ in (2.5) is identically zero.

Denote the upper and lower surfaces of the body by $z = \epsilon f_u(x)$ and $z = \epsilon f_l(x)$, where ϵ is a small parameter representing the thinness of the body. Since $\delta(S)$ can be written (see Appendix A)

$$\delta(S) = \int_{\text{surface}} \delta(x - x_0)\ \delta(z - z_0)\ dS_0,$$

where (x_0, z_0) is a point on the body surface, we can write the momentum source equation (2.8), as

$$p'\mathbf{n}\ \delta(S) = \int_{-\infty}^{+\infty} H(x_0)H(1 - x_0)p_u'\mathbf{n}_u\ \delta(x - x_0) \cdot \delta[z - \epsilon f_u(x_0)]\ \frac{dS_0}{dx_0}\,dx_0 + \int_{-\infty}^{+\infty} H(x_0)H(1 - x_0)p_l'\mathbf{n}_l\ \delta(x - x_0) \cdot \delta[z - \epsilon f_l(x_0)]\ \frac{dS_0}{dx_0}\,dx_0,$$

where the body has been normalized to unit length and p_u', p_l', $\mathbf{n}_u$, $\mathbf{n}_l$, are the pressure and normal at the upper and lower surfaces of the body. Since $dS/dx = 1 + O(\epsilon^2)$, we have, to $O(\epsilon^2)$

$$\begin{aligned} p'\mathbf{n}\ \delta(S) &= [p_u'\mathbf{n}_u\ \delta(z - \epsilon f_u(x)) + p_l'\mathbf{n}_l\ \delta(z - \epsilon f_l(x))]H(x)H(1 - x) \\ &= \{(p_u' - p_l')\mathbf{i}_z\ \delta(z) - \epsilon(p_u'f_u' - p_l'f_l')\mathbf{i}_x\ \delta(z) \\ &\quad - \epsilon(p_u'f_u - p_l'f_l)\mathbf{i}_z\ \delta'(z)\}H(x)H'(1 - x) + O(\epsilon^2), \end{aligned} \tag{2.10}$$

where $\delta'(z)$ denotes the derivative of $\delta(z)$ $\mathbf{i}_x$, $\mathbf{i}_z$ are unit vectors in the x and z directions, and $f_{u,\epsilon}'$ denotes the derivative.

The linearization of the current source, Eq. (2.9), is carried out in a similar fashion. Recalling the above remarks, this term is nonvanishing only in the aligned-fields case, and then we have $[\mathbf{B}_t'] = \mathbf{B}'$, where $\mathbf{B}'$ is the field at the body surface. We then obtain

$$\begin{aligned} (1/\mu)[\mathbf{B}'] \times \mathbf{n}\ \delta(S) &= \mathbf{i}_y[-(1/\mu)(B_{ux}' - B_{lx}')\ \delta(z) \\ &\quad + (\epsilon/\mu)(B_{ux}'f_u - B_{lx}'f_l)\ \delta'(z)[H(x)H(1 - x)] + O(\epsilon^2), \end{aligned} \tag{2.11}$$

where $\mathbf{i}_y$ is the unit vector in the y direction.

We seek a perturbation solution of the form

$$\begin{aligned} (\mathbf{u}', \mathbf{B}', p', \rho', e', \mathbf{E}', \mathbf{J}) &= (\mathbf{U}_0, \mathbf{B}_0, p_0, \rho_0, e_0, \mathbf{E}_0, 0) \\ &\quad + \epsilon(\hat{\mathbf{u}}, \hat{\mathbf{b}}, \hat{p}, \hat{\rho}, \hat{e}, \hat{\mathbf{E}}, \hat{\mathbf{j}}) + O(\epsilon^2), \end{aligned} \tag{2.12}$$

where the zero subscripts denotes the undisturbed field, and

$$\mathbf{E}_0 + \mathbf{U}_0 \times \mathbf{B}_0 = 0. \tag{2.13}$$

We define

$$a_0^2 = \left(\frac{\partial p_0}{\partial \rho_0}\right)_{T_0}, \qquad A^2 = \frac{B_0^2}{\mu \rho_0 a_0^2}, \qquad c_v = \left(\frac{\partial e_0}{\partial T_0}\right)_{\rho_0}, \qquad c_0^2 = \left(\frac{\partial p_0}{\partial \rho_0}\right)_{S_0},$$

where a_0 is the isothermal speed of sound and c_0 the adiabatic speed of sound. Next we normalize

$$\mathbf{u} = \frac{\hat{\mathbf{u}}}{a_0}, \quad \rho = \frac{\hat{\rho}}{\rho_0}, \quad T = \frac{(\partial p_0/\partial T_0)\rho_0}{\rho_0 a_0^2}\,\hat{T}, \quad \mathbf{b} = \frac{\hat{\mathbf{b}}}{|\mathbf{B}_0|};$$

$$\mathbf{U} = \frac{\mathbf{U}_0}{a_0}, \quad \mathbf{B} = \frac{\mathbf{B}_0}{|\mathbf{B}_0|}, \quad \mathbf{E} = \frac{\hat{\mathbf{E}}}{a_0\,|\mathbf{B}_0|}, \quad \mathbf{x} = \frac{\mathbf{x}'}{L};$$

$$p = \frac{\hat{p}}{\rho_0 a_0^2}, \qquad P_0 = \frac{p_0}{\rho_0 a_0^2}.$$

(L is the body length and we can, without loss of generality, take $L = 1$.) Introducing the normalization, we have to first order,

$$\nabla\cdot\mathbf{u} + \mathbf{U}\cdot\nabla\rho = 0, \tag{2.14}$$

$$\mathbf{U}\cdot\nabla\mathbf{u} + \nabla\rho + \nabla T - A^2(\nabla\times\mathbf{b})\times\mathbf{B} = \mathfrak{M}, \tag{2.15}$$

$$\mathbf{U}\cdot\nabla T + (\gamma - 1)\nabla\cdot\mathbf{u} = \mathfrak{F}, \tag{2.16}$$

$$\mathbf{U}\times\mathbf{b} + \mathbf{u}\times\mathbf{B} + \mathbf{E} = 0, \tag{2.17}$$

$$\nabla\cdot\mathbf{b} = 0, \tag{2.18}$$

$$\nabla\times\mathbf{E} = 0, \tag{2.19}$$

where we have used the relation

$$\gamma = c_p/c_v = c_0^2/a_0^2$$

which may be proved with some manipulation. (c_p is, of course, the specific heat at constant pressure.) We have left as arbitrary the gas laws $e = e(\rho, T)$, $p = p(\rho, T)$ subject only to the compatibility relation

$$p - \rho^2(\partial e/\partial\rho)_T = +T(\partial p/\partial T)_\rho.$$

The sources are

$$\begin{aligned}\mathfrak{M} &= \delta(z)\int \mathbf{M}(x_0)\ \delta(x - x_0)\ dx_0 \\ &\quad + \delta'(z)\int \bar{\mathbf{M}}(x_0)\ \delta(x - x_0)\ dx_0, \\ \mathfrak{F} &= \delta(z)\int F(x_0)\ \delta(x - x_0)\ dx_0,\end{aligned} \tag{2.20}$$

where, in the nonaligned fields case,

$$\begin{aligned}M_x &= -P_0[f_u'(x) - f_l'(x)]H(x)H(1 - x), \\ M_z &= [p_u(x) - p_l(x)]H(x)H(1 - x), \\ F &= +P_0U_x\frac{(\partial p_0/\partial T_0)_{\rho_0}}{\rho_0 c_v} \\ &\quad \cdot[f_u'(x) - f_l'(x)]H(x)H(1 - x), \\ \bar{M}_x &= 0, \\ \bar{M}_z &= -P_0[f_u(x) - f_l(x)]H(x)H(1 - x);\end{aligned} \tag{2.21}$$

and, in the aligned fields case,

$$\begin{aligned}M_x &= -P[f_u'(x) - f_l'(x)]H(x)H(1 - x), \\ M_z &= [p_u(x) - p_l(x)]H(x)H(1 - x) \\ &\quad + A^2[b_{ux}(x) - b_{lx}(x)]H(x)H(1 - x), \\ F &= P_0U_x\frac{(\partial p_0/\partial T_0)_{\rho_0}}{\rho_0 c_v}[f_u'(x) - f_l'(x)]H(x)H(1 - x), \\ \bar{M}_x &= 0, \\ \bar{M}_z &= -[f_u(x) - f_l(x)](P_0 + A^2)H(x)H(1 - x).\end{aligned} \tag{2.22}$$

The source term appears in the energy equation as a result of eliminating the mechanical work from it. The magnetic terms appear in $\mathbf{M}$ after substituting for $\mathbf{J}$ from (2.5) and (2.11).

The above system (2.14–2.19) may be symbolically written as

$$\begin{aligned}\mathbf{Lv} &= \delta(z)\int_{-\infty}^{\infty} \mathbf{S}(x_0)\ \delta(x - x_0)\ dx_0 \\ &\quad + \delta'(z)\int_{-\infty}^{\infty} \bar{\mathbf{S}}(x_0)\ \delta(x - x_0)\ dx_0,\end{aligned} \tag{2.23}$$

where $\mathbf{v}$ represents the dependent variables and $\mathbf{L}$, $\mathbf{S}$ and $\bar{\mathbf{S}}$ are defined in an obvious manner.

