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Initial and Boundary Value Problems in Dissipative Gas Dynamics

LAWRENCE SIROVICH
Division of Applied Mathematics and the Center for Fluid Dynamics, Brown University, Providence, Rhode Island
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The discontinuous forms of the full Navier–Stokes equations are derived. A general method of solving linearized initial and boundary value problems is discussed. The following specific examples are solved: initial value problem in an unbounded domain, shock reflection from an insulating wall, shock evolution in a finite shock tube with insulating walls, and reflection of a sound wave from an isothermal wall. In all cases the dissipative parameters and the gas law are left arbitrary.

1. INTRODUCTION

In this paper we consider compressible flows governed by the Navier–Stokes equations and having boundaries which are possibly moving. Solution to specific problems, however, is greatly facilitated by having a flow field defined everywhere. To achieve this we define a flow field behind a material boundary and regard the boundary surface as a distribution of sources. The derivation of the Navier–Stokes equations under such conditions is carried out in Appendix A. It is found that the source strengths have simple relations with the stress, heat conduction, etc., of the original problem.

In Sec. 2, after formulating the general problem, a perturbation solution is sought. The perturbation parameter is a Mach number, and to lowest order one is led to the linearized Navier–Stokes equations. It is then shown that the general solution to the latter can be reduced to solving an integral equation whose kernel is the solution of a comparatively simple initial value problem. Several advantages of this method now appear.

It is found that an asymptotic analysis of the kernel (referred to as the fundamental matrix) can be performed independently of any specific problem. This analysis is carried out by means of a theorem proved in Appendix B. The asymptotic fundamental matrix is obtained in terms of elementary functions in Sec. 3. It is, in fact, the case that this asymptotic solution is the best that can be obtained from the linearized Navier–Stokes equations. This is meant in the sense that any higher approximation must come from kinetic theory. The solution to any pure initial value problem is then given in terms of a single quadrature on known functions. For large times this can further reduce to a closed-form expression only involving the total mass, momentum, and energy added at the initial instant. The inviscid theory is briefly discussed and is shown to be a nonuniform limit of the dissipative theory.

Another feature of the present method is that, by defining a flow field in a complementary region in an appropriate manner, the solution to the integral equation can be considerably reduced and in certain cases made trivial. In Sec. 4, problems involving stationary insulating walls are considered. The integral equation is completely eliminated in this case, and the problems of shock reflection and a shock in a finite shock tube are solved.

In Sec. 5 the problem of a sound wave being reflected from an isothermal wall is considered. By the proper choice of the complementary flow, the problem is reduced to a single integral equation. This is solved, and it is shown that the heat flow at the wall can be obtained without a knowledge of the full flow field. It is, in fact, a general feature of the present method that one may determine quantities such as heat flow and stress at a wall without the full determination of the flow field.

A point of note in the method presented here is that no assumptions need be made on the nature of the fluid. In the general method and the problems which are worked out, the Prandtl, Reynolds, etc., numbers as well as the gas law are left arbitrary.

2. FORMULATION

We consider compressible fluid flows governed by the Navier–Stokes equations in domains with boundaries which are possibly moving. As becomes clear in the later analysis, it is advantageous to have a flow defined in all space. To accomplish this we replace all boundaries by surfaces with distributions of momentum and energy rates and also of stress and heat flow. The resulting discontinuous Navier–Stokes equations are derived in Appendix A, [(A12), (A9)] and for the case of impenetrable boundaries, (A10), are

$$(\partial/\partial t)\rho + (\partial/\partial \mathbf{x})\rho u = 0,$$

$$(\partial/\partial t)\rho \mathbf{u} + (\partial/\partial x_i)\rho \mathbf{u} u_i + (\partial/\partial \mathbf{x})p - (\partial/\partial \mathbf{x})\cdot \mathbf{P} = [p\mathbf{n} - \mathbf{P}\cdot\mathbf{n}]\ \delta(S),$$

$$(\partial/\partial t)\rho(e + \tfrac{1}{2}u^2) + (\partial/\partial \mathbf{x})\cdot \rho\mathbf{u}(e + \tfrac{1}{2}u^2) + (\partial/\partial \mathbf{x})\cdot(p\mathbf{u} - \mathbf{P}\cdot\mathbf{u}) + (\partial/\partial \mathbf{x})\cdot\mathbf{Q} = [p\mathbf{u} - \mathbf{P}\cdot\mathbf{u} + \mathbf{Q}]\cdot\mathbf{n}\ \delta(S), \quad (2.1)$$

$$(\mathbf{P})_{ij} = \mu(u_{i,j} + u_{j,i} - \tfrac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}) + \beta\nabla\cdot\mathbf{u}\ \delta_{ij} - (\mu[u_i n_j + u_j n_i - \tfrac{2}{3}\mathbf{u}\cdot\mathbf{n}\ \delta_{ij}] + \beta[\mathbf{u}\cdot\mathbf{n}\ \delta_{ij}])\ \delta(S),$$

$$\mathbf{Q} = -\kappa(\partial T/\partial \mathbf{x}) + \kappa[T\mathbf{n}]\ \delta(S).$$

The notation is discussed in the Appendix A; in short, $S = 0$ denotes the equation of the boundary, $\mathbf{n}$ the normal to $S = 0$ (directed into the region of interest), and $[F]$ the jump of F across $S = 0$. The flow field in the formerly excluded portion of space can be chosen arbitrarily (provided it satisfies the Navier–Stokes equations) and is referred to as the complementary flow. For example, if we consider flow past a finite body, S is closed and we can take a constant state in the interior of S. The conservation equations in (2.1) then can be interpreted as having sources of momentum and energy rates which are equal in magnitude and oppositely directed to the comparable effects of the fluid on the body. In practice, less simple minded complementary flows serve to simplify solution to a problem.

The conservation and constitutive equations (2.1) now have to be augmented by thermodynamic relations which specify the fluid

$$e = e(\rho, T), \qquad p = p(\rho, T), \qquad p - \rho^2(\partial e/\partial\rho)_T = -T(\partial p/\partial T)_\rho. \quad (2.2)$$

The last of these is, of course, just a compatibility relation. Relation (2.2) holds piecewise in each flow region.

It should be noted that in (2.1), (2.2) and in what follows no restriction is made on the dissipative functions μ, β, κ nor on the gas laws.

We now restrict attention to one-dimensional motions in the x direction for which $u_2 = u_3 = 0$. In one dimension, bounded flows take place between parallel planes. For simplicity we consider the boundary to be a single plane,

$$x = x_0(t), \qquad U = dx_0/dt, \quad (2.3)$$

where U is the velocity of the plane. The surface singularity is

$$\delta(S) = \delta[x - x_0(t)]. \quad (2.4)$$

Next we introduce equilibrium quantities[1]

$$\rho_0,\ T_0,\ a_0 = [(\partial p_0/\partial\rho_0)_{T_0}]^{\frac{1}{2}}, \quad C_v = (\partial e_0/\partial T_0)_{\rho_0} \quad (2.5)$$

an unspecified length scale L, and the following normalized quantities:

$$\tilde{x} = \frac{x}{L}, \qquad \tilde{t} = \frac{a_0 t}{L}, \qquad \tilde{\rho} = \frac{\rho - \rho_0}{\rho_0},$$

$$\tilde{u} = \frac{u}{a_0}, \qquad \tilde{T} = \left(\frac{C_v}{a_0^2 T_0}\right)^{\frac{1}{2}}(T - T_0), \quad (2.6)$$

$$\tilde{P}_{ij} = P_{ij}/\rho_0 a_0^2, \qquad \tilde{Q} = Q/\rho_0 a_0^2 (C_v T_0)^{\frac{1}{2}}.$$

