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Steady Gasdynamic Flows

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The steady Oseen equations for an arbitrary gas are considered. The inverse of the Oseen operator is obtained in closed form uniformly through transonic speeds in a certain limit. This limit is the same as the limit under which the Navier–Stokes equations are derivable from kinetic theory. The far field flow past an arbitrary body in terms of lift, drag, and energy dissipation of the body is obtained. As another application, the supersonic flow field past a semi-infinite flat plate is explicitly obtained. The latter problem serves as the basis for an extension of the Oseen theory which gives an approximation of skin friction for arbitrary bodies.

1. INTRODUCTION

As is well known, the gasdynamic Navier–Stokes equations follow from the application of the Chapman–Enskog procedure to the Boltzmann equation.[1] The basic assumption in this procedure is that the scale variations of the flow field be large compared with mean molecular scales (mean free path, mean time of flight). From the viewpoint of gasdynamics the condition imposed by kinetic theory is

$$M/\mathrm{Re} \ll 1, \tag{1.1}$$

where M is the characteristic Mach number and Re the characteristic Reynolds number. We use the term hydrodynamical limit to refer to the condition (1.1). From the kinetic theory viewpoint it is, therefore, only consistent (although not mandatory) to continue imposing the hydrodynamical limit (1.1) in a discussion based on the Navier–Stokes equations. This limit is consistently carried out in the present paper and in fact results in a number of significant simplifications.

A basic flaw in the above argument lies in the fact that it only applies to simple gases.[2] Our subsequent discussion, however, is meant to apply to the Navier–Stokes equations for an arbitrary gas. These equations, are based on continuum hypotheses (hence implying a much more general theory) and the basis for their validity cannot be stated with such certainty. Nevertheless we will assume (and we regard it as plausible) that the hydrodynamical limit should still be imposed in the more general framework. Therefore, the arguments of the first paragraph are regarded as motivating the use of the limit (1.1) and not as a restriction on the materials which will be considered.

We focus attention mainly on the steady compressible Oseen equations. In Sec. 3 we show that the solution to a boundary-value problem may be represented as a solution operator acting on the boundary data. This solution operator, which is formally the inverse of the Oseen operator, can be regarded as the matrix form of the fundamental solution. The latter was considered by Oseen[3] in the incompressible case; the compressible case was first considered in the pioneering report of Lagerstrom, Cole, and Trilling.[4] In the latter paper an ideal, nonconducting gas is considered. The second restriction was later lifted by Cole and Wu[5,6] who consider a Prandtl number of 0.75. Even with such specialized gases, these fundamental solutions, due to their analytical complexity, do not appear in a readily useable form. A critical discussion, of the Oseen equations, with additional references, appears in an article by Lagerstrom.[7]

From the standpoint of the present paper the above cited works leave all or a portion of the hydrodynamical limit, uncompleted. When this condition is imposed, the solution operator takes on a very simple form in terms of known functions (Sec. 4). This, in fact, is carried out for Mach numbers which range from supersonic through subsonic speeds.

An advantage in the use of the solution operator (or the fundamental solution) appears in the determination of quantities such as skin friction, heat transfer, and temperature at a body. Using a formulation of the Navier–Stokes equations introduced in

[1] See, e.g., S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1952), 2nd ed.

[2] The Chapman–Enskog procedure has, of course, been applied to gas mixtures and to gases with internal degrees of freedom.[1] This can result in departures from the usual Navier–Stokes equations. Since our paper does not bear directly on kinetic theory, this discussion will not be taken up.

[3] C. W. Oseen, *Neuere Methoden und Ergebnisse in der Hydrodynamik* (Akad. Verlag, Leipzig, 1927).

[4] P. A. Lagerstrom, J. D. Cole, and L. Trilling, Guggenheim Aeronautical Laboratory, California Institute of Technology, Monograph, 1949.

[5] J. D. Cole and T. Y. Wu, in *Proceedings of the* 1952 *Heat Transfer and Fluid Mechanics Institute* (University of California, Los Angeles, California, 1952), p. 139.

[6] J. D. Cole and T. Y. Wu, J. Appl. Mech. 19, 209 (1952).

[7] P. A. Lagerstrom, in *Theory of Laminar Flows*, F. K. Moore, Ed. (Princeton University Press, Princeton, New Jersey, 1964), p. 20.

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an earlier paper,[8] in Sec. 3 we show that such quantities may be determined without a prior determination of the flow field.

In general, the solution operator exhibits four regions of maximal asymptotic behavior. These are the structured Mach layers and two structured wakes. The latter can be identified as a thermal wake and a viscous wake. This identification can always be made in the sense that these two wakes always decouple. For subsonic flow, the Mach layers, of course, disappear while the two wakes remain. In general, the waves and wakes grow in thickness as the square root of distance from the body. In the transonic regime, however, the Mach layer is markedly thicker and grows as the two-thirds power of the distance. An interesting side remark can be made about the inviscid limit. It is found that although the dissipative solution approaches the inviscid solution, it does so nonuniformly. The neighborhood of infinity is always a dissipative "boundary layer."

As an application of the solution operator, the far field flow is considered (Secs. 3 and 5). For this case, the fundamental solution itself describes the flow field. Another application is to supersonic flow past a semi-infinite flat plate which is carried out in Sec. 6.

A point of note is that all of the above-stated results have been found for an arbitrary fluid. The bulk viscosity and Prandtl number are left unspecified, as is the form of the equation of state.[9]

With the exception of low Reynolds number flows,[10] the use of the Oseen equations remains largely unjustified. As a rule these equations are accepted as correctly describing the far field and are further regarded as a model of the true flow in the neighborhood of a body. If the Oseen equations are solved in a self-consistent manner, this statement is incorrect. If the correct boundary conditions (e.g., no slip at a wall) are applied in the Oseen theory, incorrect values of skin friction and heat-transfer coefficients are obtained. On the other hand, it is easily demonstrated (see Sec. 3) that the first-order flow at large distances from a body, is determined by the forces and total heat flow of the body. In Secs. 6 and 7 the notion of a slip velocity at a wall is introduced. It is shown that this slip velocity may be so chosen, that it leads to the correct skin friction at a wall and to the formally correct flow field at infinity. Both the skin friction and the entire flow field are explicitly obtained for the case of supersonic flow past a semi-infinite flat plate. Because of the introduction of a slip velocity, the flow field in the neighborhood of the wall is, of course, incorrect. In their study of the incompressible Oseen flow past a semi-infinite flat plate Carrier and Lewis[11] introduce an allied method for obtaining the correct skin friction. They consider Oseen equations containing a fictitious uniform velocity which is chosen to give the correct skin friction. This gives an incorrect flow at infinity; however, it is plausible that the flow in the neighborhood of a flat plate is described by their solution.

In Sec. 7, a method based on the slip velocity approach is suggested for the determination of skin friction of arbitrary bodies. In doing this one gives up the idea of approximating the true flow in the neighborhood of the body in order to obtain the correct skin friction, heat transfer, and far field. There is, however, good reason not to use the Oseen equations for a description of the neighborhood of a body since they predict the wrong boundary-layer growth for walls not parallel to the free stream. This was first pointed out by Latta[12] and is further discussed in the article by Lagerstrom.[7]

2. FORMULATION

We consider steady flow past an object whose surface we denote by S: $S(\mathbf{x}) = 0$. It proves useful to regard the interior of S as also containing a steady fluid flow. The surface itself may then be regarded as a singular surface. An earlier paper[8] contains a derivation of the Navier–Stokes equations under such conditions, and in the Appendix a different, and greatly simplified derivation, is given. For the above flow these equations take the form

$$\frac{\partial}{\partial \mathbf{x}} \cdot \rho \mathbf{u} = 0, \tag{2.1}$$

$$\frac{\partial}{\partial \mathbf{x}} \cdot (\rho \mathbf{u}\mathbf{u} + p\mathbf{I} - \mathbf{P}) = [p\mathbf{I} - \mathbf{P}] \cdot \mathbf{n}\ \delta(S), \tag{2.2}$$

[8] L. Sirovich, Phys. Fluids 10, 24 (1967).

[9] An earlier paper [L. Sirovich, in *Rarefied Gas Dynamics*, L. Talbot, Ed. (Academic Press Inc., New York, 1951), p. 283] contains the fundamental solution for an ideal gas of Prandtl number $\frac{2}{3}$. The extension of the analysis to gases of arbitrary Prandtl number was announced by L. Sirovich, in *Partial Differential Equations and Continuum Mechanics*, R. E. Langer, Ed. (University of Wisconsin Press, Madison, Wisconsin, 1961), p. 382.

[10] For low Reynolds number incompressible flows the Oseen equations reduce to the classical Stokes equations near a body, and hence furnish a uniformly valid approximation to the flow field. For low Reynolds number compressible flows, Lagerstrom (Ref. 7) shows that the "Stokes" equations are in fact nonlinear. However, in the absence of external body heating and under the limit (1.1) the classical Stokes equations are valid near a body and the Oseen equations again provide a uniformly valid approximation.

[11] J. A. Lewis and G. F. Carrier, Quart. Appl. Math. 8, 228 (1949).

