Dissipative Wave Propagation in Compressible Magnetohydrodynamics

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A complete discussion of the linearized, compressible magnetohydrodynamic equations is given. No restriction is placed on any of the dissipative mechanisms, and moreover the nature of the gas is unspecified. Fundamental solutions are obtained for arbitrary values of the pertinent dimensionless parameters as well as field orientation. Use of the fundamental solutions is illustrated by the construction of solutions for several problems. Various inviscid limits are considered, and restrictions on inviscid theory thereby result.

1. INTRODUCTION

MANY investigations of magnetohydrodynamic waves have been carried out using the Lundquist equations.[1] So far, however, little attention has been given to dissipative effects in the waves. In an early paper,[2] van de Hulst sketched the effects of dissipation on magnetohydrodynamic waves by a study of the resulting dispersion relation. In a more recent paper, Trilling[3] also considered the dispersion relation. Another approach to the same problem was taken by Whitham,[4] who described the effects of dissipation by studying the equations themselves.

In this paper the initial value problem for the dissipative magnetohydrodynamic equations is investigated. In particular, we consider the evolution of small disturbances from equilibrium. The method of attack and analysis closely parallels an earlier treatment of the same problem for the Navier–Stokes equations[5] (hereafter referred to as I). Here as in I, no restriction (other than linearization) is imposed on the problem. A compressible fluid with an arbitrary equation of state is considered. In addition to compressibility, the effects of bulk, and absolute viscosities, heat and electrical conductivities are simultaneously included—and no condition is placed on the resulting dimensionless ratios.

2. BASIC EQUATIONS

We denote equilibrium values by zero subscripts and perturbed quantities by primes. Density, velocity, temperature, pressure, internal energy, and magnetic induction are represented by

$$\hat{\rho} = \rho_0 + \rho', \qquad \hat{\mathbf{u}} = \mathbf{u}', \qquad \hat{T} = T_0 + T',$$
$$\hat{p} = p_0 + p', \qquad \hat{e} = e_0 + e', \qquad \hat{\mathbf{B}} = \mathbf{B}_0 + \mathbf{B}',$$

respectively. Further denoting space and time by $(\mathbf{x}', t')$, the linearized magnetohydrodynamic Navier–Stokes (mNS) equations are

$$(\partial/\partial t')\rho' + \rho_0 \nabla' \cdot \mathbf{u}' = 0, \tag{2.1}$$

$$\rho_0(\partial \mathbf{u}'/\partial t') + \nabla' p' + (\mathbf{B}_0/\bar{\mu}) \times (\nabla' \times \mathbf{B}') = (\beta + \tfrac{1}{3}\mu)\nabla'(\nabla' \cdot \mathbf{u}') + \mu \nabla'^2 \mathbf{u}', \tag{2.2}$$

$$\rho_0(\partial e'/\partial t') + p_0 \nabla' \cdot \mathbf{u}' = \kappa \nabla'^2 T', \tag{2.3}$$

$$(\partial/\partial t')\mathbf{B}' + \nabla' \times (\mathbf{B}_0 \times \mathbf{u}') = (1/\sigma\bar{\mu})\nabla'^2 \mathbf{B}', \tag{2.4}$$

$$\nabla' \cdot \mathbf{B}' = 0. \tag{2.5}$$

The viscosity, bulk viscosity, heat conductivity, and magnetic susceptibility μ, β, κ, $\bar{\mu}$ are taken constant, as is consistent with the linearization. Specifying

$$\hat{p} = \hat{p}(\hat{\rho}, \hat{T}), \qquad \hat{e} = \hat{e}(\hat{\rho}, \hat{T}),$$

closes off the above system of equations.

As in I we desire a symmetric system, and to this end we introduce the following normalization:

$$\mathbf{x} = \frac{\mathbf{x}'}{L}, \qquad t = \frac{t' a_0}{L}, \qquad \rho = \frac{\rho'}{\rho_0},$$
$$\mathbf{u} = \frac{\mathbf{u}'}{a_0}, \qquad T = \left(\frac{c_v}{a_0^2 T_0}\right)^{\frac{1}{2}} T', \tag{2.6}$$
$$\mathbf{b} = \mathbf{B}'/(\rho_0 a_0^2 \bar{\mu})^{\frac{1}{2}}, \qquad \tilde{\mathbf{B}}_0 = \mathbf{B}_0/|\mathbf{B}_0|,$$

where L is an unspecified length scale,

$$a_0^2 = (\partial p_0/\partial \rho_0)_{T_0} \tag{2.7}$$

is the isothermal speed of sound, and

$$c_v = (\partial e_0/\partial T_0)_{\rho_0} \tag{2.8}$$

is the specific heat at constant volume. Then introducing the eight-vector (in the interest of saving space, vectors are displayed in row form),

$$\mathbf{v} = [\rho, \mathbf{u}, T, \mathbf{b}], \tag{2.9}$$

[1] For a bibliography see A. Jeffrey and T. Taniuti, *Non-linear Wave Propagation* (Academic Press Inc., New York, 1964).
[2] H. C. van de Hulst, in *Problems of Cosmical Aerodynamics* (Central Air Documents Office, 1951).
[3] L. Trilling, J. Fluid Mech. **13**, 272 (1962).
[4] G. B. Whitham, Commun. Pure Appl. Math. **12**, 113 (1959).
[5] L. Sirovich, Phys. Fluids **10**, 24 (1967).

and the eight-by-eight symmetric matrix,

$$\mathsf{A}(\nabla) = \begin{bmatrix} 0 & \nabla\cdot & 0 & 0 \\ \nabla & -1\eta\nabla^2 - \zeta\nabla\nabla\cdot & \chi\nabla & -\theta(1\tilde{\mathbf{B}}_0\cdot\nabla - \nabla\tilde{\mathbf{B}}_0\cdot) \\ 0 & \chi\nabla\cdot & -\xi\nabla^2 & 0 \\ 0 & -\theta(1\tilde{\mathbf{B}}_0\cdot\nabla - \tilde{\mathbf{B}}_0\nabla\cdot) & 0 & -\epsilon 1\nabla^2 \end{bmatrix}, \tag{2.10}$$

the mNS system can be written as

$$(\partial/\partial t)\mathbf{v} + \mathsf{A}(\nabla)\mathbf{v} = 0. \tag{2.11}$$