This normalization is carried out in all space and the field variables $\tilde{\rho}$, $\tilde{T}$, $\tilde{u}$ are regarded as small perturbations. The latter is equivalent to a Mach number expansion. On carrying out the normalization, linearizing (i.e. carrying only the first order in Mach number) and restricting the equations to one dimension, we obtain

$$(\partial\rho/\partial t) + (\partial u/\partial x) = 0,$$

$$(\partial u/\partial t) + (\partial\rho/\partial x) + \chi(\partial T/\partial x) - (\partial P/\partial x) = [p - P]\ \delta(S),$$

$$(\partial T/\partial t) + \chi(\partial u/\partial x) + (\partial Q/\partial x) = [Q]\ \delta(S), \quad (2.7)$$

$$P = \zeta(\partial u/\partial x),$$

$$Q = -\xi(\partial T/\partial x) + \xi[T]\ \delta(S).$$

In (2.7) tildes have been dropped with the understanding that all quantities are dimensionless according to (2.6).

Also, we have defined

$$\zeta = \left(\beta + \frac{4u}{3}\right) \Big/ a_0\rho_0 L,$$

$$\xi = \kappa/\rho_0 C_v L a_0, \quad (2.8)$$

$$\chi = (\gamma - 1)^{\frac{1}{2}}$$

and use has been made of

$$\gamma = C_p/C_v = c_0^2/a_0^2, \quad (2.9)$$

which may be proved with some manipulation. c_0 is the adiabatic speed of sound.

To solve (2.7) we introduce Fourier transforms, e.g.,

$$\rho = \int \exp(ikx)\rho(x)\ dx, \quad (2.10)$$

and on combining terms we have

$$(\partial\mathbf{v}/\partial t) + \mathbf{A}\mathbf{v} = \mathbf{G}, \quad (2.11)$$

[1] Note that a_0 is the isothermal speed of sound.

with

$$\mathbf{v} = (\rho, u, T), \tag{2.12}$$

$$\mathbf{A} = \begin{bmatrix} 0 & -ik & 0 \\ -ik & \zeta k^2 & -\chi ik \\ 0 & -\chi ik & \xi k^2 \end{bmatrix}, \tag{2.13}$$

$$\mathbf{G} = \begin{bmatrix} 0 \\ [p - P] \\ [Q] + ik\zeta[T] \end{bmatrix} \exp [ikx_0(t)]. \tag{2.14}$$

Since by our formulation v is defined everywhere, there is no difficulty in defining (2.10).

To solve (2.11) we introduce the matrix $\mathbf{V}(k, t)$ which satisfies

$$(\partial \mathbf{V}/\partial t) + \mathbf{A}(-ik)\mathbf{V} = 0, \quad \mathbf{V}(t = 0) = \mathbf{1}. \tag{2.15}$$

Then writing

$$\mathbf{v}(t = 0) = \mathbf{v}_0, \tag{2.16}$$

the solution of (2.11) is

$$\mathbf{v} = \mathbf{V}\mathbf{v}_0 + \mathbf{V} \circ \mathbf{G}, \tag{2.17}$$

where the circle denotes the time convolution

$$f \circ g = \int_0^t f(t - s)g(s)\, ds.$$

Fourier inverting (2.17), we obtain

$$\mathbf{v}(x, t) = \mathbf{V}(x, t) * \mathbf{v}_0(x) + \mathbf{V}(x, t) \circledast \mathbf{G}(t), \tag{2.18}$$

where the asterisk denotes the space convolution

$$f * g = \int_{-\infty}^{\infty} f(x - y)g(y)\, dy \tag{2.19}$$

and the circle and asterisk the space–time convolution [the latter takes a simple form as can be seen from (2.14)].

We pause now to point out several advantages to the present formulation. From (2.18) we see that the solution of any problem rests on finding V, which by (2.15) is the most elementary of initial value problems. [$\mathbf{V}(x, t)$ is referred to as the fundamental matrix.] Subsequent solution of the problem then involves an integral equation in $\mathbf{G}(t)$. The solution of the latter problem then furnishes $\mathbf{G}$ without an explicit knowledge of the entire flow $\mathbf{v}(x, t)$. In many applications, only $\mathbf{G}$ (which gives wall stress and wall heat flow) is demanded from a solution. This therefore furnishes a short cut in such cases. We find that a full evaluation of (2.19) is unfeasible and asymptotics must be resorted to. In the present formulation the asymptotic analysis may be imposed on $\mathbf{V}$ alone, and this is a relatively simple procedure. Lastly, it is once again emphasized that, in the above analysis and that which follows, no special assumptions on the nature of the gas is ever made. This is true for both the dissipative parameters and for the gas law.

3. EVALUATION OF THE FUNDAMENTAL MATRIX AND SOLUTION OF THE INITIAL VALUE AND AND RELATED PROBLEMS

From (2.15) we have that

$$\mathbf{V}(k, t) = \exp [-\mathbf{A}(-ik)t] \tag{3.1}$$

and therefore

$$\mathbf{V}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp [-ikx - \mathbf{A}(-ik)t]\, dk. \tag{3.2}$$

The solution to the pure initial value problem, $\mathbf{G} = 0$, $\mathbf{v}(t = 0) = \mathbf{v}_0$, is given by

$$\begin{aligned} \mathbf{v} &= \mathbf{V} * \mathbf{v}_0 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp [-ikx - \mathbf{A}(-ik)t]\mathbf{v}_0(k)\, dk. \end{aligned} \tag{3.3}$$

The evaluation of $\mathbf{V}$ and also $\mathbf{v}$ rests on first evaluating $\exp(-\mathbf{A}t)$. As preparation for this, we demonstrate certain properties of $\mathbf{A}$.

We observe that $\mathbf{A}$ is symmetric,

$$A_{ij} = A_{ji}. \tag{3.4}$$

(Note that A is not Hermitian.) Denote the eigenvalues of $\mathbf{A}$ by $d^\mu(\mu = 1, 3)$ and the associated eigenvectors by $\mathbf{q}^\mu$. Then from (3.4)

$$(d^\mu - d^\nu)\mathbf{q}^\mu \cdot \mathbf{q}^\nu = 0. \tag{3.5}$$

Therefore $\mathbf{q}^\mu$ is orthogonal to $\mathbf{q}^\nu$ if $d^\mu \neq d^\nu$, and by construction one easily sees $d^\mu \neq d^\nu$, $\mu \neq \nu$. Note that a vector may be orthogonal to itself under the dot product of (3.5), i.e., the dot product is not an inner product. However $\mathbf{q}$, and hence $\mathbf{q}\cdot\mathbf{q}$, may be chosen to be an analytic function of $z = ik$, so that $\mathbf{q}\cdot\mathbf{q} = 0$ at most on an isolated point set. This, as is clear, is of no importance.