[12] G. E. Latta, in *Proceedings of the Eighth International Congress of Theoretical and Applied Mechanics* (Faculty of Science, Istanbul, 1953), Vol. I, p. 479.

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$$\frac{\partial}{\partial \mathbf{x}} \cdot \left[\rho \mathbf{u}\left(e + \frac{u^2}{2}\right) + p\mathbf{u} - \mathbf{P}\cdot\mathbf{u} + \mathbf{Q} \right] = [\mathbf{Q}]\cdot\mathbf{n}\ \delta(S), \tag{2.3}$$

$$\frac{P_{ij}}{\mu} = u_{i,j} + u_{j,i} + \left(\frac{\beta}{\mu} - \frac{2}{3}\right)\nabla\cdot u\ \delta_{ij}, \tag{2.4}$$

$$\frac{\mathbf{Q}}{\kappa} + \frac{\partial T}{\partial \mathbf{x}} = [T]\mathbf{n}\ \delta(S). \tag{2.5}$$

The notation is explained in the Appendix. A definition of the "one-dimensional" delta function $\delta(S)$ more convenient than that given in the Appendix is

$$\delta(S) = \int \delta(\mathbf{x} - \mathbf{y})\ dS(\mathbf{y}), \tag{2.6}$$

where

$$\delta(\mathbf{x}) = \prod_{i=1}^{N} \delta(x_i).$$

[N is the number of dimensions and $\delta(x)$ the usual one-dimensional δ function] and the integration is over the surface $S = 0$. One easily sees that Eq. (2.6) satisfies Eq. (A4) of the Appendix.

The nature of the fluid will be left arbitrary, and we write quite generally,

$$p = p(\rho, T), \qquad e = e(\rho, T),$$

$$p - \rho^2\left(\frac{\partial e}{\partial \rho}\right)_T = -T\left(\frac{\partial p}{\partial T}\right)_\rho.$$

The last relation is just the compatibility relation for the first two. Also we do not place any restrictions on the dissipative coefficients μ, κ, β, nor on their ratios.

A. Oseen Approximation

We introduce the upstream equilibrium quantities

$$p_0,\ \rho_0,\ T_0,\ \mathbf{u}_0,\ a_0 = \left[\left(\frac{\partial p_0}{\partial \rho_0}\right)_{T_0}\right]^{\frac{1}{2}},$$

$$c_v = \left(\frac{\partial e_0}{\partial T_0}\right)_{\rho_0}, \qquad c_0 = \left[\left(\frac{\partial p_0}{\partial \rho_0}\right)_{S_0}\right]^{\frac{1}{2}}, \tag{2.7}$$

and an as yet uspecified length scale L. The following dimensionless quantities will be employed:

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{L}, \qquad \tilde{\rho} = \frac{\rho - \rho_0}{\rho_0}, \qquad \tilde{\mathbf{u}} = \frac{\mathbf{u} - \mathbf{u}_0}{a_0},$$

$$\tilde{T} = \left(\frac{c_v}{a_0^2 T_0}\right)^{\frac{1}{2}}(T - T_0), \tag{2.8}$$

$$\tilde{\mathbf{P}} = \frac{\mathbf{P}}{\rho_0 a_0^2}, \qquad \tilde{\mathbf{Q}} = \frac{\mathbf{Q}}{\rho_0 a_0^2 (c_v T_0)^{\frac{1}{2}}},$$

$$\mathbf{U} = \frac{\mathbf{u}_0}{a_0}, \frac{p - p_0}{\rho_0 a_0^2} = \tilde{p}.$$

Carrying only the first order in tilde quantities (the Oseen approximation) and then dropping the tildes, we obtain

$$\begin{aligned}
&\mathbf{U}\cdot\nabla\rho + \nabla\cdot\mathbf{u} = 0, \\
&\mathbf{U}\cdot\nabla\mathbf{u} + \nabla p + \chi\nabla T - \zeta\nabla^2\mathbf{u} - \eta\nabla\nabla\cdot\mathbf{u} \\
&\quad = \mathbf{n}\cdot(p\mathbf{1} - \mathbf{P})\ \delta(S), \\
&\mathbf{U}\cdot\nabla T + \chi\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{Q} \\
&\quad = \mathbf{n}\cdot\mathbf{Q}\ \delta(S) - \chi\mathbf{U}\cdot(p\mathbf{n} - \mathbf{P}\cdot\mathbf{n})\ \delta(S), \\
&\mathbf{Q} = -\xi\nabla T + \xi\mathbf{n}T\ \delta(S),
\end{aligned} \tag{2.9}$$

where

$$\eta = (\beta + \tfrac{1}{3}\mu)/(\rho_0 a_0 L), \qquad \zeta = \mu/(a_0\rho_0 L),$$

$$\xi = \kappa/(\rho_0 c_v L a_0), \qquad \chi = (\gamma - 1)^{\frac{1}{2}}. \tag{2.10}$$

Here use has been made of

$$\gamma = \frac{c_p}{c_v} = \frac{c_0^2}{a_0^2},$$

which may be proved after some manipulation.

We remark again that our choice for the internal flow has changed the jumps into perturbed quantities evaluated at the surface. The source terms can immediately be recognized as the stress, heat conduction, and temperature at the body surface. In many instances one is only interested in obtaining these quantities and not necessarily in the entire flow field. We shall present a method which accomplishes this.

3. REDUCED PROBLEMS

We consider the problem of solving the system (2.9) for the typical external boundary-value problem. In short, uniform flow $\mathbf{U}$, past a closed body located at the origin is considered. Representing the dependent variables in Eq. (2.9) by $\boldsymbol{\omega}$, the Oseen operator by $\mathbf{L}$, and the source terms by $\mathbf{F}$, we may write Eq. (2.9) symbolically as

$$\mathbf{L}\boldsymbol{\omega} = \mathbf{F}\ \delta(S). \tag{3.1}$$

Associated with Eq. (3.1) is the equation for the fundamental matrix $\boldsymbol{\Omega}$:

$$\mathbf{L}\boldsymbol{\Omega}(\mathbf{x}) = \mathbf{1}\ \delta(\mathbf{x}), \tag{3.2}$$

where $\mathbf{1}$ represents the unit matrix. Then clearly

$$\int_S \boldsymbol{\Omega}(\mathbf{x} - \mathbf{y})\cdot\mathbf{F}(\mathbf{y})\ dS(\mathbf{y}) = \boldsymbol{\omega}(\mathbf{x}) \tag{3.3}$$

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satisfies Eq. (3.1). Assuming that $\mathbf{\Omega}$ is known, the evaluation of Eq. (3.3) at the boundary leads us to an integral equation in $\mathbf{F}$. The solution to this integral equation then furnishes the forces and heat transfer at the body. This is accomplished without first determining the flow field [which may then be obtained from Eq. (3.3)].

A. Thin Bodies

For thin bodies a simpler formulation results. Taking the length scale in Eq. (2.8) to be a characteristic length of the body, we represent the body surface by

$$z^{\pm} = \epsilon f^{\pm}(x, y). \tag{3.4}$$

Further, we denote the characteristic function of the surface projection on the $z = 0$ plane by $\theta(x, y)$, i.e., $\theta(x, y) = 1$ if (x, y) belongs to Eq. (3.4) for some z, and is zero otherwise. Turning to the definition of $\delta(S)$, Eq. (2.6), we find

$$\delta(S) = \iint_{-\infty}^{\infty} \theta(x', y')\, \delta(x - x')\, \delta(y - y') \cdot [\delta(z - \epsilon f^{+}(x', y')) + \delta(z - \epsilon f^{-}(x', y'))]\, dS' = \theta(x, y)\{\delta(z - \epsilon f^{+}) + \delta(z - \epsilon f^{-})\} + O(\epsilon^2).$$

On expanding the delta function, the source term becomes

$$\mathbf{F}\,\delta(S) = \theta(x, y) \cdot \{\mathbf{F}^{+}\,\delta(z - \epsilon f^{+}) + \mathbf{F}^{-}\,\delta(z - \epsilon f^{-})\} + O(\epsilon^2) = \theta(x, y)\{(\mathbf{F}^{+} + \mathbf{F}^{-})\,\delta(z) - \epsilon\,\delta'(z) \cdot (\mathbf{F}^{+}f^{+} + \mathbf{F}^{-}f^{-})\} + O(\epsilon^2), \tag{3.5}$$

where $\delta'(z)$ is the derivative of the δ function. As is easily seen, the normal to the body surface is given by

$$\mathbf{n}^{\pm} = \pm\mathbf{e}_z \mp \epsilon\nabla f^{\pm} + O(\epsilon^2),$$

where $\mathbf{e}_z$ denotes the unit vector in the z direction. From Eq. (2.9) we see that the source is given by[13]

$$\begin{aligned}\mathbf{F}^{\pm} &= [0, \mathbf{n}^{\pm}\cdot(p^{\pm}\mathbf{1} - \mathbf{P}^{\pm}), \mathbf{n}^{\pm}\cdot(\mathbf{Q}^{\pm} - \mathbf{U}\cdot(p\mathbf{1} - \mathbf{P})), \mathbf{n}^{\pm}T^{\pm}] \\ &= [0, \pm\,\mathbf{e}_z\cdot(p^{\pm}\mathbf{1} - \mathbf{P}^{\pm}), \pm\,\mathbf{e}_z\cdot\mathbf{Q}^{\pm} \pm \mathbf{e}_z\cdot\mathbf{P}\cdot\mathbf{U}, \pm\,\mathbf{e}_zT^{\pm}] \\ &\quad + \epsilon[0, \mp\,(p^{\pm}\mathbf{1} - \mathbf{P}^{\pm})\cdot\nabla f^{\pm}, \mp\,\mathbf{Q}^{\pm}\cdot\nabla f^{\pm} \pm \mathbf{U}\cdot(p\mathbf{1} - \mathbf{P})\cdot\nabla f^{\pm}, \mp\,T^{\pm}\nabla f^{\pm}] + O(\epsilon^2) \\ &= \mathbf{F}_{\pm}^{0} + \epsilon\mathbf{F}_{\pm}^{1} + O(\epsilon^2).\end{aligned}$$