The quantities ζ, η, ξ, ϵ, θ, χ are given by

$$\zeta = \frac{\beta + \frac{1}{3}\mu}{a_0\rho_0 L}, \qquad \eta = \frac{\mu}{L\rho_0 a_0},$$

$$\xi = \frac{\kappa}{\rho_0 c_v L a_0}, \qquad \epsilon = \frac{1}{\sigma\bar{\mu}a_0 L}, \tag{2.12}$$

$$\chi = [(c_0^2/a_0^2) - 1]^{\frac{1}{2}}, \quad \theta = (B_0^2/\mu\rho_0 a_0^2)^{\frac{1}{2}},$$

where

$$c_0^2 = (\partial p_0/\partial\rho_0)_{s_0} \tag{2.13}$$

is the adiabatic sound speed (s_0 is the entropy). In (2.10) **1** denotes the unit matrix and, for instance,

$$1\eta\nabla^2 + \zeta\nabla\nabla\cdot$$

$$= \begin{bmatrix} \eta\nabla^2 + \zeta\dfrac{\partial^2}{\partial x_1^2} & \zeta\dfrac{\partial^2}{\partial x_1\,\partial x_2} & \zeta\dfrac{\partial^2}{\partial x_1\,\partial x_3} \\ \zeta\dfrac{\partial^2}{\partial x_1\,\partial x_2} & \eta\nabla^2 + \zeta\dfrac{\partial^2}{\partial x_2^2} & \zeta\dfrac{\partial^2}{\partial x_2\,\partial x_3} \\ \zeta\dfrac{\partial^2}{\partial x_1\,\partial x_3} & \zeta\dfrac{\partial^2}{\partial x_2\,\partial x_3} & \eta\nabla^2 + \zeta\dfrac{\partial^2}{\partial x_3^2} \end{bmatrix}.$$

At this point the undetermined length scale L may be fixed as

$$L = \max\left(\frac{\beta + \frac{1}{3}\mu}{a_0\rho_0}, \frac{\mu}{\rho_0 a_0}, \frac{\kappa}{\rho_0 c_v a_0}, \frac{1}{\sigma\bar{\mu}a_0}\right). \tag{2.14}$$

The dimensionless parameters ξ, η, ζ, ϵ are then $O(1)$ or less. The quantity L may be regarded as a mean free path for the gas.

3. ONE-DIMENSIONAL PROBLEM

We now restrict attention to problems with variation only in x and t. We can then in full generality take

$$\mathbf{B}_0 = (B_0 \cos\alpha, B_0 \sin\alpha, 0). \tag{3.1}$$

From Eqs. (2.4) and (2.5), we immediately conclude that $b_x = 0$. The dependent variables are now seven in number (ρ, u_x, u_y, u_z, b_y, b_z, T) and are governed by the following equations:

$$\frac{\partial\rho}{\partial t} + \frac{\partial u_x}{\partial x} = 0,$$

$$\frac{\partial u_x}{\partial t} + \frac{\partial\rho}{\partial x} + \chi\frac{\partial T}{\partial x} + \theta_y\frac{b_y}{\partial x} - (\eta + \zeta)\frac{\partial^2}{\partial x^2}u_x = 0,$$

$$\frac{\partial u_y}{\partial t} - \theta_x\frac{\partial b_y}{\partial x} - \eta\frac{\partial^2 u_y}{\partial x^2} = 0, \tag{3.2}$$

$$\frac{\partial T}{\partial t} + \chi\frac{\partial u_x}{\partial x} - \xi\frac{\partial^2}{\partial x^2}T = 0,$$

$$\frac{\partial b_y}{\partial t} - \theta_x\frac{\partial u_y}{\partial x} + \theta_y\frac{\partial u_x}{\partial x} - \epsilon\frac{\partial^2}{\partial x^2}b_y = 0;$$

$$\frac{\partial u_z}{\partial t} - \theta_x\frac{\partial b_z}{\partial x} - \eta\frac{\partial^2}{\partial x^2}u_z = 0,$$

$$\frac{\partial b_z}{\partial t} - \theta_x\frac{\partial u_z}{\partial x} - \epsilon\frac{\partial^2}{\partial x^2}b_z = 0; \tag{3.3}$$

where $\theta_x = \theta\cos\alpha$, $\theta_y = \theta\sin\alpha$. It is immediately seen that (3.3) decouple from (3.2); the former are referred to as the transverse or Alfvén mode. Further, if $\theta_y = 0$, the (u_y, b_y) components also decouple into an Alfvén mode. The remaining set are just the gas-dynamic equations which were solved in I.

The method of solution follows that given in I. For convenience we briefly outline this method. A typical system of equations can be written as

$$[(\partial/\partial t) + \mathsf{A}(\partial/\partial x)]\mathbf{v} = 0, \tag{3.4}$$

where the matrix A and the vector $\mathbf{v}$ may be readily identified with the terms of (3.2), (3.3). Introducing Fourier transforms

$$\mathbf{v}(k) = \int_{-\infty}^{\infty} \exp(ikx)\mathbf{v}(x)\,dx, \tag{3.5}$$

the system (3.4) becomes

$$[(\partial/\partial t) + \mathsf{A}(-ik)]\mathbf{v} = 0. \tag{3.6}$$

The vector $\mathbf{v}$ is easily solved for in terms of $\mathbf{v}_0(k)$, the transformed initial data, and on inverting we have

$$\mathbf{v}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-ikx) \cdot \exp[-t\mathsf{A}(-ik)]\mathbf{v}_0(k)\,dk. \tag{3.7}$$

To avoid talking of specific initial data, we introduce the fundamental matrix

$$\mathsf{R}(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx) \cdot \exp[-t\mathsf{A}(-ik)]\, dk, \tag{3.8}$$

so that

$$\mathbf{v}(x) = \mathsf{R} * \mathbf{v}_0 = \int_{-\infty}^{\infty} \mathsf{R}(x-y, t)\mathbf{v}_0(y)\, dy. \tag{3.9}$$

Denoting the matrix of eigenvalues of $\mathsf{A}(-ik)$ by D and the corresponding matrix of eigenvectors by S, we can write

$$\mathsf{A} = \mathsf{SDS}^{-1}, \tag{3.10}$$

so that

$$\mathsf{R} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx)\mathsf{S}\exp(-t\mathsf{D})\mathsf{S}^{-1}\, dk. \tag{3.11}$$

One may easily show (see I, Appendix I) that the eigenvalues of A (for k real) are non-negative, and zero only for $k = 0$. It can then be proved (I, Appendix II) that

$$\mathsf{R} = \frac{1}{2\pi}\mathsf{S}_0\int_{-\infty}^{\infty} \exp(-ikx)\exp(-t\mathsf{D}_2)\, dk\, \mathsf{S}_0^{-1} + O(1/t) = \mathsf{R}_0 + O(1/t), \tag{3.12}$$

where

$$\mathsf{S}_0 = \mathsf{S}(k=0), \tag{3.13}$$

$$\mathsf{D}_2 = \mathsf{D}|_{k=0} + k(d\mathsf{D}/dk)|_{k=0} + \tfrac{1}{2}k^2(d^2\mathsf{D}/dk^2)|_{k=0}.$$

A discussion similar to I, Appendix II, shows that the asymptotics are valid for dimensional time $t' \gg L/a_0$.