It is useful to consider the fundamental solution $\mathbf{v}^f$, which satisfies

$$\mathbf{Lv}^f = \mathbf{G}(x_0)\ \delta(x)\ \delta(z), \tag{2.24}$$

where $\mathbf{G}(x_0)$ is only a function of x_0 and is therefore a constant with respect to the equation. To emphasize the dependence of $\mathbf{v}^f$ on $\mathbf{G}$ we write

$$\mathbf{v}^f = \mathbf{v}^f[x, z; \mathbf{G}(x_0)].$$

One easily sees then, that the solution to (2.23) is given by

$$\begin{aligned}\mathbf{v} &= \int_{-\infty}^{\infty} \mathbf{v}^f[x - x_0, z; \mathbf{S}(x_0)]\ dx_0 \\ &\quad + \frac{\partial}{\partial z}\int \mathbf{v}^f[x - x_0, z; \bar{\mathbf{S}}(x_0)]\ dx_0\end{aligned} \tag{2.25}$$

with $\mathbf{S}$ and $\bar{\mathbf{S}}$ given by (2.21) or (2.22). We therefore consider (2.24) in the next section.

It is well known that the superposition of fundamental solutions leads, in general, to the solution of integral equations involving the boundary data. However, it is evident in this case that the solution is given in closed form in terms of the body thickness and the lift distribution. Therefore for the indirect problem, i.e., when lift distribution and body thickness are given, (2.25) yields the solution immediately. However, if the problem is to be determined in terms of body shape alone, the only unknown function, M_z, can be found from the boundary condition $\mathbf{u}'\cdot\mathbf{n} = 0$ at the body surface.

III. FUNDAMENTAL SOLUTIONS

We now solve Eqs. (2.24) to obtain the fundamental solutions. We seek solutions which are independent of y. The condition $\nabla\cdot\mathbf{b} = 0$ can be satisfied by introducing a stream function φ such that

$$b_x = \frac{\partial\varphi}{\partial z}, \qquad b_z = -\frac{\partial\varphi}{\partial x}. \tag{3.1}$$

Then, in component form, the equations for the fundamental solutions are

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} + U_x \frac{\partial \rho}{\partial x} = m\ \delta(x)\ \delta(z), \qquad (3.2)$$

$$U_x \frac{\partial u_x}{\partial x} + \frac{\partial \rho}{\partial x} + \frac{\partial T}{\partial x} - A^2 B_x \frac{\partial^2 \varphi}{\partial x^2} - A^2 B_z \frac{\partial^2 \varphi}{\partial z^2} = M_x\ \delta(x)\ \delta(z), \qquad (3.3)$$

$$U_x \frac{\partial u_z}{\partial x} + \frac{\partial \rho}{\partial z} + \frac{\partial T}{\partial z} + A^2 B_x \frac{\partial^2 \varphi}{\partial z^2} + A^2 B_z \frac{\partial^2 \varphi}{\partial x^2} = M_z\ \delta(x)\ \delta(z), \qquad (3.4)$$

$$U_x \frac{\partial T}{\partial x} + (\gamma - 1) \frac{\partial u_x}{\partial x} + (\gamma - 1) \frac{\partial u_z}{\partial z} = F\ \delta(x)\ \delta(z), \qquad (3.5)$$

$$B_x u_z - B_z u_x + U_x \frac{\partial \varphi}{\partial x} = 0, \qquad (3.6)$$

$$U_x \frac{\partial u_y}{\partial x} - A^2 B_x \frac{\partial b_y}{\partial x} - A^2 B_z \frac{\partial b_y}{\partial z} = M_y\ \delta(x)\ \delta(z), \qquad (3.7)$$

$$B_x \frac{\partial u_y}{\partial x} + B_z \frac{\partial u_y}{\partial z} - U_x \frac{\partial b_y}{\partial x} = 0, \qquad (3.8)$$

where m, M_x, M_y, M_z, F are now regarded as constants. We have included sources in (3.1) and (3.7) for generality.

Equation (3.6) is the y component of the magnetic-flux equation and (3.8) is a combination of the x and z components. [Eq. (3.8) is the y component of the curl of the magnetic-flux equation.] Note that U_y does not enter into the equations, but does appear in the boundary conditions. Also (3.7) and (3.8), for the y component of the velocity and magnetic induction, decouple from the others, so that each system can be solved separately. The decoupled mode (3.7), (3.8) is considered in Section IIIC.

Introducing Fourier transforms, e.g.,

$$\rho(\mathbf{k}) = \int_{-\infty}^{+\infty} \rho(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x},$$

we obtain the transformed system

$$\begin{bmatrix} ik_1 U_x & ik_1 & ik_3 & 0 & 0 \\ ik_1 & ik_1 U_x & 0 & ik_1 & (k_1^2 + k_3^2) A^2 B_z \\ ik_3 & 0 & ik_1 U_x & ik_3 & -(k_1^2 + k_3^2) A^2 B_x \\ 0 & ik_1(\gamma - 1) & ik_3(\gamma - 1) & ik_1 U_x & 0 \\ 0 & -B_z & B_x & 0 & ik_1 U_x \end{bmatrix} \begin{bmatrix} \rho \\ u_x \\ u_z \\ T \\ \varphi \end{bmatrix} = \begin{bmatrix} m \\ M_x \\ M_z \\ F \\ 0 \end{bmatrix} \qquad (3.9)$$

or, in an obvious notation,

$$\mathbf{Mv} = \mathbf{G}.$$

We may write the inverse of $\mathbf{M}$ as

$$\mathbf{M}^{-1} = \mathbf{C}/D,$$

where $D = \det(\mathbf{M})$ and $\mathbf{C}$ is the signed transposed cofactor matrix of $\mathbf{M}$. The formal solution of the system (3.2)–(3.6) is

$$\mathbf{v}(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\mathbf{C}}{D} \mathbf{G}\, d\mathbf{k}. \qquad (3.10)$$

The determinant D is a homogeneous fifth-degree polynomial given by

$$\begin{aligned} D = {} & ik_1^5[U_x^5 - U_x^3 A^2 + \gamma U_x A^2 B_x^2 - \gamma U_x^3] \\ & + ik_1^4 k_3 [2\gamma U_x A^2 B_x B_z] \\ & + ik_1^3 k_3^2 [-\gamma U_x^3 - A^2 U_x^3 + \gamma U_x A_z^2] \\ & + ik_1^2 k_3^3 [2\gamma U_x A^2 B_x B_z] \\ & + ik_1 k_3^4 [\gamma U_x A^2 B_z^2] = i\Gamma k_1 \prod_{i=1}^{4} (k_1 - d_i k_3). \end{aligned} \qquad (3.11)$$

The characteristics are given by the real roots of the equation $D(d) = 0$, where $d = k_1/k_3$. The root $d = d_0 = 0$ can be factored, leaving a fourth-order equation for d. It can be shown[2] that two roots are always real (we denote these by d_1 and d_2) while the other two (d_3, d_4) can be either real or complex. Hence the magnetohydrodynamic flow field can be either doubly hyperbolic or a superposition of a hyperbolic and an elliptic field. The root $d_0 = 0$ corresponds to the nondissipative limit of the wake.

When $B_z = 0$, the last two terms of D are zero and the equation becomes d^3 times a quadratic in d. The extra two $d = 0$ roots correspond to the two waves (d_1 and d_2) which collapse on the axis as $B_z \to 0$. The equation $D = 0$ then has only two nonzero roots, given by

$$d_3, d_4 = \pm \left[\frac{\gamma U_x^2 + A^2 U_x^2 - \gamma A^2}{U_x^4 - U_x^2 A^2 + \gamma A^2 - \gamma U_x^2} \right]^{\frac{1}{2}}. \qquad (3.12)$$

[2] J. E. McCune and E. L. Resler, J. Aerospace Sci. **27**, 493 (1960).

These roots can be either real or imaginary, so that the flow field is therefore either hyperbolic or elliptic.

The angle the characteristic makes with the positive x axis is shown to be $\theta = \tan^{-1}(-d)$. This corresponds to the angle between the velocity vector and the tangent drawn from its tip to the second Friedrichs diagram.[3]

From the fact that the contributions to the integral (3.10) only appear at the zeros of D, a simple representation for $\mathbf{C}$ follows. For, when $D = 0$, $\mathbf{C}$ is the classical adjoint of a degenerate matrix (i.e., the rank of $\mathbf{M}$ is then equal to 4). It is then well known[4] that the rank of $\mathbf{C}$ may be written as the product of two five-vectors, i.e.

$$C_{ij}(ik_1, ik_3) = X_i(ik_1, ik_3) Y_j(ik_1, ik_3).$$

Since the entries of $\mathbf{C}$ are polynomials in ik_1 and ik_3, we may extract $\mathbf{C}$ from the integration (3.10) and write

$$v_i(\mathbf{x}) = G_j C_{ij}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\Phi(x, z),$$

where

$$\Phi = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{D}\, d\mathbf{k} = \sum_{i=0}^{4} \Phi_i(x, z).$$

The five terms Φ_i in this summation correspond to the five roots of D already discussed. We designate by Φ_0 the contribution resulting from the $k_1 = 0$ root and refer to this as the wake contribution. Further, we write

$$v_i(\mathbf{x}) = G_j C_{ij}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\Phi_0 + G_j C_{ij} \sum_{\mu=i}^{4} \Phi_\mu$$

$$= v_{wi} + G_j C_{i_j} \sum_{\mu=1}^{4} \Phi_\mu. \qquad (3.13)$$

The solution has been decomposed in this manner, since the operator $\mathbf{C}(\partial/\partial x, \partial/\partial z)$ assumes a particularly simple form when acting on either Φ_0 or $\sum_{i=1}^{4} \Phi_i$.