Introducing this into Eq. (3.5), we obtain

$$\mathbf{F}\,\delta(S) = \theta(x, y)\{(\mathbf{F}_{+}^{0} + \mathbf{F}_{-}^{0})\,\delta(z) + \epsilon(\mathbf{F}_{+}^{1} + \mathbf{F}_{-}^{1})\,\delta(z) - \epsilon(\mathbf{F}_{+}^{0}f^{+} + \mathbf{F}_{-}^{0}f^{-})\,\delta'(z)\} + O(\epsilon^2).$$

We write[14]

$$(\mathbf{F}_{+}^{0} + \mathbf{F}_{-}^{0}) = [0, \mathbf{e}_z\cdot[p\mathbf{1} - \mathbf{P}], \mathbf{e}_z\cdot[\mathbf{Q}] + \mathbf{e}_z\cdot[\mathbf{P}]\cdot\mathbf{U}, 0] + \epsilon\left[0, 0, 0, \frac{[T]}{\epsilon}\,\mathbf{e}_z\right] = \mathbf{M}^0 + \epsilon\mathbf{m}.$$

By considering heat flow across the slender body it is clear that except under unusual conditions $[T] = O(\epsilon)$, and $\mathbf{m}$ is, therefore, appropriately grouped with the $O(\epsilon)$ terms. Setting

$$\mathbf{M}^1 = \mathbf{m} + (\mathbf{F}_{+}^{1} + \mathbf{F}_{-}^{1}),$$

$$\mathbf{N}^1 = -(\mathbf{F}_{+}^{0}f^{+} + \mathbf{F}_{-}^{0}f^{-}),$$

Eq. (3.1) may be written

$$\mathbf{L}\boldsymbol{\omega} = \mathbf{M}^0\theta\,\delta(z) + \epsilon\mathbf{M}^1\theta\,\delta(z) + \epsilon\mathbf{N}^1\theta\,\delta'(z) + O(\epsilon^2)$$

and

$$\boldsymbol{\omega} = \int \mathbf{\Omega}(x-x', y-y', z)\cdot\mathbf{M}^0(x', y')\theta(x', y')\,dx'\,dy' + \epsilon\int \mathbf{\Omega}(x-x', y-y', z)\cdot\mathbf{M}^1(x', y')\theta(x', y')\,dx'\,dy' + \epsilon\int \frac{\partial\mathbf{\Omega}}{\partial z}(x-x', y-y', z)\mathbf{N}^1(x', y')\theta(x', y')dx'\,dy'.$$

It should be noted that the lowest-order solution (i.e., zero order in ϵ) is exactly the case of flow past a flat plate with $\theta(x, y)$ characteristic function. (If the thermal boundary condition is given in terms of temperature, the plate may be taken at the temperature of either the upper or lower surfaces or, more symmetrically, as the arithmetic mean of the temperatures at both surfaces.)

Since we will primarily be interested in two-dimensional flows, we briefly note the alterations which are to be made in this case. The body surface is

[13] In the interest of conserving space we display vectors in their transposed form, and we allow the entries of vectors to also be vectors.

[14] The jump operator is now used to give the jump across the upper and lower surfaces.

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given by

$$z^{\pm} = \epsilon f^{\pm}(x),$$

the characteristic function by [H(x) denotes the Heaviside function]

$$\theta(x) = \mathrm{H}(x - \tfrac{1}{2})\mathrm{H}(\tfrac{1}{2} - x),$$

and the solution by

$$\boldsymbol{\omega} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\Omega}(x - x', z)[\mathbf{M}^0(x') + \epsilon \mathbf{M}^1(x')]\, dx' + \epsilon \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial \boldsymbol{\Omega}}{\partial z}(x - x', z)\mathbf{N}^1(x')\, dx'.$$

B. Far-Field Approximation

Denote a typical body diameter by $\mathcal{L}$ and let $R \gg \mathcal{L}$ denote a large distance from the interior of the body. To consider the flow field at large distances from a body we choose the length scale of (2.8) to be R. Consider the source term[15]

$$[\mathbf{F}]\ \delta(S) = \int_S \mathbf{F}(\mathbf{x}')\ \delta(\mathbf{x} - \mathbf{x}')\ dS(\mathbf{x}').$$

On formally expanding the δ function this becomes

$$\begin{aligned}[\mathbf{F}]\ \delta(S) &= \int_S \mathbf{F}(\mathbf{x}') \sum_{n=0}^{\infty} \frac{(-x_i'\nabla_i)^n}{n!}\ \delta(\mathbf{x})\ dS' \\ &= \int_S \mathbf{F}(\mathbf{x}')\ dS'\ \delta(\mathbf{x}) \\ &\quad - \int_S \mathbf{F}(\mathbf{x}')\mathbf{x}'\ dS \cdot \nabla\ \delta(\mathbf{x}) + \cdots. \end{aligned} \tag{3.6}$$

By our choice of normalization this is an expansion in $\mathcal{L}/R \ll 1$. To lowest order the source term is

$$\begin{aligned}\int_S [0, \mathbf{n}\cdot(p\mathbf{l}-\mathbf{P}), \mathbf{n}\cdot(\mathbf{P}-p\mathbf{l})\cdot\mathbf{U} + \mathbf{n}\cdot\mathbf{Q}, \mathbf{n}T]\ dS\ \delta(\mathbf{x}), \\ = \mathcal{G}_0\ \delta(\mathbf{x}) = [0, \mathfrak{F}, \mathcal{H}, \mathfrak{Q}]\ \delta(\mathbf{x}). \end{aligned} \tag{3.7}$$

The constants $\mathfrak{F}$ and $\mathcal{H}$ are the total force of the body on the fluid, and the net energy flux into the fluid from the body. The additional constant $\mathfrak{Q}$ has no familiar interpretation, but as we see later it acts as heat conduction from a body even for an insulating body. For this reason $\mathfrak{Q}$ will be referred to as virtual heat flow. It is clear that the higher-order terms in the expansion (3.6) involve the body shape and the distribution of the sources.

The above formal expansion is independent of the linearization, and hence (3.6) or to lowest order (3.7), can be introduced into the nonlinear equations (2.1)–(2.5). This demonstrates, formally at least, that the far field may to lowest order be regarded as arising from a point source at the origin of strength $\mathbf{G}_0$, given by Eq. (3.7). Since it is also plausible that the far field is governed by the Oseen equations, the far field is to lowest order given simply by

$$\boldsymbol{\omega} \sim \boldsymbol{\Omega}(x)\cdot\mathbf{G}_0. \tag{3.8}$$

For what follows it is convenient to eliminate the heat conduction relation in Eq. (2.9) by introducing it into the energy equation of that set. Instead of the operator $\mathbf{L}\boldsymbol{\omega}$ we are then lead to

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} \mathbf{U}\cdot\nabla & \nabla\cdot & 0 \\ \nabla & \mathbf{U}\cdot\nabla - \zeta\nabla^2 - \eta\nabla\nabla\cdot & \chi\nabla \\ 0 & \chi\nabla\cdot & \mathbf{U}\cdot\nabla - \xi\nabla^2 \end{bmatrix} \begin{bmatrix} \rho \\ \mathbf{u} \\ T \end{bmatrix}. \tag{3.9}$$

A new fundamental matrix $\mathbf{V}$ is now defined by

$$\mathbf{AV} = 1\ \delta(\mathbf{x}). \tag{3.10}$$

In terms of $\mathbf{V}$ the far-field solution $\mathbf{v}_f$, is given by

$$\mathbf{v} \sim \mathbf{v}_f = \mathbf{V}\begin{Bmatrix} 0 \\ \mathfrak{F} \\ \mathcal{H} \end{Bmatrix} + \mathfrak{Q}\cdot\nabla\mathbf{V}\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{3.11}$$

[which is equivalent to Eq. (3.8)], where the constants are defined as before by Eq. (3.7).