Consider the matrix of eigenvectors

$$S_{ij} = q_i^j. \tag{3.6}$$

The inverse of S_{ij} is

$$(\mathbf{S}^{-1})_{ij} = q_j^i / \mathbf{q}^i \cdot \mathbf{q}^i.$$

Denoting the diagonal matrix of eigenvalues by $\mathbf{D}$,

$$D_{\mu\nu} = d^\mu \delta_{\mu\nu}, \tag{3.7}$$

we have

$$\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}. \tag{3.8}$$

Substituting in (3.3) we obtain easily

$$\mathbf{v}(x,t) = \sum_{\mu=1}^{3} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[-ikx - d^{\mu}(k)t\right] \frac{\mathbf{q}^{\mu}\mathbf{q}^{\mu}\cdot\mathbf{v}^{0}}{\mathbf{q}^{\mu}\cdot\mathbf{q}^{\mu}}\, dk. \tag{3.9}$$

This summation splits the solution into modes which are orthogonal under the dot product. These modes have the character of waves as seen shortly. Note that if a mode is absent in the initial data, it will be absent for all time, i.e., if $\mathbf{q}^{\mu}\cdot\mathbf{v}_0 = 0$, then $\mathbf{q}^{\mu}\cdot\mathbf{v}(k) = 0$.

Next we write

$$\mathbf{A} = i\mathbf{A}^{(1)} + \mathbf{A}^{(2)}, \tag{3.10}$$

with

$$A^{(1)} = \begin{bmatrix} 0 & -k & \\ -k & 0 & -\chi k \\ 0 & -\chi k & 0 \end{bmatrix}.$$

Choosing $\mathbf{q} = (q_1, q_2, q_3)$ such that $\mathbf{q}\cdot\mathbf{q}^* = 1$ ($\mathbf{q}^*$ denotes the complex conjugate) for the moment, and forming

$$\mathbf{q}^*\cdot\mathbf{A}\mathbf{q} = d,$$

we see, since $\mathbf{A}^1$ and $\mathbf{A}^2$ are real symmetric,

$$\mathrm{Re}\,(d) = \zeta k^2\, |q_2|^2 + \xi k^2\, |q_3|^2 \geq 0.$$

Moreover Re $(d) = 0$ if $k = 0$. Conversely, if Re $(d) = 0$, $q_i = 0$, $i = 2, 3$ and on substituting into the matrix equations this proves that $k = 0$. Hence by continuity, Re (d) is bounded away from zero for $k \neq 0$. Finally, we observe that $\mathbf{A}/ik$ can be regarded as the perturbation of a Hermitian matrix and therefore susceptible to classical perturbation analysis.[2] In fact, we find easily that d^{μ} has a convergent power series at the origin and that $\mathbf{q}^{\mu}$ can also be taken to be analytic.

For convenience we specify L of (2.12) so that ζ, (2.18) is unity. This choice yields an L which is of the order of the mean free path.[3] The only remaining dissipative parameter, ξ, is inversely proportional to a Prandtl number and hence is of $O(1)$. It is clear therefore that the series expansions of $\mathbf{q}$ and d have coefficients of $O(1)$. In fact, a straightforward analysis gives

$$\begin{aligned} d^1 &= d_2^1 + O(k^3) = \nu k^2 + O(k^3), \\ \mathbf{q}^1 &= \mathbf{q}_0^1 + O(k) = [\chi, 0, -1] + O(k), \\ d^2 &= d_2^2 + O(k^3) = -\gamma^{\frac{1}{2}} ik + \sigma k^2 + O(k^3), \\ \mathbf{q}^2 &= \mathbf{q}_0^2 + O(k) = [1, \gamma^{\frac{1}{2}}, \chi] + O(k), \\ d^3 &= d_2^3 + O(k^3) = \gamma^{\frac{1}{2}} ik + \sigma k^2 + O(k^3), \\ \mathbf{q}^3 &= \mathbf{q}_0^3 + O(k) = [-1, \gamma^{\frac{1}{2}}, -\chi] + O(k), \end{aligned} \tag{3.11}$$

with[4]

$$\sigma = 1 + [(\gamma - 1)/\gamma]\xi. \qquad \nu = \xi/\gamma. \tag{3.12}$$

From the already proved properties of d, we may use the theorem proved in the Appendix B. This permits us to replace d^{μ} by d_2^{μ} and q^{μ} by q_0^{μ} and obtain[5]

$$\begin{aligned} \mathbf{V} &= \sum_{\mu=1}^{3} \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \exp\left(-ikx - d_2^{\mu} t\right) \frac{\mathbf{q}_0^{\mu}\cdot\mathbf{q}_0^{\mu}}{\mathbf{q}_0^{\mu}\cdot\mathbf{q}_0^{\mu}}\, dk + O_\delta\left(\frac{1}{t}\right) \\ &= \mathbf{V}_{\mathrm{NS}} + O_\delta\left(\frac{1}{t}\right). \end{aligned} \tag{3.13}$$

From the above remarks it is clear that the numerical coefficient in the correction term is $O(1)$. Further, since t is physical time normalized with respect to the mean time of flight, large time in the asymptotic sense can be very small on a macroscopic scale.[6] Furthermore, $\mathbf{V}_{\mathrm{NS}}$ is the "best" asymptotic form which can be obtained from the linearized Navier–Stokes equation. For, as is discussed in Appendix B, the next approximation to $\mathbf{V}$ after $\mathbf{V}_{\mathrm{NS}}$ requires $O(k^3)$ in the expansion of d^{μ}, (3.11). From kinetic theory[7] one finds that this term is given incorrectly by the Navier–Stokes equations. Hence there is no justification for seeking a higher approximation to $\mathbf{V}$ than $\mathbf{V}_{\mathrm{NS}}$ without resort to kinetic theory. Another feature of $\mathbf{V}_{\mathrm{NS}}$ is that

$$\mathbf{V}_{\mathrm{NS}}(t = 0) = 1\delta(x). \tag{3.14}$$

We may therefore use $\mathbf{V}_{\mathrm{NS}}$ as a uniform approximation to $\mathbf{V}$, and the above discussion justifies replacing $\mathbf{V}$ by $\mathbf{V}_{\mathrm{NS}}$.

$\mathbf{V}_{\mathrm{NS}}$ may be easily integrated, and one finds

$$\begin{aligned} \mathbf{V}_{\mathrm{NS}} = {} & \frac{\exp\left(-x^2/4\nu t\right)}{(4\pi\nu t)^{\frac{1}{2}}} \frac{\mathbf{q}_0^1\mathbf{q}_0^1}{\gamma} \\ & + \frac{\exp\left[-(x - \gamma^{\frac{1}{2}}t)^2/4\sigma t\right]}{(4\pi\sigma t)^{\frac{1}{2}}} \frac{\mathbf{q}_0^2\mathbf{q}_0^2}{2\gamma} \\ & + \frac{\exp\left[-(x + \gamma^{\frac{1}{2}}t)^2/4\sigma t\right]}{(4\pi\sigma t)^{\frac{1}{2}}} \frac{\mathbf{q}_0^3\mathbf{q}_0^3}{2\gamma}. \end{aligned} \tag{3.15}$$

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[2] K. O. Friedrichs, *Perturbation of Spectra in Hilbert Space* (American Mathematical Society, Providence, Rhode Island, 1965).

[3] The term "mean free path" becomes ambiguous for nonsimple gases, and our use of this expression should only be taken as a generic terminology.

[4] For an ideal gas of Prandtl number 3/4, $\xi = \gamma$, $\nu = 1$, $\sigma = \gamma$.

[5] The quantity $O_\delta(1/t)$ has been defined in Appendix B to be a quantity which vanishes, for large t, slightly less rapidly than t^{-1}.

[6] The mean free time in a simple gas under normal conditions is $O(10^{-10})$ sec.

[7] L. Sirovich, Phys. Fluids **6**, 10, 218 (1963).