A typical term of D_2 may be written as

$$d = i\alpha k + \beta k^2,$$

with $\beta > 0$. Therefore a typical integral of R_0 is

$$I = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-ikx - i\alpha tk - \beta k^2 t)\, dk = \frac{\exp[-(x+\alpha t)^2/4\beta t]}{(4\pi\beta t)^{\frac{1}{2}}}. \tag{3.14}$$

As is implied by (3.12), the asymptotic forms are restricted to[6]

$$|x - \alpha t| < O(t^{\frac{1}{2}}). \tag{3.15}$$

[6] See I, Sec. 7, for a discussion that suggests that the Navier–Stokes equations are inaccurate outside this range.

Therefore once S_0 and D_2 are determined R_0 is immediately given by (3.12) and (3.14). Further, the solution to a problem with initial data $\mathbf{v}_0$ is then given by

$$\mathbf{v} \sim \mathsf{R}_0 * \mathbf{v}_0.$$

It is also to be noted that (3.14) (I) takes on the delta function property as $t \to 0$ and therefore that the asymptotic solution $\mathsf{R}_0 * \mathbf{v}_0 \to \mathbf{v}_0$ as $t \to 0$.

4. ALFVÉN MODE

We consider the pure Alfvén mode separately. It should again be noted that in the event that $B_{0y} = 0$ the system (3.2), (3.3) decouples into two pure Alfvén modes and the pure gas-dynamic case, the latter having been solved in I. In order to more clearly reveal the nature of the method outlined at the close of the last section, we now mimic the steps indicated there.

With an obvious change of variables, we write the basic system as

$$\frac{\partial u}{\partial t} - B\frac{\partial}{\partial x}b = \eta\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial b}{\partial t} - B\frac{\partial}{\partial x}u = \epsilon\frac{\partial^2 b}{\partial x^2}. \tag{4.1}$$

On introducing the Fourier transforms for $u(x)$ and $b(x)$, we obtain

$$\begin{bmatrix} u \\ b \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-ikx)\exp(-\mathsf{A}t)\begin{bmatrix} u_0 \\ b_0 \end{bmatrix} dk, \tag{4.2}$$

where

$$\mathsf{A} = \begin{bmatrix} \eta k^2 & iBk \\ iBk & \epsilon k^2 \end{bmatrix} \tag{4.3}$$

and $u_0(k)$, $b_0(k)$ denote the transformed initial data.

A convenient representation of the solution is in terms of the convolution product

$$\begin{bmatrix} u \\ b \end{bmatrix} = \mathsf{R} * \begin{bmatrix} u_0(x) \\ b_0(x) \end{bmatrix}. \tag{4.4}$$

For the case $\eta = \epsilon$, the solution for R is easily found to be

$$\mathsf{R}(x) = \frac{\exp[-(x+Bt)^2/4\epsilon t]}{(4\pi\epsilon t)^{\frac{1}{2}}}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \frac{\exp[-(x-Bt)^2/4\epsilon t]}{(4\pi\epsilon t)^{\frac{1}{2}}}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{4.5}$$

This is immediately seen to give a diffusing wave moving to the right with a speed B and one to the left with a speed $-B$.

When $\eta \neq \epsilon$, we must seek an asymptotic solution. The asymptotic solution only depends on the power series representation of the eigenvalues and eigenvectors. These are easily computed, and if we denote the eigenvalues by λ_+ and λ_- and the corresponding eigenvectors by q_+ and q_-, their series representations are

$$\lambda_\pm = \pm ikB + \tfrac{1}{2}(\eta + \epsilon)k^2 + O(k^3), \qquad (4.6)$$

$$\mathbf{q}_\pm = \begin{bmatrix} \pm 1 \\ 1 \end{bmatrix} + O(k). \qquad (4.7)$$

As is shown in I, the above terms are sufficient for the lowest-order calculation of $\mathsf{R}(x)$,[6] which is then easily shown to be

$$\mathsf{R}(x) \sim \frac{\exp\left[-(x+Bt)^2/2(\epsilon+\eta)t\right]}{[2\pi(\epsilon+\eta)t]^{\frac{1}{2}}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \frac{\exp\left[-(x-Bt)^2/2(\epsilon+\eta)t\right]}{[2\pi(\epsilon+\eta)t]^{\frac{1}{2}}} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \qquad (4.8)$$

This has the same form as (4.5) and for $\eta = \epsilon$ is exact. The asymptotics hold for dimensional time

$$t' \gg \max\left[(\mu/a_0^2\rho_0),\ (1/\sigma\bar{\mu}a_0^2)\right];$$
$$|x \pm Bt| < O(t^{\frac{1}{2}}).$$

The time scale is roughly the mean free path and so is quite short under normal circumstances. Moreover, (4.8) has the δ-function property as $t \to 0$, and so we recover the correct initial data. This lends support to using the solution for all time.

As an illustration of the use of (4.8), we consider the following simple problem for Eq. (3.3). Let

$$b_z(t=0) = 0, \qquad u_z(t=0) = UH(-x),$$

where U is a constant and $H(x)$ the Heaviside function. The solution therefore gives the evolution of a transverse shear discontinuity. A simple calculation, using (4.8), then gives

$$\begin{bmatrix} u_z(x,t) \\ b_z(x,t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{U}{4} \operatorname{erfc}\left(\frac{x+Bt}{[2(\epsilon+\eta)t]^{\frac{1}{2}}}\right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{U}{4} \operatorname{erfc}\left(\frac{x-Bt}{[2(\epsilon+\eta)t]^{\frac{1}{2}}}\right),$$

where erfc (x) denotes the complement of the error function. This leads to diffusing waves moving to the right and the left with Alfvén speed B. From the properties of erfc (x), i.e.,

$$\lim_{t\to 0} \operatorname{erfc}(x/t) \to 2H(-x),$$

it is easily seen that as $t \to 0$ the initial data are recovered.