Disk flow $\mathbf{v}_D$ (i.e., the lowest-order solution of flow past a thin body) is now given by

$$\mathbf{v}_D = \iint_{-\infty}^{\infty} \theta(x', y')\mathbf{V}(x - x', y - y', z) \cdot \begin{Bmatrix} 0 \\ \mathbf{e}_z\cdot[p\mathbf{l} - \mathbf{P}] \\ \mathbf{e}_z\cdot[\mathbf{Q}] - \chi\mathbf{e}_z\cdot[p\mathbf{l} - \mathbf{P}]\cdot\mathbf{U} \end{Bmatrix} dx'\ dy'.$$

4. THE ASYMPTOTIC FUNDAMENTAL MATRIX FOR TWO-DIMENSIONAL FLOW

We hence forth restrict attention to two-dimensional flows. In order to solve Eq. (3.10) for the fundamental maxtrix $\mathbf{V}$, we introduce Fourier transforms. We then have

$$\mathbf{V} = \mathbf{A}^{-1}(i\mathbf{k}), \tag{4.1}$$

where

[15] The jump operator may be removed from the integrand since we chose equilibrium for the interior flow. See the closing paragraph of the previous section.

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$$\mathbf{A} = \begin{bmatrix} ik_1U & ik_1 & ik_2 & 0 \\ ik_1 & ik_1U + \zeta k^2 + \eta k_1^2 & \eta k_1 k_2 & i\chi k_1 \\ ik_2 & \eta k_1 k_2 & ik_1 U + \zeta k^2 + \eta k_2^2 & i\chi k_2 \\ 0 & i\chi k_1 & i\chi k_2 & ik_1 U + \xi k^2 \end{bmatrix}. \tag{4.2}$$

Inverting the transforms in Eq. (4.1) yields

$$\begin{aligned} \mathbf{V}(\mathbf{x}) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{A}^{-1}\,d\mathbf{k}, \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}}\frac{\mathbf{C}}{|A|}\,d\mathbf{k}, \end{aligned} \tag{4.3}$$

where $|A| = \det(A)$ and $\mathbf{C}$ is the classical adjoint of $\mathbf{A}$. It is clear from the symmetry of $\mathbf{A}$ that $\mathbf{C}$ is also symmetric.

Clearly $\mathbf{V}$ contains no other length scales except those based on the dissipative coefficients ζ, η, and ξ. A natural choice for the length scale L is one which makes these dimensionless parameters order unity. More specifically, we fix L by the condition that max $(\zeta, \eta, \xi) = 1$. For a simple gas L can be directly identified with the mean free path. This designation will also be used for nonsimple gases. In any case L, chosen in this manner, is quite small, e.g., for air $L \sim 10^{-5}$ cm.

As already discussed in the introduction, we impose the hydrodynamical limit (1.1) in our analysis. This requires that scale variations be smooth compared with the "mean free path" L. In other words we consider $\mathbf{V}$ for $|k| \ll 1$, or in configuration space we take $|x| \gg 1$. In this limit a mathematically rigorous asymptotic analysis may be performed. To carry this out one considers Eq. (4.3) separately for $-x$, x, y, $-y \gg 1$. A theorem proven elsewhere[16] permits us to replace $k_1(k_2)$ by its power series to $O(k_2^2)$. The resulting integrals are then readily evaluated.

The above method provides the justification for another approach. This alternate method, which will be employed below, is rooted in boundary-layer analysis. We start from a knowledge of the discontinuities of the nondissipative solution. These discontinuity lines are obtained by first forming, $\mathbf{A}_0 = \mathbf{A}(\eta = \zeta = \xi = 0)$ in Eq. (4.2) and then considering

$$\det |A_0| = 0.$$

This is homogeneous of degree four in k_1 and k_2. The inclination of the discontinuities are then given by

$$\theta = \tan^{-1}\left(-\frac{k_1}{k_2}\right), \tag{4.4}$$

where (k_1/k_2) are the roots. Measuring θ from the positive x axis $(-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi)$ we have[17]

$$\theta = 0,\, 0,\, \tan^{-1}\frac{1}{(M^2-1)^{\frac{1}{2}}},\, -\tan^{-1}\frac{1}{(M^2-1)^{\frac{1}{2}}}.$$

The presence of discontinuities along the lines given by Eq. (4.4), implies that the higher-order dissipative operators become important in the neighborhood of the discontinuity. A boundary-layer analysis is, therefore, required. In regions removed from the lines given by Eq. (4.4) the nondissipative solution is regarded as correct. In this view only the wake in subsonic flow and the wake and Mach lines in supersonic flow require a boundary-layer analysis. These as we shall see, become structured.[18]

Denoting a coordinate along a discontinuity by h and across a discontinuity by t, we have

$$\begin{bmatrix} t \\ h \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \tag{4.5}$$

where θ, the Mach angle, is given by Eq. (4.4). The boundary-layer condition is that

$$\frac{\partial}{\partial t} \gg \frac{\partial}{\partial h} \tag{4.6}$$

along a Mach line. For our purposes it is more convenient to carry out the boundary-layer analysis in terms of wavenumbers. Then denoting the wavenumbers corresponding to t and h, by m and n, respectively, we have

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}. \tag{4.7}$$

Then Eq. (4.6) implies

$$|m| \gg |n|.$$

Next, since we are interested in $|\mathbf{k}| \ll 1$, we also have

[16] See Appendix 2 of Ref. 8.

[17] Note that the value $\theta = 0$ is a double root. These are not generally exhibited in inviscid theory since irrotational, isentropic flow is assumed at the outset.

[18] Each of the four structured regions can be traced to a root of det $|A| = 0$. The remaining three roots are "roots at infinity" and lead to exponentially small contributions.

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$$1 \gg |m| \gg |n|. \quad (4.8)$$

This relation is the basis for the following analysis.

Wake. In the region of the wake $\theta = 0$, and we can retain k_1 and k_2 instead of m and n.

Using Eq. (4.8), one finds that to lowest order

$$|A| \sim (i\gamma U k_1 + \xi k_2^2)(ik_1 U + \zeta k_2^2)k_2^2.$$

This immediately provides the information that in the neighborhood of the wake $k_1 = O(k_2^2)$ [and also that the wake thickness is $O(x^{\frac{1}{2}})$]. Using this we find to lowest order

$$C_{ij} \sim k_2^2(iUk_1 + \zeta k_2^2) \cdot[\chi^2 \delta_{i1} \delta_{j1} - \chi(\delta_{i4} \delta_{j1} + \delta_{i1} \delta_{j4}) + \delta_{i4} \delta_{j4}] + k_2^2(i\gamma U k_1 + \xi k_2^2) \delta_{i2} \delta_{j2}. \quad (4.9)$$

The asymptotic solution of $\mathbf{V}$ in the neighborhood of the wake is now easily obtained. After elementary integrations

$$\mathbf{V} \sim \mathbf{V}^1 + \mathbf{V}^2 = I^1\boldsymbol{\omega}^1\boldsymbol{\omega}^1 + I^2\boldsymbol{\omega}^2\boldsymbol{\omega}^2 \quad (4.10)$$

with

$$I^1 = \frac{H(x) \exp(-y^2\gamma U/4x\xi)}{(4\pi\xi\gamma Ux)^{\frac{1}{2}}}, \qquad \omega_i^1 = \chi\, \delta_{i1} - \delta_{i4}, \quad (4.11)$$

$$I^2 = \frac{H(x) \exp(-y^2 U/4x\zeta)}{(4U\zeta\pi x)^{\frac{1}{2}}}, \qquad \omega_i^2 = \delta_{i2}. \quad (4.12)$$

This is seen to exhibit two different wake structures, one based on viscosity ζ, ($\mathbf{V}_2$), and the other on heat conductivity ξ, ($\mathbf{V}_1$). This shows that the heat conduction and viscous wakes decouple completely to lowest order for an arbitrary fluid.

Mach layers. In this case $\theta = \pm\tan^{-1}1/(M^2-1)^{\frac{1}{2}}$. Then substituting Eq. (4.8) into Eq. (4.2) we find to lowest order

$$|A| \sim U^2\gamma(-2m^3n \sin\theta \cos\theta) + iUm^5 \sin\theta[\zeta + \eta + (\gamma - 1)\xi] \cdot\{-U^2\gamma \sin^2\theta n^2 m^2\}. \quad (4.13)$$

We note that the term in the curly bracket is of higher order unless $|\theta| \sim \frac{1}{2}\pi$. The evaluation of the determinant is, therefore, valid through transonic speeds. On the other hand, in the hypersonic limit, $\theta \to 0$, each term of Eq. (4.13) vanishes and neglected terms are of lower order. We will not consider the hypersonic limit but will obtain the solution valid through transonic speeds. We first consider the strictly supersonic case ($|\theta|$ strictly bounded away from 0 and $\frac{1}{2}\pi$).