The solution therefore appears as diffuse waves moving to the right and left with a speed $\gamma^{\frac{1}{2}}$ and a pure diffusion mode which remains at the origin. The speed $\gamma^{\frac{1}{2}}$ is just the adiabatic speed of sound in our normalization.

To complete the discussion of the asymptotic evaluation (3.13), we observe that the error estimate in (3.13) places a restriction on x. That is, the solution is valid only in regions for which it is of lower order than $O(1/t)$. In reference to (3.15), we have for modes 1, 2, and 3, respectively,

$$|x|,\ |x - \gamma^{\frac{1}{2}}t|,\ |x + \gamma^{\frac{1}{2}}t| < (t \ln t^{\frac{1}{2}})^{\frac{1}{2}}.$$

Further, from the remarks of the previous paragraph and Appendix B, we know that a kinetic theory analysis is necessary for larger regions. In what follows it is tacitly understood that solutions are valid in regions for which they are of lower order than $O(1/t)$.

Inviscid Limit

One may now inquire about the inviscid limit. Due to the normalization (2.16), some care must be taken. To accomplish this limit we relax the condition that the length scale L is chosen proportional to the mean free path. The inviscid limit is then found by taking ξ, $\sigma \to 0$, and we find

$$\lim_{\sigma,\xi\to 0} (\mathbf{V}_{\mathrm{NS}}) = \frac{\mathbf{q}_0^1\mathbf{q}_0^1}{\gamma}\,\delta(x) + \frac{\mathbf{q}_0^2\mathbf{q}_0^2}{2\gamma}\,\delta(x - \gamma^{\frac{1}{2}}t) + \frac{\mathbf{q}_0^3\mathbf{q}_0^3}{2\gamma}\,\delta(x + \gamma^{\frac{1}{2}}t). \tag{3.16}$$

As one may easily show this is exactly the solution of the linearized Euler equations. One should observe that the limit in (3.16) is not uniform in time, t. In other words, no matter how small the dissipation may be, for large times the solution is not inviscid.

Solution of the Initial Value Problem

The solution to an initial value problem is simply given by

$$\mathbf{v}(x, t) = \mathbf{V}_{\mathrm{NS}} * \mathbf{v}_0, \tag{3.17}$$

This furnishes a solution to the initial value problem in terms of a single quadrature and is the best solution of the linearized Navier–Stokes equations in the sense described above. In addition, from (3.14) we have that (3.17) assumes the correct form at $t = 0$.

Resolution of an Initial Discontinuity

For many forms of the initial data the quadrature in (3.17) may be carried out. For example if

$$\mathbf{v}_0 = \begin{vmatrix}\bar{\rho}\\ 0\\ \bar{T}\end{vmatrix} H(x - D) = \mathbf{h}H(x - D), \tag{3.18}$$

with $\bar{\rho}$, $\bar{T}$, constants, we obtain

$$\mathbf{v}_S(D) = \frac{1}{2\gamma}\left\{\operatorname{erfc}\left[\frac{D - x}{2(\nu t)^{\frac{1}{2}}}\right]\mathbf{q}_0^1\mathbf{q}_0^1\cdot\mathbf{h} + \operatorname{erfc}\left[\frac{D - (x - \gamma^{\frac{1}{2}}t)}{2(\sigma t)^{\frac{1}{2}}}\right]\frac{\mathbf{q}_0^2\mathbf{q}_0^2}{2}\cdot\mathbf{h} + \operatorname{erfc}\left[\frac{D - (x + \gamma^{\frac{1}{2}}t)}{2(\sigma t)^{\frac{1}{2}}}\right]\frac{\mathbf{q}_0^3\mathbf{q}_0^3}{2}\cdot\mathbf{h}\right\}. \tag{3.19}$$

This solution describes the evolution of a weak shock initially at $x = D$. For later purposes, it is convenient to consider the complementary problem having the initial data

$$\mathbf{v}_0 = \mathbf{h}H(D - x).$$

Denoting the solution of this problem by $\hat{\mathbf{v}}_S$, we have

$$\hat{\mathbf{v}}_S(D) = \frac{1}{2\gamma}\left\{\operatorname{erfc}\left[\frac{x - D}{2(\nu t)^{\frac{1}{2}}}\right]\mathbf{q}_0^1\mathbf{q}_0^1\cdot\mathbf{h} + \operatorname{erfc}\left[\frac{D - (x - \gamma^{\frac{1}{2}}t)}{2(\sigma t)^{\frac{1}{2}}}\right]\frac{\mathbf{q}_0^2\mathbf{q}_0^2}{2}\cdot\mathbf{h} + \operatorname{erfc}\left[\frac{D - (x + \gamma^{\frac{1}{2}}t)}{2(\sigma t)^{\frac{1}{2}}}\right]\frac{\mathbf{q}_0^3\mathbf{q}_0^3}{2}\cdot\mathbf{h}\right\}. \tag{3.20}$$

Period of Final Decay

A useful solution is the fundamental solution. This can be regarded as the evolution of a "point explosion" at the initial instant,

$$\mathbf{v}_0 = \begin{vmatrix}\bar{\rho}\\ \bar{u}\\ \bar{T}\end{vmatrix} \delta(x) = \mathbf{f}\ \delta(x), \tag{3.21}$$

where $\mathbf{f}$ is a constant. The fundamental solution which we denote by $\mathbf{v}_f$, is given simply by

$$\mathbf{v}_f = \mathbf{V}_{\mathrm{NS}}\mathbf{f}. \tag{3.22}$$

To find another interpretation to (3.21), we consider the solution to the initial value problem (3.17), for initial data which may be said to have finite extent. More explicitly, we consider initial data of the form[8]

$$\mathbf{v}_0 = \mathbf{v}_0(x) = 0, \qquad \text{for} \qquad |x| > \mathfrak{L}/L. \tag{3.23}$$

Recall that x has been made dimensionless with

[8] By more careful estimates, the same results hold, for example, for initial data which is of exponential type at ∞. Our choice of compact support initial data is made only for convenience.

respect to L, the mean free path. Therefore $\mathfrak{L}$ measures the extent of the initial disturbance in dimensional units. Taking the Fourier transform of this initial data, we obtain

$$\mathbf{v}_0(k) = \int_{\mathfrak{L}/L}^{\mathfrak{L}/L} \exp(ikx)\mathbf{v}_0(x)\,dx = \mathbf{f} + O(ik\,\mathfrak{L}/L), \tag{3.24}$$

with

$$\mathbf{f} = \int_{-\infty}^{\infty} \mathbf{v}_0(x)\,dx. \tag{3.25}$$

The estimate of (3.25) is obtained by straightforward expansion $\mathbf{f}$, the lowest order, is immediately identified with the total mass, momentum, and energy of the initial data. Applying the same procedure used in finding the fundamental matrix, we now have for the solution to the initial value problem (3.23),

$$\mathbf{v} = \mathbf{v}_f + (\mathfrak{L}/L)O_\delta(1/t).$$

From (3.15) we therefore have

$$\mathbf{v} \sim \mathbf{v}_f$$

for dimensional time, τ,

$$\tau > (\mathfrak{L}/L)(\mathfrak{L}/a).$$

The right-hand side is the product of the inverse of a Knudsen number based on the extent of initial disturbance, with the time it takes a sound wave to traverse the initial disturbance. τ is referred to the period of final decay. In summary, the period of final decay is described by the fundamental solution associated with the total mass, momentum, and energy addition of the initial disturbance.