An important feature of the fundamental matrix (4.8) is that as $B \to 0$ the transverse velocity and field perturbations remain coupled, whereas the same limit ($B \to 0$) taken in (4.1) uncouples these effects. The latter is correct; the nonuniformity is due to the series representation (4.6), (4.7), which is not valid as $B \to 0$. (It is, however, valid when $\epsilon \sim \eta$, as is easily seen.) The same type nonuniformity appears again in Sec. 5.

5. GENERAL ONE-DIMENSIONAL CALCULATION

From the outline of the method of solution given in Sec. 3, only the eigenvalues, λ, to $O(k^2)$ and the eigenvectors, $\mathbf{q}$, to $O(1)$ of $\mathsf{A}(-ik)$ are necessary in the asymptotic evaluation of the fundamental matrix. Symbolically, if we write

$$\lambda = ik\lambda^{(1)} + k^2\lambda^{(2)} + O(k^3), \qquad (5.1)$$

$$\mathbf{q} = \mathbf{q}^{(0)} + O(k), \qquad (5.2)$$

only $\lambda^{(1)}$, $\lambda^{(2)}$, $\mathbf{q}^{(0)}$ are necessary for all the eigenvalues and eigenvectors. The calculation of these quantities follows from standard perturbation methods (see Sec. 6 where a slight departure occurs), and we merely list the results of these calculations.

One determination leads to

$$\lambda_0 = [\xi/(\chi^2+1)]k^2 + O(k^3), \qquad (5.3)$$

$$\mathbf{q}_0 = [\chi, 0, 0, -1, 0] + O(k), \qquad (5.4)$$

$$q_0^2 = \gamma + O(k). \qquad (5.5)$$

The four remaining eigencalculations are intimately connected, and we have distinguished the above root by a zero subscript. The remaining roots are given by

$$\lambda^{(1)} = \pm\left(\tfrac{1}{2}\{\theta^2 + \gamma \pm [(\theta^2+\gamma)^2 - 4\gamma\theta_x^2]^{\frac{1}{2}}\}\right)^{\frac{1}{2}}, \qquad (5.6)$$

where we have written γ for $\chi^2 + 1$ (this can be shown to be the ratio of specific heats)

$$\lambda^{(2)} = \frac{\lambda^{(1)2}[\xi\chi^2 + (\eta+\zeta)\gamma + \zeta\theta^2\sin^2\alpha + (\eta+\epsilon)\theta^2] - \gamma(2\eta+\zeta+\epsilon)\theta^2\cos^2\alpha - \xi\chi^2\theta^2\cos^2\alpha}{2\lambda^{(1)2}(\theta^2+\gamma) - 4\gamma\theta^2\cos^2\alpha}, \qquad (5.7)$$

$$\mathbf{q} = f(\lambda^{(1)})\begin{Bmatrix} \theta\lambda^{(1)} \sin\alpha \\ \theta\lambda^{(1)2} \sin\alpha \\ -\theta(\lambda^{(1)2} - \gamma)\cos\alpha \\ \chi\theta\lambda^{(1)}\sin\alpha \\ \lambda^{(1)}(\lambda^{(1)2} - \gamma) \end{Bmatrix} + O(k). \quad (5.8)$$

Henceforth we adopt the convention that $\mathbf{q}$ represents only the lowest order.

The factor $f(\lambda^{(1)})$ has been inserted to exhibit the arbitrariness in $\mathbf{q}$ and can take on different values for different $\lambda^{(1)}$. The "inviscid eigenvalues" $\lambda^{(1)}$ are, of course, the normal wave speeds of hydromagnetics. Removing the normalization (2.6), (2.12), we can write $\lambda^{(1)}$ as

$$\lambda^{(1)} = \pm(1/a_0)(\tfrac{1}{2}\{b_0^2 + c_0^2 \pm [(b_0^2 + c_0^2)^2 - 4c_0^2 A_0^2]^{\frac{1}{2}}\})^{\frac{1}{2}}, \quad (5.9)$$

where

$$b_0^2 = B_0^2/\mu\rho_0, \qquad A_0^2 = b_0^2\cos^2\alpha. \quad (5.10)$$

Here A_0 is the so-called Alfvén speed and b_0^2 is the equilibrium magnetic energy per unit mass. It is customary to write $\lambda^{(1)}$ as

$$\lambda^{(1)} = \pm(c_f/a_0), \pm(c_s/a_0), \quad (5.11)$$

where c_s denotes the value of the slow locus and c_f the value of the fast locus. This, of course, depends on the relative magnitudes of c_0 and b_0. For our purposes we find it more convenient to write

$$\begin{aligned}\lambda_{\pm a}^{(1)} &= \pm(1/a_0)(\tfrac{1}{2}\{b_0^2 + c_0^2 + \operatorname{sgn}(c_0^2 - b_0^2) \cdot[(c_0^2 - b_0^2)^2 + 4c_0^2 b_0^2\sin^2\alpha]^{\frac{1}{2}}\})^{\frac{1}{2}} \\ &= \pm(\tfrac{1}{2}\{\theta^2 + \gamma + \operatorname{sgn}(\gamma - \theta^2) \cdot[(\gamma - \theta^2)^2 + 4\gamma\theta^2\sin^2\alpha]^{\frac{1}{2}}\})^{\frac{1}{2}},\end{aligned} \quad (5.12)$$

where sgn ( ) represents the signum function. {The branch of the square root is chosen such that $[(c_0^2 - b_0^2)^2]^{\frac{1}{2}} = |c_0^2 - b_0^2|$.} We then see that $\lambda_a^{(1)}$ reduces to the acoustic speed when $b_0 \to 0$ and when $\sin\alpha \to 0$. In the same way