In particular

$$\Phi_0 = \frac{H(x)\,|z|^3}{12\Gamma \sum_{i=1}^{4} d_i}, \qquad (3.14)$$

and

$$v_{wi} = G_j C_{ij} \Phi_0 = G_j X_i^0 Y_j^0 \Phi_0.$$

The zero signifies the reduced forms that $\mathbf{X}$ and $\mathbf{Y}$ take when acting on Φ_0. We in fact find[5]

$$\mathbf{X}_i^0 = A^2 B_z^2 (\delta_{i1} - \delta_{i4}) \frac{\partial^2}{\partial z^2},$$

$$\mathbf{Y}_i^0 = [(\gamma - 1)\, \delta_{i1} - \delta_{i4}] \frac{\partial^2}{\partial z^2}.$$

From this we obtain

$$v_{wi} = \frac{H(x) A^2}{\Gamma \sum_{i=1}^{4} d_i} B_z^2 (\delta_{i1} - \delta_{i4})[(\gamma - 1)m - E]\, \delta(z). \qquad (3.15)$$

In the wake, $T = -\rho$; the perturbation pressure vanishes. Furthermore if energy and mass are added to the flow according to the relation $E = (\gamma - 1)m$ the wake disappears.

We now deal with the remainder of the flow field.

A. Doubly Hyperbolic Regime

In carrying out the integrations for Φ_i, $i = 1, \cdots, 4$, we integrate first with respect to k_1. It is seen that the poles all lie along the real axis. If dissipation is included the poles move off the axis into the complex plane. Therefore, in the limit of zero dissipation the path of integration should be deformed under or over the poles, according to whether they move into the upper or lower half plane, respectively, when dissipation is included. This analysis has been carried on in a general dissipative theory[6]

It is possible to determine whether the path should be deformed under or over a given pole without considering dissipation. If $x > 0$, the path must be closed in the upper half-plane, and if $x < 0$, in the lower half-plane. Therefore, the path should be deformed under the pole if it corresponds to a downstream wave, and over the pole, if it corresponds to an upstream wave. It was pointed out by McCune and Resler[2] that the proper branch of a characteristic is the one for which the angle made with the positive x axis decreases with increasing Mach number.

Another method for determining whether a wave is upstream or downstream is to construct the tangents from the velocity vector to the wave pulse,

[3] W. R. Sears, Rev. Mod. Phys. **32**, 701 (1960).

[4] K. Nomizu, *Fundamentals of Linear Algebra* (McGraw-Hill Book Company, Inc., New York, 1966) p. 175.

[5] The Heaviside function appearing in (3.14) should not be operated on by C_{ij}. To see this, observe that the Heaviside function appears because the path of integration in (3.10) is passed below the $k = 0$ root (e.g. dissipation requires this). It therefore should be extracted from the integral before $C_{ij}(ik_1, ik_3)$ is taken out as $C_{ij}(\partial/\partial x, \partial/\partial z)$. We have reversed the order only as a matter of convenience. From this it is clear that the particularly simple form for C_{ij} results from the fact that $\partial/\partial x = 0$.

[6] E. P. Salathé and L. Sirovich (to be published).

or second Friedrichs diagrams, as discussed by Sears.[3]

We have, on carrying out the integration,

$$\Phi_i = \frac{\pm H(\pm x)}{\Gamma\, d_i \prod_{\substack{j=1\\ j\neq i}}^{4} (d_i - d_j)} \frac{|z + d_i\, x|^3}{12}, \qquad i = 1, \cdots, 4, \tag{3.16}$$

where the $+$ or $-$ sign is taken depending on whether the wave lies downstream or upstream, respectively.

Using the formulas for the products, sums, etc. of the roots of algebraic equations, the following relation can be derived[7]:

$$\Gamma d_\mu \sum_{\substack{l=1\\ l\neq\mu}}^{4} (d_\mu - d_l) = \Theta_\mu / \cos^3 \theta_\mu,$$

where

$$\begin{aligned}\Theta_\mu = 4 \cos \theta_\mu \sin^4 \theta_\mu U_x^5 &+ \sin \theta_\mu (\sin^2 \theta_\mu - \cos^2 \theta_\mu) 2\gamma U_x A^2 B_x B_z \\ &- 2 \sin^2 \theta_\mu \cos \theta_\mu \gamma U_x^3 (\gamma + A^2) \\ &+ 2 \sin^2 \theta_\mu \cos \theta_\mu \gamma U_x A^2 (B_x^2 - B_z^2),\end{aligned}$$

and $\theta_\mu = -\tan^{-1} d_\mu$ is the angle with respect to the positive x axis, of the correct branch of the characteristic.

The complete doubly hyperbolic solution is given by

$$\begin{aligned} v_i &= v_{wi} + G_j C_{ij}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) \sum_{\mu=1}^{4} \Phi_\mu, \\ &= v_{wi} + G_j \hat{X}_i \hat{Y}_j \sum_{\mu=1}^{4} \Phi_\mu, \end{aligned} \tag{3.17}$$

with

$$\hat{Y}_1(ik_1, ik_3) = \hat{Y}_4(ik_1, ik_3) = \frac{U_x^2}{\gamma} \{k_1^4 (U_x^2 - \gamma) - k_1^2 k_3^2 A^2\},$$

$$\hat{Y}_2(ik_1, ik_3) = -k_1^4 U_x^3 + (k_1^2 + k_3^2)[k_1^2 U_x A^2 B_z^2 + k_1 k_3 U_x A^2 B_x B_z],$$

$$\hat{Y}_3(ik_1, ik_3) = -k_1^3 k_3 U_x^3 + (k_1^2 + k_3^2)[k_1^2 U_x A^2 B_x B_z + k_1 k_3 U_x A^2 B_x^2],$$

$$\hat{Y}_5(ik_1, ik_3) = -ik_1 (k_1^2 + k_3^2) U_x^2 A^2 [k_1^2 B_z - k_1 k_3 B_x],$$

$$\hat{\mathbf{X}} = \left[1, \frac{\hat{Y}_2}{\hat{Y}_1}, \frac{\hat{Y}_3}{\hat{Y}_1}, \gamma - 1, \frac{-\hat{Y}_5}{A^2 (k_1^2 + k_3^2) \hat{Y}_1}\right].$$

Note that $T/\rho = \gamma - 1$; therefore the flow is isentropic.

We point out as in (5) that C_{ij} does not operate on $H(\pm x)$ appearing in (3.16).

Inserting the above expressions for $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ into (3.17) we find, for example,

$$\begin{aligned} u_z = \sum_{\mu=1}^{4} \frac{H(\pm x)}{\Theta_\mu} \{&[\sin^2 \theta_\mu U_x A^2 B_x B_z \\ &- \sin \theta_\mu \cos \theta_\mu U_x A^2 B_x^2 + \sin^3 \theta_\mu \cos \theta_\mu U_x^3](m + E) \\ &- [\sin^2 \theta_\mu U_x^2 A^2 B_x B_z + \gamma U_x^2 \sin^3 \theta_\mu \cos \theta_\mu] M_x \\ &+ [\sin^4 \theta_\mu (U_x^4 - \gamma U_x^2) - \sin^2 \theta_\mu U_x^2 A^2 B_x^2] M_z\} \\ &\cdot \delta(z \cos \theta_\mu - x \sin \theta_\mu). \end{aligned} \tag{3.18}$$

1. *Aligned Fields*

When the angle between the applied field and free-stream velocity approaches zero (the aligned fields case), two of the waves collapse on the x axis forming part of the wake. The solution for the remaining two waves can be found from the Φ_3 and Φ_4 contributions to (3.17) corresponding to the values of d_3, d_4 given by Eq. (3.12). To find the solution for the wake, we note that, to lowest order in B_z, the two roots corresponding to the collapsing waves are given by

$$d_{1,2} = \mp B_z P / (U \pm P),$$

where

$$P = 1 \Big/ \left[\frac{1}{A^2} + \frac{1}{\gamma}\right]^{\frac{1}{2}}.$$

Taking the limit $B_z \to 0$ in the general solution, and using the above expression for d_1, d_2, we obtain, for the wake,

$$\begin{aligned} \rho = [H(x)/(-\gamma U_x^3 - A^2 U_x^3 + \gamma U_x A^2)] &\cdot \{[-U_x^2 A^2 + (\gamma - 1)(A^2 - U_x^2)] m \\ &+ [U_x A^2] M_x + [U_x^2 - A^2] E\} \ \delta(z), \end{aligned}$$

$$\begin{aligned} u_x = [H(x)/(-\gamma U_x^3 - A^2 U_x^3 + \gamma U_x A^2)] &\cdot \{[U_x A^2] m - [\gamma + A^2] U_x^2 M_x + [U_x A^2] E\} \ \delta(z), \end{aligned}$$

$$\begin{aligned} T = [H(x)/(-\gamma U_x^3 - A^2 U_x^3 + \gamma U_x A^2)] &\cdot \{[(\gamma - 1)(U_x^2 - A^2)] m + [(\gamma - 1) U_x A^2] M_x \\ &+ [A^2 - U_x^2 - A^2 U_x^2] E\} \ \delta(z), \end{aligned}$$

$$\begin{aligned} b_x = [H(x)/(-\gamma U_x^3 - A^2 U_x^3 + \gamma U_x A^2)] &\cdot \{[U_x^2] m - [\gamma U_x] M_x + [U_x^2] E\} \ \delta(z) \end{aligned}$$

$$u_z = b_z = 0.$$

[7] E. P. Salathé, Ph.D. thesis, Brown University (1965).

The fluid pressure plus the magnetic pressure is constant across the wake. In terms of the normalized variables, this can be expressed as

$$\rho + T + 2A^2 b_x = 0$$

in the wake.

2. Gasdynamic Limit

Another interesting case is the limit as the magnetic field approaches zero. As $A \to 0$ the outer envelope of the wave pulse diagram approaches a circle of radius $\gamma^{\frac{1}{2}}$, and the cusp shrinks into the origin. The doubly hyperbolic case with U_x outside the outer envelope approaches supersonic gasdynamics. The tangents to the outer envelope become the gasdynamic Mach lines and the tangents to the cusp collapse on the x axis, forming part of the wake.