4. BOUNDARY PROBLEMS

We now restrict attention to boundary problems involving a fixed wall in which case we have

$$x_0(t) = 0. \tag{4.1}$$

Further, in such problems it is convenient to continue the problem by reflection in the fixed wall. That is, we take

$$\begin{aligned} \rho_0(-y) &= \rho_0(y), \\ T_0(-y) &= T_0(y), \\ u_0(-y) &= -u_0(y). \end{aligned} \tag{4.2}$$

An immediate consequence of this is that

$$[T] = 0 = [p - P]. \tag{4.3}$$

Using the asymptotic form of the fundamental matrix $\mathbf{V}_{NS}$ (3.15), we then have from (2.18)

$$\begin{aligned}
\begin{bmatrix}\rho\\ u\\ T\end{bmatrix} &= \frac{1}{\gamma(4\pi\nu t)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4\nu t}\right)\begin{bmatrix}\chi\\ 0\\ -1\end{bmatrix}[\chi\rho_0(y) - T_0(y)]\,dy \\
&+ \frac{1}{2\gamma(4\pi\sigma t)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp\left(-\frac{[x-y-(\gamma t)^{\frac{1}{2}}]^2}{4\sigma t}\right)\begin{bmatrix}1\\ \gamma^{\frac{1}{2}}\\ \chi\end{bmatrix}[\rho_0(y) + \gamma^{\frac{1}{2}}u_0(y) + \chi T_0(y)]\,dy \\
&\cdot\frac{1}{2\gamma(4\pi\sigma t)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y+\gamma^{\frac{1}{2}}t)^2}{4\sigma t}\right)\begin{bmatrix}-1\\ \gamma^{\frac{1}{2}}\\ -\chi\end{bmatrix}[-\rho_0(y) + \gamma^{\frac{1}{2}}u_0(y) - \chi T_0(y)]\,dy \\
&- \int_0^t \frac{\exp[-x/4\gamma(t-s)]}{\gamma[4\pi\gamma(t-s)]^{\frac{1}{2}}}\begin{bmatrix}\chi\\ 0\\ -1\end{bmatrix}[Q(s)]\,ds \\
&+ \frac{1}{2\gamma}\int_0^t\left\{\frac{\chi[Q(s)]}{[4\pi\sigma(t-s)]^{\frac{1}{2}}}\exp\left(-\frac{[x-\gamma^{\frac{1}{2}}(t-s)]^2}{4\sigma(t-s)}\right)\begin{bmatrix}1\\ \gamma^{\frac{1}{2}}\\ \chi\end{bmatrix} + \exp\left(-\frac{[x+\gamma^{\frac{1}{2}}(t-s)]^2}{4\sigma(t-s)}\right)\begin{bmatrix}1\\ -\gamma^{\frac{1}{2}}\\ \chi\end{bmatrix}\right\}ds.
\end{aligned} \tag{4.4}$$

(Note the boundary condition $u = 0$ at $x = 0$ is automatically satisfied.) If we consider an insulating wall,

$$[Q] = 0 \tag{4.5}$$

and the solution is immediate. For now the boundary terms vanish and our solution is given by a pure initial value problem by means of a reflection principle. This is illustrated by the following two problems.

Reflection of a Weak Shock off an Insulating Wall

We consider the initial data

$$\mathbf{v}_0 = \mathbf{h}[H(x-D) + H(-x-D)]. \qquad (4.6)$$

By symmetry we see that the solution describes the reflection of a shock, initially located at $x = D$, off an insulating wall at $x = 0$. This solution is easily given in terms of (3.19) (3.20),

$$\mathbf{v} = \mathbf{v}_S(D) + \hat{\mathbf{v}}_S(-D). \qquad (4.7)$$

Finite Shock Tube with Insulating Walls

The above can now be generalized to describe the gas motion in a shock tube of length L with an initial discontinuity (weak) located at D. Taking the initial data,

$$\mathbf{v}_0 = \mathbf{h} \sum_{n=-\infty}^{\infty} \{H(x - 2nL - D) - H[x - 2(n+1)L + D]\}, \qquad (4.8)$$

it is clear by symmetry that this leads to a description of the above problem; the solution is

$$\mathbf{v} = \sum_{n=-\infty}^{\infty} \{\mathbf{v}_S(x - 2nL - D) - \mathbf{v}_S[x - 2(n+1)L + D]\}. \qquad (4.9)$$

Again by symmetry the walls are insulating.

5. REFLECTION OF A SOUND PULSE FROM AN ISOTHERMAL WALL

To consider a sound pulse impinging on a wall, we regard the wave as originating from a point "explosion" at a distance D, i.e., we take initial data

$$\begin{bmatrix} \rho_0 \\ u_0 \\ T_0 \end{bmatrix} = -q \begin{bmatrix} -1 \\ \gamma^{\frac{1}{2}} \\ -\chi \end{bmatrix} \delta(x-D) + q \begin{bmatrix} 1 \\ \gamma^{\frac{1}{2}} \\ \chi \end{bmatrix} \delta(x+D), \qquad (5.1)$$

q measures the strength of the wave. This particular data give a single pulse moving to the left in the right-hand plane[9].

For comparison, we note that the solution in the case of an insulating wall is

$$\begin{bmatrix} \rho \\ u \\ T \end{bmatrix} = \frac{q}{(4\pi\sigma t)^{\frac{1}{2}}} \left\{ \exp\left(-\frac{(x - D + \gamma^{\frac{1}{2}}t)^2}{4\sigma t}\right) \begin{bmatrix} 1 \\ -\gamma^{\frac{1}{2}} \\ \chi \end{bmatrix} + \exp\left(-\frac{(x + D - \gamma^{\frac{1}{2}}t)^2}{4\sigma t}\right) \begin{bmatrix} 1 \\ \gamma^{\frac{1}{2}} \\ \chi \end{bmatrix} \right\}. \qquad (5.2)$$

Restricting our attention to the right half-plane, we see that the pulse moving to the left with the adiabatic speed $\gamma^{\frac{1}{2}}$ arrives at the wall at a time $D\gamma^{-\frac{1}{2}}$ and is then reflected. It should be noted that no residual effect is left at the origin.

In contrast to the above solution, we consider the same problem, same initial data, (5.1), but with an isothermal wall. In this case, $[Q] \neq 0$. On imposing the boundary condition that $T = 0$ at $x = 0$, we have from (4.4)

$$T(0, t) = 0 = \frac{q\chi}{(\pi\sigma t)^{\frac{1}{2}}} \exp\left(-\frac{(D - \gamma^{\frac{1}{2}}t)^2}{4\sigma t}\right) + \frac{1}{\gamma}\int_0^t \frac{[Q(s)]\,ds}{[4\pi\nu(t-s)]^{\frac{1}{2}}} + \int_0^t \frac{\chi^2 \exp[-\gamma(t-s)/4\sigma]}{\gamma[4\pi\sigma(t-s)]^{\frac{1}{2}}}[Q(s)]\,ds. \qquad (5.3)$$