$$\begin{aligned}\lambda_{\pm A}^{(1)} &= \pm(1/a_0)(\tfrac{1}{2}\{b_0^2 + c_0^2 + \operatorname{sgn}(b_0^2 - c_0^2) \cdot[(c_0^2 - b_0^2)^2 + 4c_0^2 b_0^2\sin^2\alpha]^{\frac{1}{2}}\})^{\frac{1}{2}} \\ &= \pm(\tfrac{1}{2}\{\theta^2 + \gamma + \operatorname{sgn}(\theta^2 - \gamma) \cdot[(\gamma - \theta^2)^2 + 4\gamma\theta^2\sin^2\alpha]^{\frac{1}{2}}\})^{\frac{1}{2}},\end{aligned} \quad (5.13)$$

so that $\lambda_A^{(1)}$ reduces to the Alfvén speed when $\sin\alpha \to 0$ and when $b_0 \to 0$ ($\lambda_A^{(1)} = 0$). Furthermore, we take the eigenvector normalization $f(\lambda)$ so that

$$\mathbf{q}_{\pm a} = \frac{1}{\cos[\alpha H(b_0^2 - c_0^2)]}\begin{bmatrix} \lambda_{\pm a}^{(1)} \\ \lambda_a^{(1)2} \\ -(\lambda_a^{(1)2} - \gamma)\cot\alpha \\ \chi\lambda_{\pm a}^{(1)} \\ \lambda_{\pm a}^{(1)}(\lambda_a^{(1)2} - \gamma)/\theta\sin\alpha \end{bmatrix}, \quad (5.14)$$

$$\mathbf{q}_{\pm A} = \frac{1}{\cos[\alpha H(c_0^2 - b_0^2)]}\begin{bmatrix} \lambda_{\pm A}^{(1)}\sin\alpha \\ \lambda_A^{(1)2}\sin\alpha \\ -(\lambda_A^{(1)2} - \gamma)\cos\alpha \\ \chi\lambda_{\pm A}^{(1)}\sin\alpha \\ \lambda_{\pm A}^{(1)}(\lambda_A^{(1)2} - \gamma)/\theta \end{bmatrix}, \quad (5.15)$$

where $H(\ )$ represents the Heaviside function. We also require the eigenvector magnitudes which are given by

$$(q_{\pm A})^2 = \frac{2[\lambda_A^{(1)2}(\theta^2 - \gamma\cos 2\alpha) + (\gamma - \theta^2)\gamma\cos^2\alpha]}{\cos^2[\alpha H(c_0^2 - b_0^2)]}, \quad (5.16)$$

$$(q_{\pm a})^2 = \frac{2[\lambda_a^{(1)2}(\theta^2 - \gamma\cos 2\alpha) + (\gamma - \theta^2)\gamma\cos^2\alpha]}{\sin^2\alpha\cos^2[\alpha H(b_0^2 - c_0^2)]}. \quad (5.17)$$

The choice of normalization was dictated by the desire to obtain representations which are uniformly valid for all α and θ.

As an illustration of the general one-dimensional solution, we consider the simple problem of weak-shock initial data. Let

$$\mathbf{v}_0 = \boldsymbol{\delta} H(-x),$$

where $\boldsymbol{\delta}$ is a constant vector giving the strength of the jump. Then, from (3.9), (3.12), and (3.14), we have

$$\mathbf{v} = \frac{\boldsymbol{\delta}\cdot\mathbf{q}_0\mathbf{q}_0}{(q_0)^2}\frac{1}{2}\operatorname{erfc}\frac{x}{[4(\xi/\gamma)t]^{\frac{1}{2}}} + \sum_\mu \frac{\boldsymbol{\delta}\cdot\mathbf{q}_\mu\mathbf{q}_\mu}{(q_\mu)^2}\frac{1}{2}\operatorname{erfc}\frac{x + \lambda_\mu^{(1)}t}{(4\lambda_\mu^{(2)}t)^{\frac{1}{2}}}; \quad \mu = \pm a, \pm A,$$

where q_μ, $\lambda_\mu^{(1)}$, and $\lambda_\mu^{(2)}$ are given in (5.7) and (5.12)–(5.17). This solution represents four diffusing waves traveling at the normal wave speeds of nondissipative magnetohydrodynamics and a residual effect at the origin.

In the limit $\alpha \to 0$, the solution breaks up into the Alfvén mode discussed in Sec. 4 and the conventional gas-dynamic mode discussed in I. When the applied magnetic field is normal to the direction of variation, there are only two propagating waves. These are given by the general solution in the limit $\alpha \rightarrow \frac{1}{2}\pi$. The other two waves no longer propagate, but form a residual effect at the origin which is not given by the limit of the general solution as $\alpha \rightarrow \frac{1}{2}\pi$. This case is discussed in the next section.

6. PERPENDICULAR APPLIED FIELD

When the steady magnetic field is normal to the direction of variation, several unique features present themselves. As already mentioned, this calculation must be performed separately in as much as it is not given by taking the limit of the solution in Sec. 5.

First we note that the five equations (3.2) decouple[7] as follows:

$$[(\partial/\partial t) - ik\mathsf{A}_1 + k^2\mathsf{A}_2]\mathbf{v} = 0, \tag{6.1}$$

with

$$\mathbf{v} = \begin{bmatrix} \rho \\ u_x \\ T \\ b_y \end{bmatrix}, \tag{6.2}$$

$$\mathsf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \chi & \theta \\ 0 & \chi & 0 & 0 \\ 0 & \theta & 0 & 0 \end{bmatrix}, \tag{6.3}$$

$$\mathsf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \zeta + \eta & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \epsilon \end{bmatrix}, \tag{6.4}$$

$$(\partial u_y/\partial t) + \eta k^2 u_y = 0. \tag{6.5}$$

The latter is just the diffusion equation, and its solution is, of course, immediate. We now concentrate on solving (6.1).