The leading terms for the roots corresponding to the tangents to the cusps are

$$d_{1,2} = \pm(A/U_x)B_z.$$

The roots corresponding to the Mach lines are

$$d_{3,4} = \pm[\gamma/(U_x^2 - \gamma)]^{\frac{1}{2}} = \pm\tan\theta.$$

The solution for the Mach lines is

$$\begin{aligned}\rho = {} & [H(x)/2U_x \cos\theta] \\ & \cdot\left\{\left[\frac{1}{\gamma} m - \frac{1}{U_x} M_x + \frac{\cot\theta}{U_x} M_z + \frac{1}{\gamma} E\right]\right. \\ & \cdot\delta(z\cos\theta - x\sin\theta) \\ & + \left[\frac{1}{\gamma} m - \frac{1}{U_x} M_x - \frac{\cot\theta}{U_x} M_z + \frac{1}{\gamma} E\right] \\ & \left.\cdot\delta(z\cos\theta + x\sin\theta)\right\},\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} u_x \\ u_z \end{bmatrix} = {} & \frac{-\gamma H(x)}{2U_x^2} \\ & \cdot\left\{\left[\frac{1}{\gamma} m - \frac{1}{U_x} M_x + \frac{\cot\theta}{U_x} M_z + \frac{1}{\gamma} E\right]\right. \\ & \cdot\begin{bmatrix} \sec\theta \\ -\csc\theta \end{bmatrix} \delta(z\cos\theta - x\sin\theta) \\ & + \left[\frac{1}{\gamma} m - \frac{1}{U_x} M_x - \frac{\cot\theta}{U_x} M_z + \frac{1}{\gamma} E\right]\begin{bmatrix} \sec\theta \\ \csc\theta \end{bmatrix} \\ & \left.\cdot\delta(z\cos\theta + x\sin\theta)\right\},\end{aligned}$$

$$T = (\gamma - 1)\rho,$$

$$p = \gamma p,$$

and the solution for the wake is

$$\begin{aligned}\rho_w &= \frac{1}{\gamma U_x}[(\gamma - 1)m - E]H(x)\,\delta(z), \\ T_w &= -\rho_w, \\ u_{xw} &= (M_x/U_0)H(x)\,\delta(z).\end{aligned} \qquad (3.19)$$

These solutions have been obtained previously.[8]

B. Hyperliptic

In this case the equation $D = 0$ has one zero root d_0, two real roots d_1, d_2, and two complex roots d_3, $d_4 = d_r \pm i\, d_i$. The solution may be written as

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_w + \mathbf{G}\cdot\hat{\mathbf{Y}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\hat{\mathbf{X}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\sum_{i=1}^{4}\Phi_i, \\ &= \mathbf{v}_w + \mathbf{G}\cdot\hat{\mathbf{Y}}\hat{\mathbf{X}}\sum_{i=1}^{2}\Phi_i + \mathbf{v}_E,\end{aligned}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are as given above and hence $\mathbf{v} - \mathbf{v}_w$ is isentropic. The second term of the sum is the same as the corresponding term in (3.17) except that we have only two waves instead of four. $\mathbf{v}_E$, the elliptic portion, is the contribution of the complex roots.

The integral (3.10) can as before be evaluated in terms of distributions. The elliptic part $\mathbf{v}_E$ is given by

$$\mathbf{v}_E = \mathbf{G}\cdot\hat{\mathbf{Y}}\hat{\mathbf{X}}\Phi_E,$$

with

$$\begin{aligned}\Phi_E = \Phi_3 + \Phi_4 = {} & G\left\{[\tfrac{1}{12} d_i^2 x^3 - \tfrac{1}{4} d_i x(z + d_r x)^2]\ln r\right. \\ & \left.+ [\tfrac{1}{4} d_i^2 x^2(z + d_r x) - \tfrac{1}{12}(z + d_r x)^3]\tan^{-1}\frac{d_i x}{z + d_r x}\right\} \\ & + H\left\{[\tfrac{1}{12}(z + d_r x)^3 - \tfrac{1}{4} d_i^2 x^2(z + d_r x)]\ln r\right. \\ & \left.+ [\tfrac{1}{12} d_i^3 x^3 - \tfrac{1}{4} d_i x(z + d_r x)^2]\tan^{-1}\frac{d_i x}{z + d_r x}\right\},\end{aligned}$$

where

$$r = [(d_i x)^2 + (z + d_r x)^2]^{\frac{1}{2}},$$

and G, H are constants given by

$$G = \frac{1}{\pi\Gamma}\frac{d_i^2 - d_r(3d_r - 2d_1 - 2d_2) - d_1 d_2}{(d_i^2 + d_r^2)\prod_{j=1}^{2}[(d_r - d_j)^2 + d_i^2]},$$

$$H = \frac{1}{\pi\Gamma}\frac{d_r(d_r - d_1)(d_r - d_2) - d_i^2(3d_r - d_1 - d_2)}{d_i(d_i^2 + d_r^2)\prod_{j=1}^{2}[(d_r - d_j)^2 - d_i^2]}.$$

[8] L. Sirovich, in *Rarefied Gas Dynamics*, L. Talbot, Ed. (Academic Press, Inc., New York, 1961) p. 283.

For example we obtain,

$$u_{zE} = \Big\{(m+E)\Big[(U_xA^2B_xB_z)\frac{\partial^4}{\partial x^4} + (U_xA^2B_z^2 - U_x^3)\frac{\partial^4}{\partial x^3\,\partial z} + (U_xA^2B_xB_z)\frac{\partial^4}{\partial x^2\,\partial z^2} + (U_xA^2B_z^2)\frac{\partial^4}{\partial x\,\partial z^3}\Big] - M_x\Big[(U_xA^2B_xB_z)\frac{\partial^4}{\partial x^4} - (U_x^2\gamma)\frac{\partial^4}{\partial x^3\,\partial z} + (U_x^2A^2B_xB_z)\frac{\partial^4}{\partial x^2\,\partial z^2}\Big] + M_z\Big[(U_x^4 - U_x^2A^2B_z^2 - \gamma U_x^2)\frac{\partial^4}{\partial x^4} - (A^2U_x^2B_z^2)\frac{\partial^4}{\partial x^2\,\partial z^2}\Big]\Big\}\Phi_E. \tag{3.20}$$

The above solutions are given in terms of the mixed partial derivatives of Φ_E. For convenience we list these derivatives—

$$\partial^4\,\Phi_E/\partial x^4 = G\,\frac{1}{r^2}\Big[x\Big(\frac{d_i^5}{2} - d_i^3d_r^2 - \tfrac{3}{2}\,d_id_r^4\Big) + z(2\,d_id_r[d_i^2 - d_r^2])\Big] + H\,\frac{1}{r^2}\Big[z\Big(\frac{d_i^4}{2} - \tfrac{3}{2}\,d_i^2d_r^2 + \tfrac{1}{2}\,d_r^4\Big) + x(\tfrac{1}{2}\,d_r^5 - \tfrac{3}{2}\,d_i^4d_r - d_i^2d_r^3)\Big],$$

$$\frac{\partial^4\,\Phi_E}{\partial x^3\,\partial z} = G\,\frac{1}{r^2}\Big[z\Big(\frac{d_i^3}{2} - \tfrac{3}{2}\,d_id_r^2\,z\Big) - x(d_i^3d_r - d_id_r^3)\Big] + H\,\frac{1}{r^2}\,[z(\tfrac{1}{2}\,d_r^3 - \tfrac{3}{2}\,d_i^2d_r) + x\tfrac{1}{2}(d_r^4 - d_i^4)],$$

$$\frac{\partial^4\,\Phi_E}{\partial x^2\,\partial z^2} = G\,\frac{1}{r^2}\,[(-z\,d_id_r) - x\tfrac{1}{2}\,d_i(d_i^2 + d_r^2)] + H\,\frac{1}{r^2}\Big[z\tfrac{1}{2}(d_r^2 - d_i^2) + \frac{x}{2}\,d_r^2\Big],$$

$$\frac{\partial^4\,\Phi_E}{\partial x\,\partial z^3} = \frac{-G\,d_i\,z}{2r^2} + H\,\frac{1}{r^2}\Big[\frac{z\,d_r}{2} + x\tfrac{1}{2}(d_i^2 + d_r^2)\Big],$$

$$\frac{\partial^4\,\Phi_E}{\partial z^4} = \frac{Gx\,d_i}{2r^2} + H\,\frac{1}{r^2}\,[\tfrac{1}{2}z + x(\tfrac{1}{2}\,d_r)].$$

1. Gasdynamic Limit

In the limit $A \to 0$, the hyperbolic portion collapses on the axis forming part of the wake, as described in the previous section, and the remaining flow field becomes pure elliptic. Then, since $d_1 = d_2 = d_r = 0$ the flow field becomes

$$\rho = \rho_w + \frac{1}{2\pi U_x(U_x^2 - \gamma)}\Big\{\Big[\frac{U_x^2}{\gamma}\Big]m - [U_x]M_x + \Big[\frac{U_x(U_x^2-\gamma)}{\gamma}\,\frac{z}{x}\Big]M_z + \Big[\frac{U_x^2}{\gamma}\Big]E\Big\}\,\frac{dx}{d^2\,x^2 + z^2},$$

$$T - T_w = (\gamma - 1)(\rho - \rho_w),$$

$$u_x = u_{wx} + \frac{1}{2\pi U_x(U_x^2 - \gamma)}\Big\{[-U_x]m + [\gamma]M_x + \Big[(\gamma - U_x^2)\,\frac{z}{x}\Big]M_z - [U_x]E\Big\}\,\frac{dx}{d^2\,x^2 + z^2},$$

$$u_z = \frac{1}{2\pi U_x(U_x^2 - \gamma)}\Big\{\Big[\frac{U_x(U_x^2 - \gamma)}{\gamma}\Big]m + [\gamma - U_x^2]M_x + \Big[(U_x^2 - \gamma)\,\frac{x}{z}\Big]M_z + \Big[\frac{U_x(U_x^2-\gamma)}{\gamma}\Big]E\Big\}\,\frac{dz}{d^2\,x^2 + z^2},$$

where

$$d^2 = d_i^2 = -\gamma/U_x^2 - \gamma.$$

The solution in the wake $\mathbf{v}_w$ is given by (3.19). These give the elliptic portion of the subsonic gasdynamic solutions and were obtained previously.[8]