Strictly supersonic flow. The lowest-order determinant in this case is

$$|A| \sim -2m^3nU^2\gamma \sin\theta\cos\theta + iUm^5[\eta + \zeta + (\gamma - 1)\xi] \sin\theta. \quad (4.14)$$

It is clear as before that $n = O(m^2)$ [also that the thickness of the Mach layer is $O(h)^{\frac{1}{2}}$]. Introducing this into $\mathbf{C}$, we find, to the lowest order,

$$C_{ij}(\theta) \sim \frac{im^3}{U \sin\theta}\omega_i\omega_j, \quad (4.15)$$

with

$$\omega_i = [U\sin\theta, \gamma\sin\theta, \gamma\cos\theta, \chi U\sin\theta]. \quad (4.16)$$

The fundamental matrix in the neighborhood of a Mach line is, therefore, given by

$$\mathbf{V} \sim I\omega_i\omega_j, \quad (4.17)$$

where

$$I = \frac{i}{(2\pi)^2 U\sin\theta}\iint_{-\infty}^{\infty} \frac{e^{imt+inh}\, dm\, dn}{-2nU^2\gamma\sin\theta\cos\theta + iUm^2\{\eta + \zeta + (\gamma-1)\xi\}\sin\theta}. \quad (4.18)$$

The integral I is easily evaluated, and we find

$$I = \frac{H(x\cos\theta + y\sin\theta)\exp[-(x\sin\theta - y\cos\theta)^2/4\pi\Theta(x + y\tan\theta)]}{U\sin\theta[U^2\gamma\sin 2\theta][4\pi\Theta(x + y\tan\theta)]^{\frac{1}{2}}}, \quad (4.19)$$

where we have substituted for t and h from (4.5), and defined

$$\Theta = \frac{[\eta + \zeta + (\gamma - 1)\xi]}{2U\gamma}. \quad (4.20)$$

The above asymptotic form (4.19) may be further reduced. In fact, it is clear that Eq. (4.19) is small outside of the region determined by

$$y = x\tan\theta + O(x^{\frac{1}{2}}).$$

If this is employed, then to lowest order

$$I = \frac{H(x)\exp\{-(x\sin\theta - y\cos\theta)^2/[4\Theta x(1 + \tan^2\theta)]\}}{\gamma U^3\sin\theta\sin 2\theta[4\pi\Theta(1 + \tan^2\theta)]^{\frac{1}{2}}} + O\left(\frac{1}{x}\right). \quad (4.21)$$

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A careful examination shows that already neglected terms are $O(1/x)$, and in fact the rigorous analysis outlined at the beginning of this section furnishes this exponential term as the leading term of the expansion, and the error as given in Eq. (4.21).

Denoting the lowest-order contributions to $\mathbf{V}[\theta = \pm \tan^{-1}(M^2 - 1)^{-\frac{1}{2}}]$ by $\mathbf{V}^{3,4}$, we may write

$$\mathbf{V}^{3,4} = I^{3,4}\boldsymbol{\omega}^{3,4}\boldsymbol{\omega}^{3,4}, \tag{4.22}$$

where

$$I^{3,4} = \frac{M^2H(x)\exp\{-[y \mp x/(M^2-1)^{\frac{1}{2}}]^2/\alpha x\}}{2\gamma^2U(M^2-1)(\pi\alpha x)^{\frac{1}{2}}}, \tag{4.23}$$

$$\boldsymbol{\omega}^{3,4} = \left[\pm\gamma^{\frac{1}{2}}, \mp\frac{\gamma}{M}, \frac{\gamma(M^2-1)^{\frac{1}{2}}}{M}, \pm\chi\gamma^{\frac{1}{2}}\right], \tag{4.24}$$

$$\alpha = \frac{2M^3[\eta + (\gamma-1)\xi + \zeta]}{(M^2-1)^2\gamma^{\frac{1}{2}}}. \tag{4.25}$$

Then denoting the lowest order of $\mathbf{V}$ by $\mathbf{V}^{NS}$ we may write

$$\mathbf{V}^{NS} = \sum_{i=1}^{4}\mathbf{V}^i \tag{4.26}$$

since the Mach layers and wake layers contribute negligibly in each others dominant regions.

A. Non-Dissipative Limit

If we set $\zeta = \eta = \xi = 0$ in (4.2), the nondissipative theory is obtained. Denoting the nondissipative fundamental matrix by $\mathbf{W}$, the evaluation of Eq. (4.3) (for the supersonic case) yields

$$W_{ij} = \frac{H(x)\,\delta(y)}{U}\left[\frac{\omega_i^1\omega_j^1}{\gamma} + \omega_i^2\omega_j^2\right] + \frac{H(x)M^2}{2\gamma^2U(M^2-1)}\left[\omega_i^3\omega_j^3\,\delta\left(y - \frac{x}{(M^2-1)^{\frac{1}{2}}}\right) + \omega_i^4\omega_j^4\,\delta\left(y + \frac{x}{(M^2-1)^{\frac{1}{2}}}\right)\right], \tag{4.27}$$

where $\boldsymbol{\omega}^i$ are the same as defined by Eqs. (4.11), (4.12), and (4.24). It is clear that in the application of $\mathbf{W}$ to the external flow problem, the mass, drag, and energy sources vanish. For this reason the first term in Eq. (4.27) does not contribute. The remaining portion takes the form usually found in linearized inviscid flow.

Regarding V^{NS}, (4.26), and taking the nondissipative limit $\xi, \eta, \zeta \to 0$,[19] we see that

$$\lim_{\zeta\eta\xi\to}\mathbf{V}^{NS} = \mathbf{W} \tag{4.28}$$

for fixed (x, y). [We use the well-known representation of the delta function $\lim_{\epsilon\to 0} e^{-x^2/\epsilon}/(\epsilon\pi)^{\frac{1}{2}} = \delta(x)$].

A study of the limit taken in obtaining (4.28) indicates that it is not uniform in distance from the origin. Irrespective of how small the dissipative parameters may be, for large distances the exponentials in (4.26) are not approximated by delta functions. We must conclude that the far field is a dissipative regime.

B. Transonic Mach Layers

We first note that for the sonic case, Eq. (4.13) states that $n^2 = O(m^3)$, and hence the Mach wave is considerably thickened, now being $O(h^{\frac{1}{3}})$. Therefore, n^2 varies from $O(m^3)$–$O(m^4)$ in passing through sonic speeds to supersonic speeds. A simple analysis then shows that $C_{ij}(\theta)$ to lowest order is still given by Eq. (4.15) as $|\theta| \to \frac{1}{2}\pi$. (An immediate effect of this is that u_2 vanishes to lowest order as $|\theta| \to \frac{1}{2}\pi$.) The fundamental matrix for transonic as well as supersonic regimes is given by Eq. (4.17) where now

$$I = -\frac{i}{(2\pi)^2\gamma^{5/2}}\int\frac{e^{imt+inh}m\,dm\,dn}{n^2 + 2mn\cot\theta - 2im^3\Theta\cos\theta} \tag{4.29}$$

instead of Eq. (4.18) [the latter is a consequence (4.29) when $|\theta| < \frac{1}{2}\pi$]. To evaluate Eq. (4.29), we set

$$n = \frac{4\nu}{\Theta}, \quad h = \frac{\Theta}{4}h', \quad m = \frac{2\mu}{\Theta}, \quad t = \frac{\Theta}{2}t' \tag{4.30}$$

from which

$$I = -\frac{i}{(2\pi)^2\Theta\gamma^{5/2}}\int\frac{e^{i\mu t'+i\nu h'}\mu\,d\mu\,d\nu}{\nu^2 + \mu\nu\cot\theta - i\mu^3\csc\theta}.$$

The roots of the denominator in the integrand are

$$\nu\pm = -\frac{\mu[\cos\theta \pm (4i\mu\sin\theta + \cos^2\theta)^{\frac{1}{2}}]}{2\sin\theta}.$$

The complex μ plane, therefore, has a branch point

$$\mu_0 = \frac{i\cos^2\theta}{4\sin\theta}.$$

Restricting attention to $\theta > 0$, μ_0 lies on the positive imaginary axis. Placing a branch cut from μ_0 to $+i\infty$, we easily see that ν^- and ν^+ lie in the upper-

[19] To be consistent we should first remove the normalization which set ζ, ξ, η to $O(1)$. The end effect is, however, the same.

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and lower-half planes, respectively, for real μ. Closing the path of integration appropriately we obtain

$$I = I^3 - I^- = \frac{\sin\theta H(h')}{\Theta\gamma^{5/2}(2\pi)}\int\frac{e^{i\mu t'+i\nu^- h'}\,d\mu}{(4i\mu\sin\theta+\cos^2\theta)^{1/2}} - \frac{\sin\theta H(-h')}{\Theta\gamma^{5/2}(2\pi)}\int\frac{e^{i\mu t'+i\nu^+ h'}\,d\mu}{(4i\mu\sin\theta+\cos^2\theta)^{1/2}}.$$

The second integral I^-, (which is exponentially small except for $\theta \sim \frac{1}{2}\pi$) is due to the extraneous root ν^+ introduced by the boundary-layer analysis and we neglect it.