It is advantageous to briefly discuss the inviscid solution gotten by setting $\mathsf{A}_2 = 0$. The fundamental matrix for the equations is given by (3.8), with $\mathsf{A} = -ik\mathsf{A}_1$. A straightforward calculation shows that the eigenvalues of A_1 are

$$\lambda = 0, 0, -\nu, +\nu, \tag{6.6}$$

where

$$\nu = (\theta^2 + \gamma)^{\frac{1}{2}}, \tag{6.7}$$

and the corresponding eigenvector matrix is

$$\mathsf{S}_\mathrm{I} = \begin{bmatrix} \theta & \chi & 1 & 1 \\ 0 & 0 & -\nu & \nu \\ \chi\theta & -1 & \chi & \chi \\ -\gamma & 0 & \theta & \theta \end{bmatrix}, \tag{6.8}$$

where the columns are the eigenvectors. On carrying out the integration, we have for the fundamental matrix

$$\mathsf{R}_\mathrm{I} = \mathsf{S}_\mathrm{I} \begin{bmatrix} \delta(x) & 0 & 0 & 0 \\ 0 & \delta(x) & 0 & 0 \\ 0 & 0 & \delta(x + \nu t) & 0 \\ 0 & 0 & 0 & \delta(x - \nu t) \end{bmatrix} \mathsf{S}_\mathrm{I}^{-1}, \tag{6.9}$$

where S_I^{-1} is the inverse matrix of S_I. This then leads to waves moving to the right and left with speed ν, as well as a residual effect at the origin. If, for example, we take as initial data[8]

$$\mathbf{v}_0 = \begin{bmatrix} \delta \\ 0 \\ 0 \\ 0 \end{bmatrix} H(-x), \tag{6.10}$$

then the solution is

$$\mathbf{v}_I = \delta\mathsf{S}_I \begin{bmatrix} \theta H(-x)/\gamma\nu^2 \\ (\chi/\gamma)H(-x) \\ (1/2\nu^2)H(-x - \nu t) \\ (1/2\nu^2)H(\nu t - x) \end{bmatrix}. \tag{6.11}$$

The expression for density, for example, is

$$\rho = \delta\,\frac{(\nu^2 - 1)}{\nu^2}\,H(-x) + \frac{\delta}{2\nu^2}\,H(-x - \nu t) + \frac{\delta}{2\nu^2}\,H(-x + \nu t). \tag{6.12}$$

The evolution of density is indicated in Fig. 1.

An important feature of this calculation is that $\lambda = 0$ is a double eigenvalue. Therefore, aside from normalization the corresponding two eigenvectors may be chosen in an infinite variety of ways. This has no bearing on the inviscid solution but is ex-

[7] Note also that the transverse mode (3.3) also decouples into two diffusion equations.

[8] This corresponds to a density discontinuity in a shock tube.

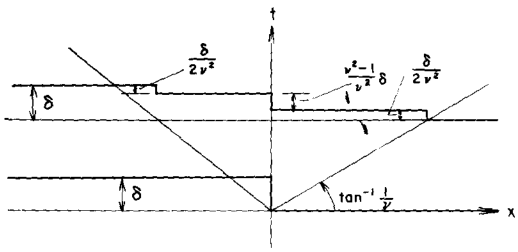

FIG. 1. Evolution of a density discontinuity for perpendicular applied field—nondissipative case.

tremely important in the dissipative solution as we now show.

The fundamental matrix of (6.1) is

$$\mathsf{R} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-ikx) \cdot \exp[(ik\mathsf{A}_1 - k^2\mathsf{A}_2)t]\, dk. \qquad (6.13)$$

To carry out the integration we need the eigentheory of the matrix $-ik\mathsf{A}_1 + k^2\mathsf{A}_2$. Using the methods of I we may show that the eigenvalues of this matrix are non-negative and are zero only for $k = 0$. Then the theorem of I states that the power series representation is sufficient for the asymptotic evaluation of R.

Expanding the eigenvalues

$$\lambda = i\alpha k + \beta k^2 + O(k^3), \qquad (6.14)$$

and the eigenvectors

$$\mathbf{q} = \mathbf{x} + ik\mathbf{y} + O(k^2), \qquad (6.15)$$

we obtain the following equations in the first two orders:

$$(\mathsf{A}_1 + \alpha)\mathbf{x} = 0, \qquad (6.16)$$

$$(\mathsf{A}_1 + \alpha)\mathbf{y} = (\beta - \mathsf{A}_2)\mathbf{x}. \qquad (6.17)$$

Equation (6.16) has already been solved in (6.6), (6.7), and (6.9). One point remains to be discussed in connection with the degenerate eigenvalue $\lambda = 0$. For the inviscid theory, any two eigenvectors spanning the null space of A_1 could be chosen. For the dissipative case now under study, we have the additional condition that $(\beta - \mathsf{A}_2)\mathbf{x}$ must be orthogonal to the null space of A_1 (since A_1 is symmetric). In this case we must therefore write

$$\mathbf{x} = [0, 0, \theta, -\chi] + z[\chi, 0, -1, 0] \qquad (6.18)$$

and determine z by the orthogonality condition.[9] The algebra is straightforward, and we are led to

$$\beta_\pm = \xi\big[((\theta^2 + 1) + \gamma P_m \mp \{[(1 - \theta^2) - \gamma P_m]^2 + 4\theta^2(1 - P_m)\}^{\frac{1}{2}})/2(\theta^2 + \gamma)\big], \qquad (6.19)$$

$$z_\pm = \big[(\gamma P_m + (\theta^2 - 1) \pm \{[(1 - \theta^2) - \gamma P_m]^2 + 4\theta^2(1 - P_m)\}^{\frac{1}{2}})/2\theta\big], \qquad (6.20)$$

for the two roots $\alpha = 0$. The quantity P_m is defined by

$$P_m = \epsilon/\xi = \rho_0 c_v/\sigma\bar{\mu}\kappa \qquad (6.21)$$

and it is referred to as the magnetic Prandtl number.

The calculation of β for the remaining two eigenvalues is more straightforward, and in an obvious notation

$$\beta_{\pm\nu} = [\nu^2(\zeta + \eta) + \chi^2\xi + \epsilon\theta^2]/2\nu^2. \qquad (6.22)$$

The expressions (6.18)–(6.22) furnish all the terms of S_0, S_0^{-1}, and D_2, and with (3.12), (3.14) the present problem is solved. Inasmuch as the expressions under study are unwieldy, the explicit form of the solution is not given in the general case. Instead, we consider three special cases. This is sufficient to show the nature of the solutions in general and their relation to the inviscid case (6.12), (6.14).