We now obtain the incompressible limit of these gasdynamic solutions. This limit is given by $a_0 \to \infty$, or due to the normalization, $U_x \to 0$, $\gamma \to 1$.[9] Also, from the normalization in Sec. II we have $m \sim U_x$; M_x, $M_z \sim U_x^2$; $E \sim U_x^3$. The incompressible solution is

$$\frac{u_x}{U_x} = \frac{1}{2\pi}\Big\{\frac{m}{U_x} - \frac{M_x}{U_x^2} - \frac{z}{x}\,\frac{M_z}{U_x^2}\Big\}\,\frac{x}{x^2 + z^2}, \qquad \frac{u_z}{U_x} = \frac{1}{2\pi}\Big\{\frac{m}{U_x} - \frac{M_x}{U_x^2} + \frac{x}{z}\,\frac{M_z}{U_x^2}\Big\}\,\frac{z}{x^2 + z^2}. \tag{3.21}$$

and the wake, given by Eq. (3.19) vanishes.

2. Aligned Fields

We saw in the last section that when $B_z \to 0$ two of the Mach lines collapse onto the x axis. In the hyperliptic case, it is possible for one of the waves to collapse upstream, yielding both an upstream and a downstream wake. In fact, this occurs whenever

$$U_x < P = 1\Big/\Big[\frac{1}{A^2} + \frac{1}{\gamma}\Big]^{\frac{1}{2}},$$

as can easily be seen from Fig. 1. The leading terms of the roots corresponding to the waves collapsing downstream and upstream, respectively, are

$$d_1 = -\frac{PB_z}{P + U_x}, \qquad d_2 = -\frac{PB_z}{P - U_x}.$$

The upstream wake is given by

[9] This is the liquid limit. Another incompressible limit is the slow-flow limit ($U_x \to 0$, a_0 fixed). In this case, body thickness and m must be small compared to U_x; this leads to (3.21) with $m = 0$.

$$\rho = \frac{1}{2\Lambda}\left\{\left[\frac{A^2U_x}{P}(P+U_x)\right]M_x - \left[\frac{A^2U_x}{\gamma}(P+U_x)\right](m+E)\right\}\delta(z),$$

$$u_x = \frac{1}{2\Lambda}\left\{\left[\frac{A^2U_x}{P}(P+U_x)\right](m+E) - [U_x(A^2+\gamma)(P+U_x)]M_x\right\}\delta(z),$$

$$b_x = \frac{1}{2\Lambda}\left\{[U_x(P+U_x)](m+E) - \left[\frac{\gamma U_x}{P}(P+U_x)\right]M_x\right\}\delta(z),$$

$$T = (\gamma-1)\rho,$$

where

$$\Lambda = -\gamma U_x^3 - A^2U_x^3 + \gamma U_x A^2.$$

The downstream wake consists of both the wake given in the general solution, i.e.,

$$\rho_w = (1/\gamma U)[(\gamma-1)M - E],$$
$$T_w = -\rho_w,$$

and the wave that collapses downstream—

$$\rho = \frac{1}{2\Lambda}\left\{\left[\frac{U_xA^2}{P}(P-U_x)\right]M_x + \left[\frac{A^2U_x}{\gamma}(P-U_x)\right](m+E)\right\}\delta(z),$$

$$u_x = \frac{1}{2\Lambda}\left\{\left[\frac{U_xA^2}{P}(P-U_x)\right](m+E) + [U_x(\gamma+A^2)(P-U_x)]M_x\right\}\delta(z),$$

$$b_x = \frac{1}{2\Lambda}\left\{-[U_x(P-U_x)](m+E) + \left[\frac{\gamma U_x}{P}(P-U_x)\right]M_x\right\}\delta(z),$$

$$T = (\gamma-1)\rho,$$

It is easily verified that fluid plus magnetic pressure is constant across each wake.

Next the incompressible limit of the magnetohydrodynamic flow field is obtained[9] when the applied field is parallel to the free-stream velocity. It is seen from the normalization that $A^2 = O(U_x^2)$ and the flow field is given by

$$\frac{u_x}{U_x} = \frac{1}{2\pi}\frac{U_x^2}{U_x^2-A^2}\left\{\frac{m}{U_x} - \frac{M_x}{U_x^2} - \frac{z}{x}\frac{M_z}{U_x^2}\right\}\frac{x}{x^2+z^2},$$

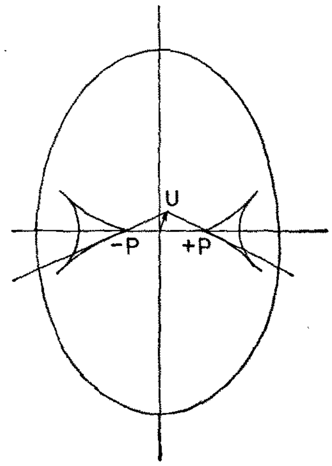

FIG. 1. One wave collapses upstream as $B_z \to 0$ when $U < P$.

$$\frac{u_z}{U_x} = \frac{1}{2\pi}\frac{U_x^2}{U_x^2-A^2}\left\{\frac{m}{U_x} - \frac{M_x}{U_x^2} + \frac{x}{z}\frac{M_z}{U_x^2}\right\}\frac{z}{x^2+z^2},$$

$$b_x = u_x/U_x,$$
$$b_z = u_z/U_x.$$

The incompressible limit of the wake can be obtained similarly, and lies either entirely downstream ($U_x > A$) or has both a downstream and an upstream portion ($U_x < A$). Again, from the normalization, we obtain $\gamma - 1 \sim U_x^2$, $P = A \sim U_x$ so that when the wake lies entirely downstream it is given by

$$b_x = H(x) \cdot \{-[mU_x/(U_x^2 - A^2)] + [M_x/(U_x^2 - A^2)]\}\,\delta(z),$$

and when there are both downstream and upstream portions

$$b_x = -\frac{1}{2}H(x)\left\{\frac{m}{A+U_x} + \frac{M_x}{A(A+U_x)}\right\}\delta(z) + \frac{1}{2}H(-x)\left\{\frac{m}{A-U_x} - \frac{M_x}{A(A-U_x)}\right\}\delta(z).$$

The other flow variables vanish in the wake.

We see, therefore, that for an incompressible fluid the magnetohydrodynamic fundamental solutions with $B_z = 0$ is identical to the hydrodynamic fundamental solution, and that the magnetic lines of force are distorted in such a way that they coincide with the stream lines. We see that the only difference is the existence of a surface current at the body in the magnetohydrodynamic case. This observation was made previously by Sears and Resler.[10]

[10] W. R. Sears and E. L. Resler, J. Fluid Mech. 5, 257 (1959).

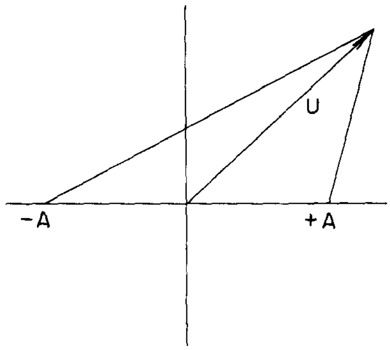

FIG. 2. Wave pattern for decoupled mode.

C. Decoupled Mode

In this section, the equations (3.7) and (3.8), governing the perturbations of velocity and magnetic field in the y direction, are discussed.

The transformed equations are

$$\begin{bmatrix} ik_1U_x & -ik_1B_x - ik_3B_z \\ -ik_1A^2B_x - ik_3A^2B_z & ik_1U_x \end{bmatrix} \begin{bmatrix} b_y \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ M_y \end{bmatrix}.$$

The determinant of the matrix is

$$D = -[(U_x + AB_x)k_1 + AB_zk_3] \cdot [(U_x - AB_x)k_1 - AB_zk_3],$$

and the solution is

$$u_y = \sum_{\mu=1}^{2} [H(\pm x)] \frac{M_yU_x \sin\theta_\mu}{\Omega(d_\mu - d_\nu)} \delta(z\cos\theta_\mu - x\sin\theta_\mu),$$

$$b_y = \sum_{\mu=1}^{2} [H(\pm x)] \frac{M_y}{\Omega(d_\mu - d_\nu)} \cdot(\sin\theta_\mu B_x - \cos\theta_\mu B_z)\ \delta(z\cos\theta_\mu - x\sin\theta_\mu),$$

where

$$\Omega = -(U_x^2 - A^2B_x^2).$$

The d_i are the roots of $D(k_1/k_3) = 0$, where D is given above and $\theta_\mu = -\tan^{-1}d_\mu$. This solution represents the two Mach lines drawn through the points $\pm A$ on the **B** axis (Fig. 2). One of the Mach lines can extend upstream, as the $\pm$ in the argument of H indicates.