We introduce the following transformation of variables in I^3:

$$(\cos^2\theta + 4\mu i\sin\theta)^{\frac{1}{2}} = \frac{2\sin^{\frac{2}{3}}\theta}{h'^{\frac{1}{3}}}\left[\frac{s}{3^{\frac{1}{3}}} - \frac{1}{3}\left(\frac{t'\sin^{\frac{1}{3}}\theta}{h^{\frac{2}{3}}} - \frac{h^{\frac{1}{3}}\cos\theta}{2\sin^{\frac{4}{3}}\theta}\right)\right]$$

from which we obtain

$$I^3 = \frac{H(h)A_i(X)\sin^{2/3}\theta}{\Theta\gamma^{5/2}h'^{1/3}3^{1/3}} \cdot\exp\left(-\frac{2t'^3\sin\theta}{27h'^2} - \frac{1}{9}\frac{t'^2\cos\theta}{h'} - \frac{1}{9}\frac{t'\cos^2\theta}{\sin\theta} + \frac{2}{27}\frac{h'\cos^3\theta}{\sin^2\theta}\right),$$

where

$$X = \frac{1}{3^{4/3}}\left(\frac{t'^2\sin^{2/3}\theta}{h'^{4/3}} - \frac{t'\cos\theta}{h'^{1/3}\sin^{1/3}\theta} + \frac{h'^{2/3}\cos^2\theta}{\sin^{4/3}\theta}\right)$$

and

$$A_i(x) = \int_C \exp\left[(\tfrac{1}{3}s^3) - xs\right]ds$$

is the Airy integral[20] with the path of integration indicated in Fig. 1.

To remove the earlier normalization (4.30) it is convenient to define new variables

$$\tau = \frac{t}{2h}, \qquad \sigma = \frac{2h\sin\theta}{9\theta}.$$

We then have

$$I^3 = \frac{H(h)A_i(X)\sin\theta}{3\theta\gamma^{5/2}\sigma^{1/3}} \cdot\exp\left[-\sigma(\tfrac{2}{3}\tau^3 + \tau^3\cot\theta + \tau\cot^2\theta - \tfrac{2}{3}\cot^3\theta)\right] \tag{4.31}$$

with

$$X = \sigma^{2/3}(\tau^2 - \tau\cot\theta + \cot^2\theta). \tag{4.32}$$

The expression (4.31) applies to the Mach layer centered at $\theta = \tan^{-1}(M^2+1)^{-\frac{1}{2}}$. We denote the corresponding integral for $\theta = -\tan^{-1}(M^2+1)^{-\frac{1}{2}}$ by I^4, then from an inspection of Eq. (4.29) we easily see that

$$I^4 = -I^3(-\theta). \tag{4.33}$$

The properties of A_i are well known,[20] and for our purposes we need

$$A_i(x) = \frac{1}{3^{\frac{2}{3}}(-\frac{1}{3})!}\left(1 + \frac{1}{3!}x^3 + \cdots\right) - \frac{1}{3^{\frac{1}{3}}(-\frac{2}{3})!}\left(x + \frac{2}{4!}x^4 + \cdots\right) \tag{4.34}$$

$$A_i(x) \sim \frac{\exp(-\frac{2}{3}x^{\frac{3}{2}})}{2\pi^{\frac{1}{2}}x^{\frac{1}{4}}}\left(1 - \frac{15}{216}\frac{3}{2x^{\frac{3}{2}}}\cdots\right). \tag{4.35}$$

If we take $|\theta| < \frac{1}{2}\pi$, we may apply the latter in Eqs. (4.31) and (4.33) to recover Eq. (4.22).

Using Eqs. (4.31) and (4.33) in Eq. (4.26) instead of Eq. (4.22) gives us the flow field uniformly from supersonic through sonic regimes.

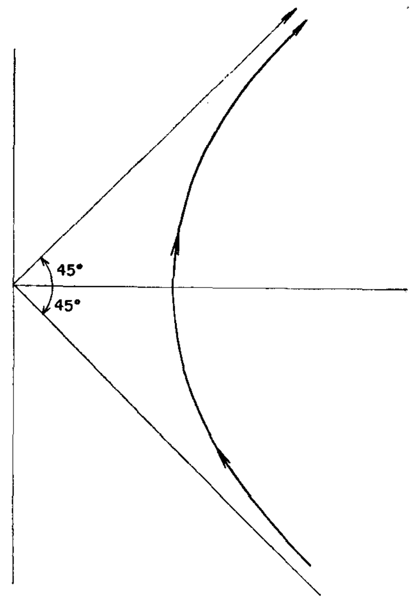

FIG. 1. Path of integration for $A_i(x)$.

5. STRUCTURE OF THE FAR FIELD

We may conveniently discuss the results of the previous section by applying them to the far field. The asymptotic solution for the far field has already been discussed in Sec. 3. In the notation of Sec. 4 we may now write, for Eq. (3.11),

$$\mathbf{v}_f \sim \sum_{\mu=1}^{4}\boldsymbol{\omega}^{\mu}\boldsymbol{\omega}^{\mu}\cdot\begin{bmatrix}0\\ \mathfrak{F}\\ \mathcal{H}\end{bmatrix}I^{\mu} + \sum_{\mu=1}^{4}\boldsymbol{\omega}^{\mu}\boldsymbol{\omega}^{\mu}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}\mathfrak{Q}\cdot\nabla I^{\mu}, \tag{5.1}$$

[20] J. C. P. Miller, *The Airy Integral* (Cambridge University Press, London, 1946), Math. Tables Pt. B.

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where the $\boldsymbol{\omega}^\mu$ and I^μ are given by Eqs. (4.11), (4.12), (4.24), (4.31), and (4.33). The fundamental solution (5.1) applies only to transonic and supersonic flows. [For subsonic flows the portions of Eq. (5.1) for $\mu = 3, 4$ combine and leads to terms involving $\ln [x^2/(1 - M^2)^{\frac{1}{2}} + y^2]$. (We do not consider this case, although it is straightforward.)

A number of properties of the solution follow directly from the vectors $\boldsymbol{\omega}^\mu$. We first point out that, in our normalization, the condition for constant pressure is

$$\rho + \chi T = 0 \tag{5.2}$$

and the condition for constant entropy is[21]

$$T - \chi\rho = 0. \tag{5.3}$$

From this we see that the Mach layers $\mu = 3, 4$ are isentropic. This, of course, is due to the fact that these are weak shocks. On the other hand, $\mu = 1$ is an entropy wake and has a constant pressure. The last layer, $\mu = 2$, is isentropic, isobaric, and carries only a velocity defect (this is a vorticity wake) in the direction of flow. We point out again that the $\mu = 1$ and $\mu = 2$ wakes are structured by heat conductivity and viscosity, respectively. Although this decomposition appears to be very natural, it is remarkable mathematically that these two coefficients are not intertwined. In fact, in the comparable magnetogasdynamic case,[22] these dissipative coefficients become intertwined in a complicated manner.

We note that the heat conduction wake appears even in the absence of heat conduction from the body.[23] This is due to two effects. There is the virtual heat conduction $\mathfrak{Q}$ which is actually a higher-order term since it involves a derivative. However, under high-speed conditions $\mathfrak{Q}(= \int \mathbf{n}T\, dS)$ itself may be large and this term has been retained. The other contributing factor to the heat conduction wake is $\boldsymbol{\mathfrak{F}}\cdot\mathbf{U}$ in $\mathfrak{H}$. This is the rate of work done on the fluid and hence is an entropy source.

To examine the far-field solution for simplicity we consider only u_1, i.e., the perturbation velocity in the direction of upstream flow. This does not enter into the heat conduction wake and hence what will be seen below will only include the $\mu = 2, 3, 4$ modes. Also for simplicity we drop virtual heat conduction $\mathfrak{Q}$.

For strictly supersonic flow ($|\theta| < \frac{1}{2}\pi$),

$$u_1 \sim \frac{\mathfrak{F}_1 H(x)}{(4\pi U\zeta x)^{\frac{1}{2}}} \exp\left(-\frac{y^2\gamma^{\frac{1}{2}}}{4x\zeta}\right) + \frac{H(x)[\gamma\mathfrak{F}_1 - \gamma(M^2-1)^{\frac{1}{2}}\mathfrak{F}_2 - \chi\gamma^{\frac{1}{2}}M\mathfrak{H})}{2\gamma^{\frac{3}{2}}M(M^2-1)(\pi\alpha x)^{\frac{1}{2}}} \cdot \exp - \left(\frac{[y - x/(M^2-1)^{\frac{1}{2}}]^2}{\alpha x}\right) + \frac{H(x)(\gamma\mathfrak{F}_1 + \gamma(M^2-1)^{\frac{1}{2}}\mathfrak{F}_2 - \chi\gamma^{\frac{1}{2}}M\mathfrak{H})}{2\gamma^{\frac{3}{2}}M(M^2-1)(\pi\alpha x)^{\frac{1}{2}}} \cdot \exp - \left(\frac{[y + x/(M^2-1)^{\frac{1}{2}}]^2}{\alpha x}\right).$$

Note that asymmetry is only due to $\mathfrak{F}_2$.