Case I, $P_m = 1$ $(\epsilon = \xi)$.

$$\beta_+ = \epsilon, \qquad z_+ = 0,$$

$$\beta_- = \epsilon/\nu^2, \quad z_- = (\nu^2 - 1)/\theta, \qquad (6.23)$$

$$\beta_{\pm\nu} = [\nu^2(\zeta + \eta) + (\nu^2 - 1)\epsilon]/2\nu^2 = \beta_\nu;$$

$$\mathsf{S}_0 = \begin{bmatrix} 0 & \nu^2 - 1 & 1 & 1 \\ 0 & 0 & -\nu & \nu \\ \theta & -\chi & \chi & \chi \\ -\chi & -\theta & \theta & \theta \end{bmatrix}.$$

S_0^{-1} is obtained by transposing S_0 and normalizing each row of the resulting matrix.

$$\mathsf{D}_2 = \begin{bmatrix} \epsilon k^2 & 0 & 0 & 0 \\ 0 & \epsilon k^2/\nu^2 & 0 & 0 \\ 0 & 0 & -i\nu k + \beta_\nu k^2 & 0 \\ 0 & 0 & 0 & +i\nu k + \beta_\nu k^2 \end{bmatrix}.$$

The asymptotic fundamental matrix R_0 is therefore

$$\mathsf{R}_0 = \mathsf{S}_0 \begin{bmatrix} \frac{e^{-x^2/4\epsilon t}}{(4\pi\epsilon t)^{\frac{1}{2}}} & 0 & 0 & 0 \\ 0 & \frac{\nu e^{-\nu^2x^2/4\epsilon t}}{(4\pi\epsilon t)^{\frac{1}{2}}} & 0 & 0 \\ 0 & 0 & \frac{e^{-(x+\nu t)^2/4\beta_\nu t}}{(4\pi\beta_\nu t)^{\frac{1}{2}}} & 0 \\ 0 & 0 & 0 & \frac{e^{-(x-\nu t)^2/4\beta_\nu t}}{(4\pi\beta_\nu t)^{\frac{1}{2}}} \end{bmatrix} \mathsf{S}_0^{-1}. \qquad (6.24)$$

[9] Choosing $z = \theta/\gamma$ and $z = \infty$ (i.e., $1/z = 0$) gives the previous choice of eigenvectors corresponding to $\lambda = 0$.

Applying this to the weak-shock initial data (6.10) we obtain

$$\mathbf{v} = \delta \mathbf{S}_0 \begin{bmatrix} 0 \\ \frac{1}{2\nu^2}\operatorname{erfc}\left[\frac{x\nu}{(2\epsilon t)^{\frac{1}{2}}}\right] \\ \frac{1}{4\nu^2}\operatorname{erfc}\left[\frac{x+\nu t}{(2\beta_\nu t)^{\frac{1}{2}}}\right] \\ \frac{1}{4\nu^2}\operatorname{erfc}\left[\frac{x-\nu t}{(2\beta_\nu t)^{\frac{1}{2}}}\right] \end{bmatrix}, \tag{6.25}$$

and, in particular, the density is given by

$$\rho = \delta\left[\frac{\nu^2-1}{2\nu^2}\operatorname{erfc}\left(\frac{x\nu}{(\epsilon t)^{\frac{1}{2}}}\right) + \frac{1}{4\nu^2}\operatorname{erfc}\left(\frac{x+\nu t}{(2\beta_\nu t)^{\frac{1}{2}}}\right) + \frac{1}{4\nu^2}\operatorname{erfc}\left(\frac{x-\nu t}{(2\beta_\nu t)^{\frac{1}{2}}}\right)\right]. \tag{6.26}$$

The evolution of density is sketched in Fig. 2.

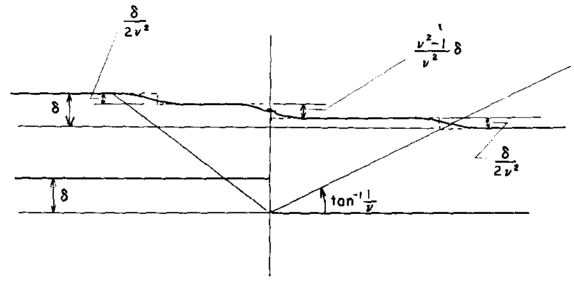

FIG. 2. Evolution of a density discontinuity for perpendicular applied field—dissipative case with magnetic Prandtl number $P_m = 1$.

Case II, $P_m \to 0$.

$$\beta_+ \sim \epsilon/(\theta^2+1), \qquad \beta_- \sim \xi[(\theta^2+1)/\nu^2],$$
$$z_\pm \sim \theta,\ -(1/\theta), \tag{6.27}$$
$$\beta_{\pm\nu} \sim [\nu^2(\zeta+\eta)+\chi^2\xi]/2\nu^2 = \beta_\nu.$$

In obtaining these expressions, we have retained terms of $O(\xi P_m) = O(\epsilon)$. This is necessary, as shown momentarily, to ensure uniform results.[10] In addition, we have regarded ξ, ζ, η as being of the same order, as is the case for most real gases. An expression for $\mathbf{S}_0$ is

$$\mathbf{S}_0 = \begin{bmatrix} \theta & -\chi & 1 & 1 \\ 0 & 0 & -\nu & \nu \\ 0 & \theta^2+1 & \chi & \chi \\ -1 & -\chi\theta & \theta & \theta \end{bmatrix}.$$

The asymptotic fundamental matrix $\mathbf{R}_0$ is easily obtained, and if it is applied to the weak-shock initial data (6.10), we obtain

$$\mathbf{v} \sim \delta\mathbf{S}_0 \begin{bmatrix} \frac{\theta}{2(\theta^2+1)}\operatorname{erfc}\left[x\left(\frac{\theta^2+1}{2\epsilon t}\right)^{\frac{1}{2}}\right] \\ \frac{-\chi}{2\nu^2(1+\theta^2)}\operatorname{erfc}\left(\frac{x\nu}{[2\xi(\theta^2+1)t]^{\frac{1}{2}}}\right) \\ \frac{1}{4\nu^2}\operatorname{erfc}\left(\frac{x+\nu t}{(2\beta_\nu t)^{\frac{1}{2}}}\right) \\ \frac{1}{4\nu^2}\operatorname{erfc}\left(\frac{x-\nu t}{(2\beta_\nu t)^{\frac{1}{2}}}\right) \end{bmatrix}. \tag{6.28}$$

[10] The limit $P_m \to 0$ must be viewed as $\epsilon \to 0$, as is required by the comment at the close of Sec. 2 in regard to L.