To solve this equation for $[Q(t)]$, we employ the Laplace transform. Denoting the transformed variables by the same letter we find

$$[Q(p)] = -\frac{2\gamma q\chi(\nu p)^{\frac{1}{2}} \exp(D\gamma^{\frac{1}{2}}/2\sigma)}{\sigma^{\frac{1}{2}}[p + (\gamma/4\sigma)]^{\frac{1}{2}} + \chi^2(\nu p)^{\frac{1}{2}}} \cdot \exp\left[-\frac{D}{\sigma^{\frac{1}{2}}}\left(p + \frac{\gamma}{4\sigma}\right)^{\frac{1}{2}}\right], \qquad (5.4)$$

where p is the transform variable. Inverting the transform

$$[Q(t)] = -\frac{\gamma q\chi}{2\pi i}\left\{\int_\uparrow \frac{(\sigma/\nu)^{\frac{1}{2}}\{p[p + (\gamma/4\sigma)]\}^{\frac{1}{2}} e^{-(D/\sigma)[p+(\gamma/4\sigma)]^{\frac{1}{2}} + pt + (D\gamma^{\frac{1}{2}}/2\sigma)}}{[(\sigma/\nu) - \chi^4]p + (\gamma/4\nu)}\,dp - \int_\uparrow \frac{\chi^2 p e^{-(D/\sigma^{\frac{1}{2}})[p+(\gamma/4\sigma)]^{\frac{1}{2}} + pt + (D\gamma^{\frac{1}{2}}/2\sigma)}}{[(\sigma/\nu) - \chi^4]p + (\gamma/4\nu)}\,dp\right\} = [Q(t)]_{\mathrm{I}} + [Q(t)]_{\mathrm{II}}. \qquad (5.5)$$

[9] This initial data are chosen so as to exclude the modes of pure diffusion and motion to the right. This is done purely for simplicity since no complication arises in the more general situation.

The second integral of the bracket may be inverted directly.[10] However, a much simplified expression is obtained by recognizing that D is large in any meaningful situation (since it is normalized with respect to the mean free path). In order to evaluate $[Q]_{\text{I}}$, branch cuts between $-\gamma/4\sigma$ and 0 and between $-\infty$ and $-\gamma/4\sigma$ are made. The second integral, $[Q]_{\text{II}}$, only requires the latter cut, and, we may move the path of integration to the left of the origin so that the integral is exponentially small for large times. It therefore suffices to consider times

$$t = \alpha D$$

with $\alpha = O(1)$. The evaluation of $[Q(t)]_{\text{II}}$ then lends itself to the saddle-point method. Denoting the saddle point by p_0, and the coefficient of D by E, we have

$$p_0 = \frac{1}{4\alpha^2\sigma} - \frac{\gamma}{4\sigma},$$

$$E(p_0) = -\frac{1}{4\sigma\alpha}(\alpha\gamma^{\frac{1}{2}} - 1)^2,$$

$$\left.\frac{d^2E}{dp^2}\right|_{p_0} = 2\sigma\alpha^3,$$

and after some manipulation we find

$$[Q]_{\text{II}} \sim \frac{\chi^3 q\gamma D(D^2-\gamma t^2)\exp[-(1/4\sigma t)(D-t\gamma^{\frac{1}{2}})^2]}{\{[(\sigma/\nu)-\chi^4)](D^2-\gamma t^2)+\gamma(\sigma/\nu)t^2\}(\pi t^3\sigma)^{\frac{1}{2}}}. \tag{5.6}$$

The analysis of $[Q]_{\text{I}}$ is the same as above when $t \leq D/\gamma^{\frac{1}{2}}$. For larger times the saddle lies on the branch cut, and as a consequence leads to a higher order result. The contribution from the branch cut is readily evaluated by asymptotic means. To the lowest order for all times we obtain

$$[Q]_{\text{I}} = \frac{D^2\gamma q\chi H(D-\gamma^{\frac{1}{2}}t)(D^2-\gamma t^2)^{\frac{1}{2}}\exp[-(1/4\sigma t)(D-t\gamma^{\frac{1}{2}})^2]}{(\pi t^3\nu)^{\frac{1}{2}}\{[(\sigma/\nu)-\chi^4][D^2-\gamma t^2)+(\gamma\sigma/\nu)t^2\}} + 2\chi q\left(\frac{\gamma\nu}{\pi t^3}\right)^{\frac{1}{2}}H(\gamma^{\frac{1}{2}}t-D)\left\{\operatorname{erf}\left(\frac{\gamma t^2-D^2}{4\sigma t}\right)^{\frac{1}{2}} - \left(\frac{\gamma t^2-D^2}{\sigma t\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{t^2\gamma-D^2}{4\sigma t}\right)\right\}. \tag{5.7}$$

Combining (5.6) and (5.7), we obtain $[Q]$. This quantity, however, is just twice the normalized heat flow at the origin. We therefore see that the heat flow at the wall has been determined without knowledge of the full flow field. To obtain the flow field, the form for $[Q]$ just obtained is substituted into (4.4). From the latter one sees that after the pulse has been reflected a residual layer remains at the wall and further that the reflected pulse is modified. We mention in closing that the interaction of an arbitrary pulse with an isothermal wall is obtained from the above result by convolution with the suitably formed initial data.

ACKNOWLEDGMENTS

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APPENDIX A: DISCONTINUOUS NAVIER–STOKES EQUATIONS

Consider a closed surface

$$F(x, t) = 0$$

and a scalar field $\rho(x, t)$, then as is well known

$$\frac{d}{dt}\int_F \rho\, d\mathbf{x} = \int_F \frac{\partial\rho}{\partial t}\, d\mathbf{x} + \int_F \rho\mathbf{u}\cdot\mathbf{n}\, dF.$$

The volume integral is over the interior of $F = 0$, and the normal velocity $\mathbf{u}\cdot\mathbf{n}$ given by

$$-\frac{\partial F/\partial t}{|\nabla F|} = \mathbf{u}\cdot\mathbf{n}$$

($\mathbf{n}$ the outward normal of F, i.e., parallel to ∇F). If, further, $\mathbf{u}$ is defined and differentiable everywhere,

$$\frac{d}{dt}\int_F \rho\, d\mathbf{x} = \int_F \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{u}\right) d\mathbf{x}.$$

We now consider a surface

$$S(x, t) = 0$$

not necessarily closed. Further, $\rho(x, t)$ and $\mathbf{u}(x, t)$ are differentiable up to the surface $S = 0$ but may be discontinuous across this surface. Now let $F = 0$ include a volume cut by $S = 0$, and consider

$$I = \frac{d}{dt}\int_F \rho\, d\mathbf{x}.$$

Denoting the volume cut out by $F = 0$ and $S = 0$ by F_1 and F_2, then we may write

$$I = \frac{g}{dt}\int_{F_1} \rho\, d\mathbf{x} + \frac{d}{dt}\int_{F_1} \rho\, d\mathbf{x}.$$

[10] Bateman Manuscript Project, *Table of Integral Transforms* (McGraw-Hill Book Company, Inc., New York, 1954), Vol. I, p. 246.

Then if

$$\mathbf{U}\cdot\mathbf{n} = -\frac{\partial S/\partial t}{|\nabla S|},$$

we have on adding and subtracting the appropriate surface quantities

$$I = \int_{F_1}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{u}\right)d\mathbf{x} + \int_{F_2}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{u}\right)d\mathbf{x} - \int_{S^0}[\rho(\mathbf{U}-\mathbf{u})]\cdot\mathbf{n}\,dS, \qquad \text{(A1)}$$

where S^0 is the portion of $S = 0$ intercepted by the interior of F and [ ] denotes the jump across $S = 0$.