Because of the complexity of Eq. (4.31) we consider only the just sonic regime, $|\theta| = \frac{1}{2}\pi$. Then after some simplification we find

$$u_1 \sim \frac{\mathfrak{F}_1 H(x)}{(4\pi U\zeta x)^{\frac{1}{2}}} \exp\left(-\frac{y^2\gamma^{\frac{1}{2}}}{4x\zeta}\right) + \left\{\mathfrak{F}_1 - \frac{\chi\mathfrak{H}}{\gamma^{\frac{1}{2}}}\right\} \frac{\exp(-\frac{2}{3}s^2)A_i(s^2)}{\gamma^{\frac{1}{2}}[12\,|y|\,\Theta^2]^{\frac{1}{3}}} \tag{5.4}$$

with

$$s^3 = \frac{x^3}{18\Theta\,|y|^2}.$$

The $\mu = 3$ and $\mu = 4$ modes have been combined to give the second term of Eq. (5.4). However, this portion of the solution may only be applied for large $|y|$. The product $\exp(-\frac{2}{3}s^2)A_i(s^2)$ is plotted in Fig. 2. It will be noted that this shows a strong upstream influence. To see this, observe that from Eq. (4.35), for large s^2

$$A_i(s^2) \sim \frac{\exp(-2\,|s|^3/3)}{2\pi^{\frac{1}{2}}\,|s|^{\frac{1}{2}}}.$$

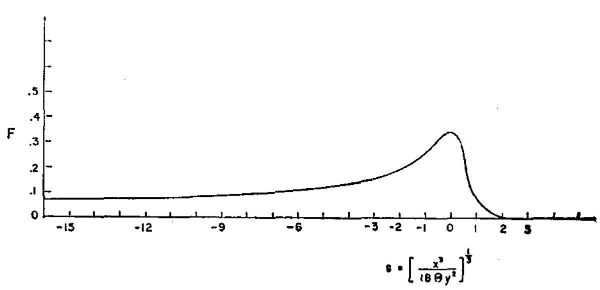

Fig. 2. Plot of $F(s) = e^{-\frac{2}{3}s^3}A_i(s)$.

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[21] These conditions can be verified directly for an ideal gas, but require some manipulation of the thermodynamic relations in the general case.

[22] E. P. Salathé and L. Sirovich, J. Fluid Mech. (to be published).

[23] Since we have restricted attention to impermeable bodies, no mass addition appears in the source terms. As is easily seen, these would also contribute to a heat conduction wake.

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(This is already within 10% of the correct value when $s^2 \sim 1$.) Therefore, the downstream portion falls off exponentially while the upstream falls off only algebraically. Qualitatively, the upstream flow resembles subsonic flow, while the downstream part is more nearly like supersonic flow. The functional forms are, however, quite different from the comparison flows.

This can be illustrated by considering the thickness of the Mach layer in going from supersonic to subsonic flows. Using the local coordinates (4.5) the similarity variable governing the Mach layer in supersonic flow is t^2/h; for sonic flow this is t^3/h^2. The transition from one to the other is not simple as can be seen from Eqs. (4.31) and (4.32). If we let λ be a quantity representative of the mean free path, and denote dimensional variables by circumflexes, the Mach layer thickness in each regime is

$$\hat{t}_{\text{supersonic}} \approx (\lambda \hat{h})^{\frac{1}{2}},$$

$$\hat{t}_{\text{sonic}} \approx (\hat{h}^2 \lambda)^{\frac{1}{3}}.$$

For application to the far field, $\hat{h} \gg \mathcal{L}$, where $\mathcal{L}$ is typical of the body size. The Mach layer thickness for sonic flow is much thicker than for the supersonic case. This is further fortified by the strong upstream influence already noted for sonic flow.

As a sidelight to these remarks we observe that the differential operator governing the supersonic-transonic Mach layer is, from Eq. (4.13),

$$L = \cos\theta \frac{\partial^2}{\partial t\, \partial h} + \frac{\sin\theta}{2} \frac{\partial^2}{\partial h^2} - \Theta \frac{\partial^3}{\partial t^3}. \qquad (5.5)$$

In the purely supersonic regime,

$$L \sim \frac{\partial}{\partial t}\left(\cos\theta \frac{\partial}{\partial h} - \Theta \frac{\partial^2}{\partial t^2}\right)$$

and hence this regime is governed by the diffusion operator, while in the near sonic regime

$$L \sim \frac{1}{2} \frac{\partial^2}{\partial h^2} - \Theta \frac{\partial^3}{\partial t^3}. \qquad (5.6)$$

(In the latter, equality holds at just sonic conditions.) The operators in Eqs. (5.5) and (5.6) are of no standard type. Their fundamental solutions are given by Eqs. (4.31) and (5.3), respectively.

A point of some interest is the inviscid limit at sonic conditions. The sonic inviscid solution for u_1 is given by

$$u_1 = \frac{\mathfrak{F}_1}{U} H(x)\, \delta(y) - \frac{1}{U}\left[\mathfrak{F}_1 - \frac{\chi \mathcal{K}}{\gamma^{\frac{1}{2}}}\right] \frac{\delta'(x)\, |y|}{2}. \quad (5.7)$$

As is well known, this solution leads to an infinite drag.[24] In addition it is clear that the appearance of the linear term $|y|$ must ultimately lead to large values for the perturbation flow and hence violate the basis for the linearization. This paradox has been resolved by the introduction of nonlinear terms.[24] However this begs the question, since at large distances from a body the flow perturbations should be small enough for linearization to be valid. If we return to Eq. (5.4) and take the inviscid limit (it is sufficient to formally allow $\zeta \to 0$ and $\Theta \to 0$), one recovers Eq. (5.7). Again as in the supersonic case the inviscid limit is not uniform. No matter how small the dissipative parameters the far field is dissipative in structure. Therefore, provided dissipation is retained, the flow field remains self-consistantly linear.

6. FLOW PAST A SEMI-INFINITE FLAT PLATE

A. Skin Friction

As an application we consider flow past a semi-infinite flat plate at zero angle of attack. As in Sec. 3 we use the square bracket to denote the jump in a quantity across the zero-thickness body. By symmetry we have

$$[T] = 0 = [p - p_{22}]. \qquad (6.1)$$

We shall further assume that the body is not heat conducting. The source term is, therefore, given by

$$[0, \mathfrak{F}, 0, -\chi \mathfrak{F} U].$$

The momentum source is directly related to the skin friction C_f; in fact,

$$\mathfrak{F} = -\gamma M^2 C_f. \qquad (6.2)$$

[If τ is the dimensional shear stress at the wall, then $C_f = \tau/(\frac{1}{2}\rho u_0^2)$.] In terms of $\mathfrak{F}$ the solution to the problem is given by

$$v_i = \omega_i^1 \int_0^\infty I^1(y, x - x') U \mathfrak{F}(x')\, dx' + \omega_i^2 \int_0^\infty I^2(y, x - x') \mathfrak{F}(x')\, dx' \cdot \left\{-\omega_i^3 \int_0^\infty I^3(y, x - x') \mathfrak{F}(x')\, dx' + \omega_i^4 \int_0^\infty I^4(y, x - x') \mathfrak{F}(x')\, dx'\right\} \cdot \frac{\gamma}{M} \{1 + M^2(\gamma^2 - 1)\}, \qquad (6.3)$$

[24] T. von Kármán, in *General Theory of High Speed Aerodynamics*, W. R. Sears, Ed. (Princeton University Press, Princeton, New Jersey, 1954), Sec. A, p. 7.

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where the ω^i are given by Eqs. (4.11), (4.12), and (4.24), $I^{(1)}$ and $I^{(2)}$ by (4.11) and (4.12) and for strictly supersonic flow $I^{(3)}$ and $I^{(4)}$ by (4.23). Introducing these into Eq. (6.3), the perturbation velocity in the direction of flow, for example, is given by

$$u_1 = \int_0^x \frac{\exp[-y^2U/4\zeta(x-x')]\mathfrak{F}(x')\,dx'}{[4\pi\zeta U(x-x')]^{\frac{1}{2}}} + \frac{1+M^2(\gamma-1)}{2U(M^2-1)} \cdot \left[\int_0^x \frac{\exp\{-[y-(x-x')/(M^2-1)^{\frac{1}{2}}]^2/\alpha(x-x')\}}{[\pi\alpha(x-x')]^{\frac{1}{2}}}\mathfrak{F}(x')\,dx' + \int_0^x \frac{\exp\{-[y+(x-x')/(M^2-1)^{\frac{1}{2}}]^2/\alpha(x-x')\}}{[\pi\alpha(x-x')]^{\frac{1}{2}}}\mathfrak{F}(x')\,dx'\right]. \quad (6.4)$$

In any case one may easily see that for Eq. (6.3) the boundary conditions

$$u_2(y=0) = \frac{\partial T}{\partial y}(y=0) = 0, \qquad x > 0$$

are automatically satisfied.