IV. MAGNETOHYDRODYNAMIC BOUNDARY-VALUE PROBLEMS.

We consider the flow past an infinite cylindrical body whose generators are parallel to the y axis, in which the free-stream velocity is in the x-y plane and the applied magnetic field is in the x-z plane.

A. Doubly Hyperbolic Case

1. *Fields not aligned*

Using Eq. (2.25) and the fundamental solutions (3.17), we have for the solution (excluding the wake, since it vanishes, as is shown):

$$v_i(x, z) = \sum_{\mu=1}^{4} \int_{-\infty}^{\infty} H[\pm(x - x_0)] \cdot A^{\mu}_{ij}S_j(x_0)\ \delta[z\cos\theta_\mu - (x - x_0)\sin\theta_\mu]\,dx_0$$
$$+ \sum_{\mu=1}^{4} \frac{\partial}{\partial z} \int_{-\infty}^{+\infty} H[\pm(x - x_0)] \cdot A^{\mu}_{ij}\bar{S}_j(x_0)\ \delta[z\cos\theta_\mu - (x - x_0)\sin\theta_\mu]\,dx_0,$$
$$= \sum_{\mu=1}^{4} H\left(\pm\frac{z}{\tan\theta_\mu}\right) \frac{A_{ij}}{|\sin\theta_\mu|} S_j\left(x - \frac{z}{\tan\theta_\mu}\right)$$
$$+ \sum_{\mu=1}^{4} H\left(\pm\frac{z}{\tan\theta_\mu}\right) \frac{A^{\mu}_{ij}}{|\sin\theta_\mu|} \cdot\left(-\frac{1}{\tan\theta_\mu}\right)\bar{S}'_j\left(x - \frac{z}{\tan\theta_\mu}\right). \quad (4.1)$$

The A_{ij} are easily determined from the X_iY_i in the fundamental solutions; S_j and $\bar{S}_j$ are given by Eqs. (2.21) and (2.23); and a prime denotes differentiation. We recall that the $(+)$ or $(-)$ sign in the Heaviside function is chosen depending on whether the given wave lies downstream or upstream.

Equation (4.1) expresses the solution in closed form in terms of the body shape $[f'_u(x) - f'_l(x)]$ and the lift distribution $[p_u(x) - p_l(x)]$. Therefore, the indirect problem (i.e., the solution for given body thickness and lift distribution) is given directly by (4.1). However, the direct problem requires determination of $[p_u(x) - p_l(x)]$. This can be obtained by imposing the boundary condition

$$u_z^+(x, 0) + u_z^-(x, 0) = U_x[f'_u(x) + f'_l(x)]. \quad (4.2)$$

This, however, merely leads to an algebraic equation for $(p_u - p_l)$.

One of the features of the present method is that the solution is expressed entirely in terms of one unknown, the lift distribution. Therefore, (4.2) is the only boundary condition we need to impose. In particular, the other velocity condition, $u_z^+ - u_z^- = U_x(f'_u - f'_l)$ is redundant (it is, in fact, satisfied identically in the crossed- or aligned-fields case, or for gasdynamics). Further, the boundary conditions on the magnetic field are also identically satisfied by our solutions and do not need to be imposed as boundary conditions.

The wake has not been included in (4.1), since it vanishes. To see this, consider for example, from (3.15)

$$\rho_w(x, z) = -\int_{-\infty}^{+\infty} A^2 B_z^2 \frac{H(x - x_0)}{\Gamma \prod_{l=1}^{4} d_l} p_0 U_x [f_u'(x_0) - f_l'(x_0)] H(x_0) H(1 - x_0)\, \delta(z)\, dx_0 \tag{4.3}$$

$$= 0; \quad \text{for} \quad x < 0 \quad \text{or} \quad x > 1 \quad \text{or} \quad z \neq 0,$$

because $(f_u - f_l) = 0$ at $x = 0$ and $x = 1$. Since $T = \gamma \rho = 0$, the wake vanishes.

Supersonic gasdynamics is also obtained as a special case of (4.1) by using the gasdynamic fundamental solutions given in Sec. III and taking the sum over $\mu = 1, 2$. Again, it may be verified that the wake vanishes in this case.

2. Fields aligned

The solution in this case is obtained by using the aligned-fields fundamental solutions in Eq. (2.25) with the sources given by (2.22). The form of the solution is again given by (4.1) but with the summation taken over only two waves. As an example, we exhibit the solution explicitly for u_z—

$$u_z(x, z) = H(z)\Big\{\tfrac{1}{2} U_x \tau'(x - z/\tan\theta) + \frac{\tan\theta (U_x^4 - \gamma U_x^2)}{2(\gamma U_x^3 + A^2 U_x^3 - \gamma U_x A^2)} \cdot \Big[\Delta p\Big(x - \frac{z}{\tan\theta}\Big) + A^2\, \Delta b_x\Big(x - \frac{z}{\tan\theta}\Big)\Big]\Big\}$$
$$+ H(-z)\Big\{-\tfrac{1}{2} U_x \tau'(x + z/\tan\theta) + \frac{\tan\theta (U_x^4 - \gamma U_x^2)}{2(\gamma U_x^3 + A^2 U_x^3 - \gamma U_x^2 A^2)} \cdot \Big[\Delta p\Big(x + \frac{z}{\tan\theta}\Big) + A^2\, \Delta b_x\Big(x + \frac{z}{\tan\theta}\Big)\Big]\Big\},$$
$$= H(z) U_x f_u'\Big(x - \frac{z}{\tan\theta}\Big) + H(-z) U_x f_l'\Big(x + \frac{z}{\tan\theta}\Big) \tag{4.4}$$

where θ is the angle the wave in the upper half-plane makes with the positive x axis, $\tau = f_u - f_l$, $\Delta p = p_u - p_l$, and $\Delta b_x = b_{ux} - b_{lx}$. Note that although Δp and Δb_x occur, there is only one unknown function. The final form for u_z is easily obtained by imposing the boundary condition (4.2).

As in the nonaligned case, it can be easily verified that the wake vanishes (this is the triple wake, consisting also of the two collapsed Mach lines).

B. Hyperliptic Case

1. Fields not aligned

The fundamental solutions have been obtained for the general hyperliptic case and can be written in the form

$$v_i^f = \sum_{\mu=1}^{2} H(\pm x) A_{ij}^{\mu} G_j\, \delta(z\cos\theta_\mu - x\sin\theta_\mu) + \Big[C_{ij}\frac{x}{r^2} + D_{ij}\frac{z}{r^2}\Big] S_j, \tag{4.5}$$

where $r^2 = d_i^2 x^2 + (z + d_r x)^2$. A_{ij}^{μ}, C_{ij}, D_{ij} may be found directly from the fundamental solution for this case. As before the wake vanishes and we have suppressed this portion in (4.5).

Therefore, the solution for the flow past a body is given by

$$v_i = \sum_{\mu=1}^{2} H\Big(\pm\frac{z}{\tan\theta_\mu}\Big) \frac{A_{ij}^{\mu}}{|\sin\theta_\mu|} S_j(x - z/\tan\theta_\mu) + \sum_{\mu=1}^{2} H\Big(\pm\frac{z}{\tan\theta_\mu}\Big)\frac{A_{ij}^{\mu}}{|\sin\theta_\mu|} \cdot \Big(-\frac{1}{\tan\theta_\mu}\Big)\bar{S}_j'(x - z/\tan\theta_\mu)$$
$$+ C_{ij}\int_0^1 S_j(x_0)\frac{(x - x_0)\, dx_0}{d_i^2(x - x_0)^2 + [z + d_r(x - x_0)]^2} + D_{ij}\int_0^1 S_j(x_0)\frac{z\, dx_0}{d_i^2(x - x_0)^2 + [z + d_r(x - x_0)]^2}$$
$$+ C_{ij}\frac{\partial}{\partial z}\int_0^1 \bar{S}_j(x_0)\frac{(x - x_0)\, dx_0}{d_i^2(x - x_0)^2 + [z + d_r(x - x_0)]^2} + D_{ij}\frac{\partial}{\partial z}\int_0^1 \bar{S}_j(x_0)\frac{z\, dx_0}{d_i^2(x - x_0)^2 + [z + d_r(x - x_0)]^2}. \tag{4.6}$$

This represents a closed-form solution for the complete hyperliptic problem in terms of the body thickness and lift distribution. Therefore, the indirect problem is again given immediately. To solve the direct problem we are led to the solution of an integral equation for $(p_u - p_l)$. The integral equation is obtained by applying the boundary condition (4.2). We first note (see Appendix B) that

$$\lim_{z\to 0^\pm}\int_0^1 S_j(x')\frac{(x - x')\, dx'}{d_i^2(x - x')^2 + [z + d_r(x - x')]^2} = \frac{1}{d^2}\int_P \frac{S_j(x')\, dx'}{x - x'} \mp \frac{d_r}{d_i d^2} S_j(x), \tag{4.7}$$

where $d^2 = d_i^2 + d_r^2$, and P denotes a principal-value integral. Also (Appendix B),

$$\lim_{z\to 0\pm} \int_0^1 S_i(x') \frac{z\,dx'}{d_i^2(x-x')^2 + [z + d_r(x-x')]^2} = \pm \frac{\pi}{d_i} S_i(x). \qquad (4.8)$$

Since $(p_u - p_l)$ is the only unknown, the integral equation is then, using (4.7) and (4.8),

$$\alpha S(x) = \int_0^1 \frac{S(x')}{x' - x}\,dx' + f(x), \qquad (4.9)$$

where we have written $S(x)$ for $(p_u - p_l)$; $f(x)$ is a known function and α a known constant—they can be determined from (4.6). Cumberbatch, Sarason, and Weitzner[11] solved the hyperliptic case by a different method and also reduced the problem to an integral equation of the type (4.9). Equation (4.9) is a Cauchy integral equation and can be solved by standard techniques (see, for example, Muskhelishvili[12]). We continue the equation into the complex plane and define

$$\Phi = \frac{1}{2} \int_c \frac{S(\zeta)}{\zeta - z}\,d\zeta, \qquad (4.10)$$

where c is the interval [0, 1] on the real line.