We now employ (A1) in deriving the discontinuous forms of the Navier–Stokes equations. Also we allow $S = 0$ to have source rates of mass, momentum, and energy σ, $\boldsymbol{\pi}$, η, respectively. We then have, using the above notation,

$$\frac{d}{dt}\int_F \rho\,d\mathbf{x} = \int_S \sigma\,dS,$$

$$\frac{d}{dt}\int_F \rho\mathbf{u}\,d\mathbf{x} = \int_F(-p\mathbf{n} + \mathbf{P}\cdot\mathbf{n})\,dF + \int_{S^0}\boldsymbol{\pi}\,dS,$$

$$\frac{d}{dt}\int_F \rho(e + \tfrac{1}{2}u^2)\,d\mathbf{x} = \int_F(-\mathbf{Q} - p\mathbf{u}\cdot\mathbf{n} + \mathbf{u}\cdot\mathbf{P}\cdot\mathbf{n})\,dF + \int_{S^0}\eta\,dS. \qquad \text{(A2)}$$

We also need an integral form of the constitutive equations, and this is taken as

$$\int_F \frac{P_{ij}}{\mu}\,d\mathbf{x} = \int_F\left(u_i n_j + u_j n_i - \tfrac{2}{3}\mathbf{u}\cdot\mathbf{n}\,\delta_{ij} + \frac{\beta}{\mu}\mathbf{u}\cdot\mathbf{n}\,\delta_{ij}\right)dF + \int_{S^0}\frac{S_{ij}}{\mu}\,dS,$$
$$\int_F \frac{\mathbf{Q}}{\kappa}\,dx = -\int_F T\mathbf{n}\,dF + \int_{S^0}\frac{\mathbf{h}}{\kappa}\,dS. \qquad \text{(A3)}$$

In the last two equations, account has been taken of a surface distribution of stress, S_{ij}, and heat production $\mathbf{h}$ on $S = 0$. The ratio of viscosities β/μ is taken as constant. The formula (A1) may now be applied to each integral, and we obtain

$$\left(\int_{F_1} + \int_{F_2}\right)\left(\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\mathbf{x}}\cdot\rho\mathbf{u}\right)d\mathbf{x} = \int_{S^0}\{\sigma + [\rho(\mathbf{U}-\mathbf{u})]\cdot\mathbf{n}\}\,dS, \qquad \text{(A4)}$$

$$\left(\int_{F_1} + \int_{F_2}\right)\left(\frac{\partial}{\partial t}\rho\mathbf{u} + \frac{\partial}{\partial x_j}\rho\mathbf{u}u_j + \frac{\partial}{\partial\mathbf{x}}p - \frac{\partial}{\partial\mathbf{x}}\cdot\mathbf{P}\right)d\mathbf{x} = \int_{S^0}\{\boldsymbol{\pi} - [(p\mathbf{n} - \mathbf{P}\cdot\mathbf{n}) - \rho\mathbf{u}(\mathbf{U}-\mathbf{u})\cdot\mathbf{n}]\}\,dS, \qquad \text{(A5)}$$

$$\left(\int_{F_1} + \int_{F_2}\right)\left\{\frac{\partial}{\partial \mathbf{t}}\rho(e + \tfrac{1}{2}u^2) + \frac{\partial}{\partial\mathbf{x}}\cdot\rho\mathbf{u}(e + \tfrac{1}{2}u^2) + \frac{\partial}{\partial\mathbf{x}}\cdot(p\mathbf{u} - P\cdot\mathbf{u} + \mathbf{Q})\right\}d\mathbf{x}$$
$$= \int_{S^0}\{\eta - [p\mathbf{u} - P\cdot\mathbf{n} - \rho(e + \tfrac{1}{2}u^2)(\mathbf{U}-\mathbf{u}) + \mathbf{Q}]\cdot\mathbf{n}\}\,dS, \qquad \text{(A6)}$$

$$\left(\int_{F_1} + \int_{F_2}\right)\left(\frac{P_{ij}}{\mu} - u_{i,j} - u_{j,i} + \tfrac{2}{3}\delta_{ij}\,\nabla\cdot\mathbf{u} - \frac{\beta}{\mu}\nabla\cdot\mathbf{u}\,\delta_{ij}\right)d\mathbf{x}$$
$$= \int_{S^0}\left\{\frac{S_{ij}}{\mu} + \left[u_i n_j + u_j n_i - \tfrac{2}{3}\mathbf{u}\cdot\mathbf{n}\,\delta_{ij} + \frac{\beta}{\mu}\mathbf{u}\cdot\mathbf{n}\,\delta_{ij}\right]\right\}dS, \qquad \text{(A7)}$$

$$\left(\int_{F_1} + \int_{F_2}\right)\left(\frac{\mathbf{Q}}{\kappa} + \frac{\partial T}{\partial\mathbf{x}}\right)d\mathbf{x} = \int_S\left\{\frac{\mathbf{h}}{\kappa} - [T]\mathbf{n}\right\}dS. \qquad \text{(A8)}$$

The integral form of the equations hold in arbitrary domains and in particular in F_1 and F_2 so that the left-hand sides of (A4)–(A8) vanish, and we have

$$\sigma = -[\rho(\mathbf{U}-\mathbf{u})\cdot\mathbf{n}],$$
$$\boldsymbol{\pi} = [p\mathbf{n} - \mathbf{P}\cdot\mathbf{n} - \rho\mathbf{u}(\mathbf{U}-\mathbf{u})\cdot\mathbf{n}],$$
$$\eta = [p\mathbf{u} - \mathbf{P}\cdot\mathbf{u} - \rho(e + \tfrac{1}{2}u^2)(\mathbf{U}-\mathbf{u}) + \mathbf{Q}]\cdot\mathbf{n}, \qquad \text{(A9)}$$
$$S_{ij} = -\mu[u_i n_j + u_j n_i - \tfrac{2}{3}\mathbf{u}\cdot\mathbf{n}\,\delta_{ij}] - \beta[\mathbf{u}\cdot\mathbf{n}\,\delta_{ij}],$$
$$\mathbf{h} = \kappa[T\mathbf{n}].$$

This has the following interpretation: Given the source strengths on a singular surface and values on one side of a singular surface, the values on the other side are determined by (A9). In the work of this paper we adopt the alternate view, that the jumps determine the source strengths.

In applying the above formulation, we replace material boundaries by singular surfaces, and further, we consider only impenetrable boundaries; hence, we take

$$(\mathbf{U}-\mathbf{u})\cdot\mathbf{n} = 0. \qquad \text{(A10)}$$

In order to obtain the differential expression of (A2) and (A3), we define

$$\delta(S) = \int \prod_{i=1}^{N} \delta(x_i - y_i)\, dS(\mathbf{y}) \qquad \text{(A11)}$$

(where N is the number of dimensions). As is easily seen, this has the property

$$\int f(x)\ \delta(S)\, d\mathbf{x} = \int_S f\, dS,$$

where the volume integral is over all space. The differential expressions then are

$$(\partial/\partial t)\rho + (\partial/\partial \mathbf{x})\cdot\rho\mathbf{u} = 0,$$

$$(\partial/\partial t)\rho\mathbf{u} + (\partial/\partial x_i)\rho\mathbf{u}u_i + (\partial/\partial\mathbf{x})p - (\partial/\partial\mathbf{x})\cdot\mathbf{P} = \pi\ \delta(S),$$

$$\frac{\partial}{\partial t}\rho(e + \tfrac{1}{2}u^2) + \frac{\partial}{\partial\mathbf{x}}\cdot\rho\mathbf{u}(e + \tfrac{1}{2}u^2) + \frac{\partial}{\partial\mathbf{x}}\cdot(p\mathbf{u} - \mathbf{P}\cdot\mathbf{u}) + \frac{\partial}{\partial\mathbf{x}}\cdot\mathbf{Q} = \eta\ \delta(S), \qquad \text{(A12)}$$

$$P_{ij} = \mu(u_{i,j} + u_{j,i} - \tfrac{2}{3}\ \delta_{ij}\ \nabla\cdot\mathbf{u}) + \beta\ \nabla\cdot\mathbf{u}\ \delta_{ij} + S_{ij}\ \delta(S),$$

$$\mathbf{Q} = -\kappa(\partial/\partial\mathbf{x})T + \mathbf{h}\ \delta(S).$$

As a word of caution, the usual reduction of the energy equation should not be carried out since it leads to products of distributions which, in general, are not defined.