In particular, the expression for density is given by

$$\rho \sim \delta\left\{\frac{\theta^2}{2(\theta^2+1)}\operatorname{erfc}\left[x\left(\frac{\theta^2+1}{2\epsilon t}\right)^{\frac{1}{2}}\right] + \frac{\chi^2}{2\nu^2(1+\theta^2)}\operatorname{erfc}\left(\frac{x\nu}{[2\xi(\theta^2+1)t]^{\frac{1}{2}}}\right) + \frac{1}{4\nu^2}\operatorname{erfc}\left(\frac{x+\nu t}{(2\beta_\nu t)^{\frac{1}{2}}}\right) + \frac{1}{4\nu^2}\left(\frac{x-\nu t}{(2\beta_\nu t)^{\frac{1}{2}}}\right)\right\}. \tag{6.29}$$

It is now clear that ϵ cannot be set to zero since it occurs in combination with t. We have, however, that for t bounded

$$\lim_{\epsilon\to 0}\operatorname{erfc}\left[x\left(\frac{\theta^2+1}{2\epsilon t}\right)^{\frac{1}{2}}\right] \to 2H(-x). \tag{6.30}$$

The evolution of density is shown in Fig. 3.

It should be noted that the "wave" which remains at the origin exhibits a discontinuity within the structure of the wave. For long times this portion exhibits two different structured regions interior to the wave.

Case III, $P_m \to \infty$.

$$\beta_+ \sim \xi/\gamma, \qquad \beta_- \sim \gamma\epsilon/\nu^2, \qquad 1/z_+ \sim 0,$$
$$1/z_- \sim \gamma/\theta, \qquad \beta_{\pm\nu} \sim \epsilon\theta^2/2\nu^2. \tag{6.31}$$

In correspondence to the previous case,[10] this limiting case is to be viewed as $\xi \to 0$. Again we have regarded ξ, ζ, η as being of the same order,

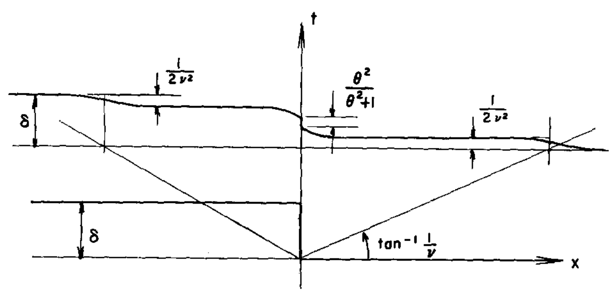

FIG. 3. Evolution of a density discontinuity; $P_m = 0$.

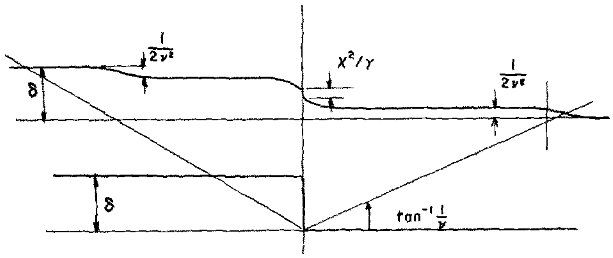

FIG. 4. Evolution of a density discontinuity; $P_m = \infty$.

which accounts for their absence in $\beta_{\pm\nu}$. An expression for S_0 in this case is

$$\mathsf{S}_0 = \begin{bmatrix} \theta & \chi & 1 & 1 \\ 0 & 0 & -\nu & \nu \\ \chi\theta & -1 & \chi & \chi \\ -\gamma & 0 & \theta & \theta \end{bmatrix}.$$

The asymptotic fundamental matrix R_0 is easily obtained, and on being applied to the weak-shock initial data (6.10), we obtain

$$v \sim \delta \mathsf{S}_0 \begin{bmatrix} \dfrac{\theta}{2\gamma\nu^2} \operatorname{erfc}\left(\dfrac{\nu x}{(2\gamma\epsilon t)^{\frac{1}{2}}}\right) \\ \dfrac{\chi}{2\gamma} \operatorname{erfc}\left[x\left(\dfrac{\gamma}{2\xi t}\right)^{\frac{1}{2}}\right] \\ \dfrac{1}{4\nu^2} \operatorname{erfc}\left(\dfrac{2\nu(x+\nu t)}{\theta(2\epsilon t)^{\frac{1}{2}}}\right) \\ \dfrac{1}{4\nu^2} \operatorname{erfc}\left(\dfrac{2\nu(x-\nu t)}{\theta(2\epsilon t)^{\frac{1}{2}}}\right) \end{bmatrix}. \qquad (6.32)$$

In particular, the expression for density is given by

$$\rho \sim \delta\left\{\frac{\theta^2}{2\gamma\nu^2} \operatorname{erfc}\left(\frac{x\nu}{(2\gamma\epsilon t)^{\frac{1}{2}}}\right) + \frac{\chi^2}{2\gamma} \operatorname{erfc}\left[x\left(\frac{\gamma}{2\xi t}\right)^{\frac{1}{2}}\right] + \frac{1}{4\nu^2} \operatorname{erfc}\left(\frac{2\nu}{\theta}\frac{x+\nu t}{(2\epsilon t)^{\frac{1}{2}}}\right) + \frac{1}{4\nu^2}\left(\frac{2\nu}{\theta}\frac{x-\nu t}{(2\epsilon t)^{\frac{1}{2}}}\right)\right\}. \qquad (6.33)$$

For t bounded we may proceed to the limit $\xi \to 0$,

$$\lim_{\xi \to 0} \operatorname{erfc}\,[x(x/2\zeta t)^{\frac{1}{2}}] \to 2H(-x).$$

Again we mention that this limit is not uniform in time. No matter how small ξ may be, for sufficiently large t the wave is diffuse.

The evolution of density in this case is shown in Fig. 4. The "wave" at the origin again exhibits a discontinuity in its interior, and for long times two different structured regions appear.

In both limiting cases ($P_m \to 0$, $P_m \to \infty$), the wave structure which resides at the origin contains a portion which is "stiff" for a long time on the scale of the diffusing portion. Inspection of the coefficient in (6.28) and (6.32) shows that these are of a different form in each of the cases. For all situations it is easily seen that the inviscid solution (6.11) is recovered formally by allowing the dissipation parameters (ξ, η, ϵ, ζ) to vanish. This limiting procedure is not uniform in time, and ultimately the diffusing solution takes over. As the two cases $P_m \to 0$, $P_m \to \infty$ show, a further nonuniformity occurs when the dissipative parameters are of different orders. We then find that a portion of the solution remains "stiff" (retains an inviscid form) while the other parts of the solution diffuse. The part that remains "inviscid" cannot be forecast from an inviscid theory as the analysis of this section has shown.

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