Using the Plemelj formulas, we can then write the integral equation as

$$[(\alpha/\pi i) - 1]\Phi^+ - [(\alpha/\pi i) + 1]\Phi^- = f(x),$$

where Φ^+ and Φ^- are the limits of Φ as z approaches the interval [0, 1] from the top and bottom, respectively. Equations (4.10) and (4.11) constitute a Hilbert boundary problem for Φ. Once Φ is determined, the function $S(x)$ is given, from the Plemelj formulas, by

$$S(x) = (1/\pi i)(\Phi^+ - \Phi^-). \qquad (4.11)$$

The solution of the Hilbert problem is

$$\Phi(z) = \frac{X(z)}{2(\alpha - \pi i)} \int_c \frac{f(\zeta)\,d\zeta}{X^+(\zeta)(\zeta - z)} + P(z)X(z), \qquad (4.12)$$

where

$$X(z) = [(z-1)/z]^{\hat{\alpha}/\pi}, \qquad (4.13a)$$

is the fundamental solution, and $\hat{\alpha} = \arg(\alpha + \pi i)$. Equation (4.13a) is only one form of the fundamental solution, since it can be multiplied by any entire function with singularities only at the end-points of [0, 1]. Two other forms are

$$X(z) = [-z/(1-z)]^{1-\hat{\alpha}/\pi} \qquad (4.13b)$$

and

$$X(z) = [-(1/z)]^{\hat{\alpha}/\pi}[1/(1-z)]^{1-\hat{\alpha}/\pi}. \qquad (4.13c)$$

All these forms are consistent with the requirement $\Phi(z) \to 0$ as $z \to \infty$. Solution (4.13a) has a singularity at the leading edge, (4.13b) at the trailing edge, and (4.13c) at both the leading and trailing edges. There is no solution satisfying $\Phi(z) \to 0$ as $z \to \infty$ which has no singularity. By choosing either (4.13a) or (4.13b), we must take $P(z) = 0$ and the solution is unique. However, with (4.13c), $P(z) = $ const, and the solution is no longer unique, representing the arbitrariness in the circulation about the body. In gasdynamics, the Kutta–Joukowski condition selects (4.13a) as the correct fundamental solution. In magnetohydrodynamics, there is no evidence at this time upon which a choice may be based, although several suggestions have been put forward by Cumberbatch, Sarason, and Weitzner,[11] and Stewartson.[13]

As an example we exhibit the solution for u_x when the applied field is normal to the free-stream velocity ($\theta = \theta_1 = -\theta_2$ is the angle of the Mach line in the upper half plane):

$$\begin{aligned} u_x(x,z) = {} & \frac{2U_x}{\Theta}\{\sin^2\theta\cos\theta(A^2U_x^3 + \gamma U_x A^2) - \cos\theta\gamma U_x A^2\} \frac{1}{2\pi}\int_0^1 \frac{d_i\, z\tau'(x_0)\,dx_0}{d_i^2(x-x_0)^2 + z^2} \\ & + \sum_{\mu=1}^{2} H((-1)^{\mu+1}z)\frac{U_x(-1)^{\mu+1}}{\Theta}\{\sin^2\theta\cos\theta(\gamma U_x^3) - \cos\theta\gamma U_x A^2\}\tau'\left(x - \frac{z}{\tan\theta}(-1)^{\mu+1}\right) \\ & + \frac{2}{\Theta}[A^2U_x^2\sin^2\theta\cos\theta - \gamma A^2\cos^3\theta]\frac{1}{2\pi}\int_0^1 \frac{\Delta p(x_0)\,d_i(x-x_0)\,dx_0}{d_i^2(x-x_0)^2 + z^2} \\ & + \sum_{\mu=1}^{2} H((-1)^{\mu+1}z)\frac{1}{\Theta\sin\theta}\{\sin^4\theta(U_x^4 - \gamma U_x^2) - \sin^2\theta U_x^2A^2\}\,\Delta p\left(x - \frac{z}{\tan\theta}(-1)^{\mu+1}\right), \end{aligned} \qquad (4.14)$$

where $\Theta = \Theta_1 = -\Theta_2$ are defined in Sec. III.

[11] E. Cumberbatch, L. Sarason, and H. Weitzner, AIAA J. 1, 679 (1963).
[12] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordorf, Ltd., Groningen, The Netherlands, 1946).
[13] K. Stewartson, Z. Angew. Math. Phys. 12, 261 (1961).

Applying the boundary condition (4.2) we obtain the integral equation (4.9) with

$$\alpha = -\pi\, d_i \tan\theta \frac{(U_x^4 - \gamma U_x^2)\sin^2\theta - U_x^2 A^2}{A^2 U_x^2 \sin^2\theta - \gamma A^2},$$

$$f(x) = -\frac{\pi\Theta\, d_i\, \tau'(x)}{2[A^2 U_x^2 \sin^2\theta \cos\theta - \gamma A^2 \cos\theta]}.$$

The solution to (4.9), using (4.10), (4.11), 4.12), and the fundamental solution (4.13c), is

$$\Delta p(x) = \frac{\alpha}{\alpha^2 + \pi^2} f(x) + \frac{1}{\alpha^2 + \pi^2} J(x) \int_0^1 \frac{f(x_0)\, dx_0}{J(x_0)(x_0 - x)} + PJ(x),$$

where P is a constant and

$$J(x) = (1 - x)^{\hat{\alpha}/\pi - 1}/x^{\hat{\alpha}/\pi}. \tag{4.15}$$

If the fundamental solutions (4.13a, b) are used we have $P = 0$,

$$J(x) = [(1 - x)/x]^{\hat{\alpha}/\pi},$$

or

$$J(x) = [x/(1 - x)]^{1 - \hat{\alpha}/\pi},$$

respectively.

2. Fields Aligned

In the aligned-fields case, the waves collapse on the axis and the field is pure elliptic. As in the crossed fields case, $d_r = 0$, so that $r^2 = d_i^2(x - x_0)^2 + z^2$. The computations are considerably reduced and we obtain, for example,

$$u_z = U_x \frac{d_i}{2\pi} \int_0^1 \frac{\tau'(x_0) z\, dx_0}{d_i^2(x - x_0)^2 + z^2} + \frac{(U_x^3 - \gamma U_x)}{(U_x^4 - U_x^2 A^2 + \gamma A^2 - \gamma U_x^2)} \cdot \left\{\frac{d_i}{2\pi} \int_0^1 \frac{(\Delta p + A^2\, \Delta b_x)(x - x_0)\, dx_0}{d_i^2(x - x_0)^2 + z^2}\right\},$$

where Δp and Δb_x are the jump in p and b_x across the body. If $\Delta p + A^2 \Delta b_x$ are to be determined, we are led to the integral equation (4.9) with $\alpha = 0$, and the solution is given by

$$\Delta p(x) + A^2\, \Delta b(x) = \frac{1}{\pi^2} J(x) \int_0^1 \frac{f(x_0)\, dx_0}{J(x_0)(x_0 - x)} + PJ(x),$$

where

$$f(x) = \frac{[f_u'(x) + f_l'(x)]\pi\, d_i}{\gamma - U_x^2} (U_x^4 - U_x^2 A^2 + \gamma A^2 - \gamma U_x^2),$$

and $J(x)$ is given by (4.15) with $\hat{\alpha} = \frac{1}{2}\pi$.

V. SUMMARY

The basis of the present method of solving the linearized magnetohydrodynamic problems rests first in finding the discontinuous forms of the governing equations and secondly in reducing the calculation to finding the fundamental solution.

The advantage of having the discontinuous forms of the equations is partly due to being able to regard the flow as taking place in all space and, therefore, allowing the use of Fourier transforms. It is also due to having the boundary conditions appear directly in the equations themselves. As we have seen, this allows the solution of any problem to be obtained by the imposition of a single boundary condition. All other boundary conditions are built directly into the equations and become redundant.

By introducing the fundamental solution, the solution to a problem is reduced to a superposition of sources. In a manner of speaking the fundamental solution is the essence of the governing equations. Once having the fundamental solution it is no longer necessary to deal with the differential equations. As we have shown, the fundamental solutions of two-dimensional magnetohydrodynamics can be obtained in closed form in all regimes.

On superposing fundamental solutions, the solution to any problem is obtained in terms of a single unknown function. For the indirect problem (given lift and thickness distribution) there is no unknown function and the solution is given immediately (the same is also true for three-dimensional problems). The direct problem leads to a single integral equation which is easily solved in all situations and, hence, all two-dimensional problems are solved by the present method.

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APPENDIX A

In considering the flow past a body, we replace the body surface by a surface distribution of sources. This is accomplished by defining the flow over all space and relating the source strength to the jumps in the variables across the surface. Consequently, we must obtain the discontinuous forms of the equations. We have already carried this out for gasdynamics[14] and give here the extension to magnetohydrodynamics.