With the above derivation the boundary conditions appear explicitly in the equations themselves. Although the notion of surface discontinuities has proved useful in perfect fluid studies, their use does not seem to have entered other branches of fluid mechanics. To our knowledge the above derivation is novel.

APPENDIX B: ASYMPTOTIC EVALUATION OF THE BASIC INTEGRAL

The theorem, proved below, is the one-dimensional form a result which holds in any number of dimensions.[11] Because the one-dimensional form is of basic importance in the calculations of this paper, we include a proof.

Theorem: Consider

$$I = \int_{-\infty}^{\infty} \exp\left[-\mu(k)t - ikx\right]p(k)\, dk,$$

with

(1) $\int_{-\infty}^{\infty} |p(k)|\, dk < M < \infty$,

(2) $\max_k |p(k)| < M$,

(3) $\mathrm{Re}\,(\mu) \geq 0$,

(4) $\mu(k) = 0$ if and only if $k = 0$;

(5) $\mu(k) = +i\alpha k + \beta k^2 + O(|k|^3)$, α, β real, $\beta \neq 0$;

(6) $\mu(k) \in C$.

Then

$$I = \int_{-\infty}^{\infty} \exp\left(-\beta k^2 t - i\alpha kt - ikx\right)p(k)\, dk + O_\delta\left(\frac{1}{t}\right). \qquad \text{(B1)}$$

The symbol $O_\delta(1/t)$ signifies a quantity of $O(1/t^{1-\delta})$, where $\delta > 0$ is small.

Proof: Denote the real part of μ by μ_r. From (3), (4), and (6) we have $\mu_r > \mu_0 > 0$ for $|k| > k_0 > 0$, i.e., μ_r is bounded away from zero for $|k|$ bounded away from zero.

From (3) we have that $\beta > 0$.

Consider all $\bar{\epsilon} > 0$ such that

$$\mu_r - \tfrac{1}{2}\beta\bar{\epsilon}^2 > 0$$

for $|k| > \bar{\epsilon}$. This set is nonempty, since

$$\mu_r - \tfrac{1}{2}\beta\bar{\epsilon} = \tfrac{1}{2}\beta k^2 + \tfrac{1}{2}\beta(k^2 - \bar{\epsilon}^2) + O(|k|^3).$$

Denote by ϵ_1 the maximum such $\bar{\epsilon}$ (which may be infinite).

Next from (5) we have

$$\lim_{k\to 0} \frac{\mu - i\alpha k - \beta k^2}{|k|^3} = C',$$

from which we obtain

$$\frac{|\mu - i\alpha k - \beta k^2|}{|k|^3} < 1 + |C'| = C$$

for

$$|k| < \epsilon_2.$$

Set

$$\epsilon_0 = \min\,(\epsilon_1, \epsilon_2).$$

Choose $\epsilon_0 > \epsilon > 0$ and write

$$I = \int_{|k|>\epsilon} \exp\left[-ikx - \mu(k)t\right]p(k)\, dk + \int_{|k|<\epsilon} \{\exp\left[-\mu(k)t\right] - \exp\left[i\alpha kt - \beta k^2 t\right]\}\cdot\exp\,(ikx)p(k)\, dk + \int_{|k|<\epsilon} \exp\left(-ikx - i\alpha kt - \beta k^2 t\right)p(k)\, dk$$

$$= I_1 + I_2 + I_3.$$

[11] L. Sirovich (to be published).

Then

$$I_1 = \exp(-\tfrac{1}{2}\beta\epsilon^2 t) \cdot \int_{|k|>\epsilon} \exp\{-ikx - [\mu(k) - \tfrac{1}{2}\beta\epsilon^2]t\}p(k)\, dk,$$

$$|I_1| \le \exp(-\tfrac{1}{2}\beta\epsilon^2 t)M.$$

Considering I_2,

$$|I_2| \le M \int_{|k|\le\epsilon} |\exp[-(\mu - \beta k^2 - i\alpha k)t] - 1|\, dk$$

$$\le 2M \sum_{n=1}^{\infty} \frac{C^n \epsilon^{3n+1} t^n}{n!\,(3n+1)}.$$

The only restriction on ϵ thus far is

$$0 < \epsilon < \epsilon_0.$$

We now choose a $\delta > 0$, small, and set

$$\epsilon = 1/t^{-\delta/2}$$

for

$$t > \epsilon_0^{-[2/(1-2\delta)]}.$$

From this we have

$$|I_1| = O[\exp(-t^{2\delta})], \quad |I_2| = O(t/t^{2-4\delta}) = O_\delta(1/t),$$

which proves (B1). (An exponentially decaying term is neglected there.)

It is clear from the above proof that if an expansion of $p(k)$ is made only the leading term enters. This, however, may lead to a large numerical coefficient in the correction term, and therefore it is best not to perform this expansion without first considering the value of this numerical coefficient. For example, for the case under study in Sec. 3, $p(k) = q(k)v^0(k)$, where $q(k)$ results from the eigenvectors and v^0 from the initial data. An expansion of $q(k)$ leads only to a numerical coefficient of $O(1)$, whereas the similar term from $v^0(k)$ may be very large for highly oscillatory initial data. In this case, it is appropriate to expand $q(k)$ [and retain only $q(k = 0)$] and not expand $v^0(k)$. This point is further amplified in the contents of this paper.

A noteworthy feature of the calculation is that the restriction on x is not explicit. Only after the evaluation of the integral of (B1) can the restriction on x be found. That is, restriction to those x is made for which the correction term is of higher order. Examples of this are to be found in calculations (see remarks in Sec. 3).

It is clear from the analysis of this section that more terms in the expansion of $\mu(k)$ could have been taken. This results in a higher-order error estimate in (B1), e.g., one finds that if $O(k^3)$ is included the error estimate becomes $O_\delta(t^{-\frac{3}{2}})$. However, this sort of generality is tempered by the fact that one must be able to evaluate the resulting approximate integral. In general, we are only guaranteed the evaluation of integrals with cubic exponents. We do not consider this case since $\mu(k)$ to $O(k^3)$ is not given correctly by the Navier–Stokes equations (see remarks in connection with Ref. 7). For completeness, however, we include the evaluation of the cubic integral.

Consider

$$B = \int_{-\infty}^{\infty} \exp(-iak - bk^2 + ick^3)\, dk$$

with a, b, c real. Defining

$$u = [a + (b^2/3c)], \qquad v = (2b^3/27c^2) + (ab/3c),$$

we find

$$B = (3c)^{-\frac{1}{3}} \exp(v) A_0[u/(3\,|c|)^{\frac{1}{3}}].$$

A_0 is an Airy integral.[12]

[12] J. C. P. Miller, *The Airy Integral* (Cambridge University Press, New York, 1958).