If $S(\mathbf{x}, t) = 0$ represents a surface (not necessarily closed) across which the variables may jump, the equations of conservation of mass, momentum, and energy have the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = s\ \delta(S), \tag{A1}$$

$$\frac{\partial}{\partial t} \rho u_i + \frac{\partial}{\partial x_j} \rho u_i u_j + \frac{\partial}{\partial x_i} p - \frac{\partial}{\partial x_j} P_{ij} - (\mathbf{J} \times \mathbf{B})_i = \pi_i\ \delta(S), \tag{A2}$$

$$\frac{\partial}{\partial t} \rho\left(e + \frac{u^2}{2}\right) + \frac{\partial}{\partial x_j}\left\{\rho\left(e + \frac{u^2}{2}\right)u_j + p u_j - P_{ij} u_i + Q_j\right\} - J_i E_i = \eta\ \delta(S), \tag{A3}$$

where

$$\delta(S) = \int \prod_{i=1}^{N} \delta(x_i - y_i)\ dS(\mathbf{y}). \tag{A4}$$

(N is the number of dimensions.) As is easily seen, this has the property

$$\int f(x)\ \delta(S)\ d\mathbf{x} = \int f\ dS.$$

P, **Q**, and **J** are the stress and heat flow and current given by the consitutive relations

$$\begin{aligned} P_{ij} &= \bar{\mu}(u_{i,j} + u_{j,i}) \\ &\quad + (\beta - \tfrac{2}{3}\bar{\mu})\nabla \cdot \mathbf{u}\ \delta_{ij} + S_{ij}\ \delta(S), \\ Q_i &= -\kappa(\partial T/\partial x_i) + h_i\ \delta(S), \\ \mathbf{J} &= \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \end{aligned} \tag{A5}$$

Here $\bar{\mu}$, β, κ, σ are the viscosity, bulk viscosity, thermal conductivity, and electrical conductivity, respectively. The source strengths are given by

$$s = [\rho(\mathbf{u} - \mathbf{U})] \cdot \mathbf{n}, \tag{A6}$$

$$\boldsymbol{\kappa} = [p\mathbf{n} - \mathbf{P} \cdot \mathbf{n} + \rho \mathbf{u}(\mathbf{u} - \mathbf{U}) \cdot \mathbf{n}], \tag{A7}$$

$$\eta = [p\mathbf{u} - \mathbf{P} \cdot \mathbf{u} + \rho(e + \tfrac{1}{2}u^2)(\mathbf{u} - \mathbf{U}) + \mathbf{Q}] \cdot \mathbf{n}, \tag{A8}$$

$$S_{ij} = -[\bar{\mu}(u_i n_j + u_j n_i) + (\beta - \tfrac{2}{3}\bar{\mu})\mathbf{u} \cdot \mathbf{n}\ \delta_{ij}], \tag{A9}$$

$$\mathbf{h} = [\kappa T]\mathbf{n}, \tag{A10}$$

where [ ] denotes the jump across $S = 0$, $\mathbf{n}$ is the normal to $S = 0$, and

$$\mathbf{U} \cdot \mathbf{n} = -(\partial S/\partial t)/|\nabla S|.$$

The derivation of these equations is completely analogous to the gasdynamic case.[14] For an impermeable body, the above simplify since $(\mathbf{u} - \mathbf{U}) \cdot \mathbf{n} = 0$.

In a similar manner we obtain the equation

$$\nabla \cdot \mathbf{B} = [\mathbf{B}] \cdot \mathbf{n}\ \delta(S). \tag{A11}$$

Let C be a closed curve which intersects $S = 0$, and let $A(\mathbf{x}, t) = 0$ be a surface with boundary C. Then Ampere's law is

$$\int_C \frac{1}{\mu} \mathbf{B} \cdot \mathbf{T}\ dl = \int_A \mathbf{J} \cdot \mathbf{n}'\ dA + \int_L \lambda\ dl, \tag{A12}$$

where L is the intersection of $A = 0$ and $S = 0$, $\mathbf{T}$ the tangent vector along C, and n' the normal to $A = 0$. λ is a line source distribution whose interpretation will become clear later. The left-hand side of (A.12) can be rewritten, by adding and subtracting appropriate line integrals,

$$\int_C \frac{1}{\mu} \mathbf{B} \cdot \mathbf{T}\ dl = \left(\int_{A_1} + \int_{A_2}\right) \cdot \left(\frac{1}{\mu} \nabla \times \mathbf{B} \cdot \mathbf{n}'\right) dA - \int_L \left[\frac{1}{\mu} \mathbf{B}\right] \cdot \mathbf{T}\ dl,$$

where A_1 and A_2 are the two portions of $A = 0$ separated by $S = 0$. Therefore, we can write (A1) as

$$\left(\int_{A_1} + \int_{A_2}\right)\left(\frac{1}{\mu} \nabla \times \mathbf{B} - \mathbf{J}\right) \cdot \mathbf{n}'\ dA = \int_L \left\{\lambda + \left[\frac{1}{\mu} \mathbf{B}\right] \cdot \mathbf{T}\right\} dl. \tag{A13}$$

The left side of (A13) vanishes and we conclude that

$$\lambda = -\left[\frac{1}{\mu} \mathbf{B}\right] \cdot \mathbf{T} = -\left[\frac{1}{\mu} \mathbf{B}\right] \times \mathbf{n} \cdot \mathbf{n}' \csc \theta.$$

We have used $\mathbf{T} = \mathbf{n} \times \mathbf{n}' \csc \theta$, where θ is the angle between the surfaces $A = 0$ and $S = 0$. In order to arrive at the differential form of Ampere's law we use the relation

$$\int_L \left\{\left[\frac{1}{\mu} \mathbf{B}\right] \times \mathbf{n} \cdot \mathbf{n}'\right\} \csc \theta\ dl = \int_A \left\{\left[\frac{1}{\mu} \mathbf{B}\right] \times \mathbf{n} \cdot \mathbf{n}'\right\} \delta(S)\ dA$$

[14] L. Sirovich, Phys. Fluids 10, 24 (1967). (See Appendix A.)

to obtain

$$(1/\mu)\nabla \times \mathbf{B} - J = -\left[\frac{1}{\mu}\mathbf{B}\times\mathbf{n}\right]\delta(S). \quad \text{(A14)}$$

In the same way, Faraday's law takes the form

$$\nabla \times \mathbf{E} + \partial\mathbf{B}/\partial t = -[\mathbf{E}\times\mathbf{n}]\,\delta(S). \quad \text{(A15)}$$

The solution to particular problems can be simplified by an appropriate choice of the interior, or complementary, flow. The only restriction on the complementary flow is that it satisfy the same equations as the exterior flow, i.e., the magnetohydrodynamic equations. For the steady inviscid flow past a closed body we take for the complementary flow $\mathbf{u} = 0$, $p = 0$, $T = 0$, $\rho = 0$ and obtain

$$\nabla \cdot \rho\mathbf{u} = 0,$$

$$(\partial/\partial x_j)(\rho u_i u_j) + (\partial p/\partial x_i) - (\mathbf{J}\times\mathbf{B})_i = pn_i\ \delta(S) \quad \text{(A16)}$$

$$(\partial/\partial x_i)\{\rho(e + \tfrac{1}{2}u^2)u_i + pu_i] - J_iE_i = 0.$$

In addition, if the body is an insulator we can take the complementary electromagnetic field to be that which occurs in the actual body. This is possible because in this case the actual field satisfies the magnetohydrodynamic equations. Because of this we have the relations $[\mathbf{B}]\cdot\mathbf{n} = 0$ and $[\mathbf{E}]\times\mathbf{n} = 0$. We see that this allows the solution of the external flow field to be obtained without solving for the interior electromagnetic field, and without requiring the assumption of a thin body.

APPENDIX B

We provide here the derivation of Eqs. (4.7) and (4.8). The result is obtained from the Plemelj formulas

$$\lim_{z\to x^{+,-}} \frac{1}{2\pi i}\int_0^1 \frac{\varphi(x_0)\,dx_0}{x_0 - z} = \pm\frac{\varphi(x)}{2} + \frac{1}{2\pi i}\int_{0P}^1 \frac{\varphi(x_0)\,dx_0}{x_0 - x} \quad \text{(B1)}$$

by letting z approach x along a path given by

$$z = x + (y/d)e^{i\theta}. \quad \text{(B2)}$$

(The notation $x \to x^{+,-}$ means z approaches x while remaining in the upper or lower half plane, respectively.) Using (B2) the left-hand side of (B1) becomes

$$\lim_{y\to 0^{+,-}} \frac{1}{2\pi i}\int_0^1 \frac{\varphi(x_0)\,dx_0}{x_0 - x - (y/d)e^{-i\theta}} = \lim_{y\to 0^{+,-}} \frac{1}{2\pi i}$$

$$\cdot\int_0^1 \frac{\varphi(x_0)(\alpha - i\beta)[(x_0 - x)\alpha - (y/d) + i\beta(x_0 - x)]}{[(x_0 - x)\alpha - (y/d)]^2 + \beta^2(x_0 - x)^2}\,dx_0$$

where $\alpha = \cos\theta$, $\beta = \sin\theta$. Separating this into real and imaginary parts and using (B1) we obtain from the real part

$$\lim_{y\to 0^{+,-}} \int_0^1 \frac{\varphi(x_0)y\,dx_0}{(\beta d)^2(x - x_0)^2 + [y + \alpha\,d(x - x_0)]^2} = \pm\frac{\pi\varphi(x)}{\beta d} \quad \text{(B3)}$$

and, from the imaginary part, using (B3),

$$\lim_{y\to 0^{+,-}} \int_0^1 \frac{\varphi(x_0)(x - x_0)\,dx_0}{(\beta d)^2(x - x_0)^2 + [y + \alpha\,d(x - x_0)]^2} = \frac{1}{d^2}\int_{0P}^1 \frac{\varphi(x_0)\,dx_0}{x - x_0} \mp \frac{\alpha}{\beta d^2}\varphi(x).$$

Letting $\beta = d_i/d$, $\alpha = d_r/d$, $d = (d_i^2 + d_r^2)^{\frac{1}{2}}$, we obtain (4.7) and (4.8).