We restrict attention to purely supersonic flows. Imposing the remaining boundary condition on (6.4), we obtain

$$-U = \int_0^x \frac{\mathfrak{F}(x')\,dx'}{[4\pi\zeta(x-x')U]^{\frac{1}{2}}} + \frac{1+M^2(\gamma-1)}{U(M^2-1)} \cdot \int_0^x \frac{\exp[-x-x'/\alpha(M^2-1)]}{[\pi\alpha(x-x')]^{\frac{1}{2}}}\mathfrak{F}(x')\,dx'. \quad (6.5)$$

It is now convenient to take L such that $\zeta = 1$ (this, of course, results in no loss of generality). Also we define

$$K^2 = \frac{\alpha\gamma^{\frac{1}{2}}(M^2-1)^2}{M^3} \quad (6.6)$$

and

$$d = 1 + (\gamma-1)M^2,$$

$$r = 4d^2 - K^2M^4. \quad (6.7)$$

Equation (6.5) may be solved by transform techniques, and we leave out the details. The solution is

$$\mathfrak{F}(x) = 2U^{\frac{1}{2}}M^2K\left[-\frac{2d}{r}\left\{\frac{1}{(\pi x)^{\frac{1}{2}}}\exp\left(-\frac{\gamma^{\frac{1}{2}}(M^2-1)x}{K^2M^3}\right) + \frac{2d}{KM}\left(\frac{\gamma(M^2-1)}{Ur}\right)^{\frac{1}{2}}\exp\left(\frac{U(M^2-1)x}{r}\right)\operatorname{erf}\left[\frac{2d}{KM}\left(\frac{x\gamma(M^2-1)}{Ur}\right)^{\frac{1}{2}}\right]\right\} + \frac{KM^2}{r}\left\{\frac{1}{(\pi x)^{\frac{1}{2}}} + \frac{4d^2\gamma(M^2-1)}{K^2M^2Ur}\left(\frac{r}{U(M^2-1)}\right)^{\frac{1}{2}}\exp\left(\frac{U(M^2-1)}{r}x\right)\operatorname{erf}\left[\left(\frac{U(M^2-1)}{r}x\right)^{\frac{1}{2}}\right]\right\}\right]. \quad (6.8)$$

Holding M fixed and considering $\mathfrak{F}(x)$ for large x, we have from the asymptotic expansion of the error function,[25]

$$\mathfrak{F}(x) \sim -\frac{2UU^{\frac{1}{2}}}{(\pi x)^{\frac{1}{2}}}. \quad (6.9)$$

From (6.2) this leads to[26]

$$C_f^0 = 2\left(\frac{\mu}{\pi\rho_0u_0x}\right)^{\frac{1}{2}} \quad (6.10)$$

which is a well-known result of the Oseen equations.[27] In our discussion of the flow field we will return to Eq. (6.10) for a discussion of C_f^0. (The superscript 0 signifies that this is the Oseen result.)

We observe that expansions used in obtaining Eq. (6.9) are not uniform in the Mach number. This is best illustrated by considering $\mathfrak{F}$ for $M \gg 1$. In this case

$$d \sim (\gamma-1)M^2,$$

$$r \sim [4(\gamma-1)^2 - K^2]M^4 = \delta M^4,$$

δ defined above may be of either sign depending on the nature of the gas. We have then

[25] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, Inc., New York, 1966), Chap. 8.

[26] Equation (6.10) is exactly what results if the Mach layer terms of Eq. (6.4) are neglected.

[27] G. B. Whitham, in *Laminar Boundary Layers*, L. Rosenhead, Ed. (Clarendon Press, Oxford, 1963), p. 138.

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$$\mathfrak{F}(x) \sim \frac{2UU^{\frac{1}{2}}K}{\delta}\left[-2(\gamma-1)\left\{\frac{\exp(-x\gamma^{\frac{1}{2}}/K^2M)}{(\pi x)^{\frac{1}{2}}} + \left(\frac{4(\gamma-1)^2\gamma}{K^2\,\delta U}\right)^{\frac{1}{2}}\exp\left(\frac{Ux}{\delta M^2}\right)\operatorname{erf}\left[\left(\frac{4(\gamma-1)^2\gamma x}{K^2\,\delta U}\right)^{\frac{1}{2}}\right]\right\} + K\left\{\frac{1}{(\pi x)^{\frac{1}{2}}} + \frac{4(\gamma-1)^2\gamma M}{K^2U\,\delta}\left(\frac{\delta}{U}\right)^{\frac{1}{2}}\exp\left(\frac{Ux}{\delta M^2}\right)\operatorname{erf}\left[\left(\frac{Ux}{\delta M^2}\right)^{\frac{1}{2}}\right]\right\}\right]. \tag{6.11}$$

In order to obtain Eq. (6.9) from Eq. (6.11) we must impose the additional condition that

$$x \gg M$$

which is far more severe than the basis for our asymptotic expansion, i.e., $x \gg 1$. Therefore, for a region along the flat plate given roughly by

$$x < 10M,$$

the skin friction departs from Eq. (6.10). It is interesting to note that after normalizing the length by means of the viscosity the skin friction depends on the dissipative parameters only through K^2.

B. The Flow Field

The Oseen flow field may now be obtained by inserting Eq. (6.8) into Eq. (6.3). The resulting integrations prove too difficult to carry out and in order to examine the flow field it is plausible to use instead the asymptotic form for $\mathfrak{F}$ given by Eq. (6.9). For moderate Mach numbers, at least, this expression is valid (within the Oseen framework) for $x \gg 1$, which by our normalization only states that x be large compared with the mean free path. In fact for smaller scales, results from fluid mechanics cannot be assumed correct. Using Eq. (6.9) in Eq. (6.3) would be self-consistent within the framework of the Oseen theory. However, in Sec. 3 we found that the far field is determined by the actual forces and heat transfer at the body. The value of C_f given by Eq. (6.10), $C_f^0 \approx 1.128(\mu/\rho_0u_0x)^{\frac{1}{2}}$, is considerably different from the accepted value of $C_f \approx 0.664(\mu/\rho_0u_0x)^{\frac{1}{2}}$.[28] The far-field solution will, therefore, be incorrectly given by the use of $\mathfrak{F}(x)$ as determined from the Oseen equations. To overcome this we merely use the accepted value of C_f, i.e., we use for $\mathfrak{F}(x)$

$$\mathfrak{F}(x) \approx -0.664\left(\frac{U^3}{x}\right)^{\frac{1}{2}} = \frac{C}{x^{\frac{1}{2}}} \tag{6.12}$$

instead of Eq. (6.9).

Inserting Eq. (6.12) into Eq. (6.3) and performing a number of transformations, we have, for example for the u_1 velocity,

$$u_1 = C\left(\frac{\pi}{4U}\right)^{\frac{1}{2}}\operatorname{erfc}\left(\frac{|y|}{2x^{\frac{1}{2}}}\right) + \frac{[1+M^2(\gamma-1)]C}{2U(M^2-1)(\pi\alpha)^{\frac{1}{2}}}J, \tag{6.13}$$

where

$$J = \beta e^{-2\eta}\int_\beta^\infty \frac{\exp\left[-\eta\left(\tau+\frac{1}{\tau}\right)\right]}{\tau(\tau-\beta)^{\frac{1}{2}}}\,d\tau \tag{6.14}$$

with

$$\eta = \frac{|y|}{\alpha(M^2-1)^{\frac{1}{2}}}, \qquad \beta = \frac{|y|\,(M^2-1)^{\frac{1}{2}}}{x}. \tag{6.15}$$

From our normalization η is large and an asymptotic evaluation of the integral in Eq. (6.14) is indicated. It should be noted that the stationary point of the exponential in this integral may or may not lie in the interval of integration, and also that there is a algebraic singularity at the lower limit of integration. For this reason we distinguish between the following three flow regions:

Downstream of the leading edge Mach line, $\beta < 1$;

$$J \sim \left\{\frac{\pi\alpha\,|y|\,(M^2-1)^{\frac{3}{2}}}{x[x-|y|\,(M^2-1)^{\frac{1}{2}}]}\right\}, \tag{6.16}$$

Neighborhood of the leading edge Mach line, $\beta = O(1)$;

$$J \sim \Gamma\left(\frac{1}{4}\right)\left\{\frac{|y|\,\alpha^{\frac{1}{2}}(M^2-1)}{2x^{\frac{1}{2}}[x+|y|\,(M^2-1)^{\frac{1}{2}}]}\right\}^{\frac{1}{2}} \cdot\exp\left\{-\frac{[|y|\,(M^2-1)^{\frac{1}{2}}-x]^2}{x(M^2-1)\alpha}\right\}, \tag{6.17}$$

Upstream of the leading edge Mach line, $\beta > 1$;

$$J \sim \left[\frac{\alpha\,|y|\,\pi(M^2-1)^{\frac{1}{2}}}{y^2(M^2-1)-x^2}\right]^{\frac{1}{2}} \cdot\exp\left\{-\frac{[|y|\,(M^2-1)^{\frac{1}{2}}-x]^2}{x(M^2-1)\alpha}\right\}. \tag{6.18}$$

Each of these calculations follows from standard asymptotic methods.[29] It is important to note that the forms for J in the three regions, Eqs. (6.16), (6.17), and (6.18) do not approach one another. Following a method used by Bleistein,[30] a uniform asymptotic form may be obtained, this is given by

[28] N. Curle, *The Laminar Boundary Layer Equations* (Oxford University Press, Oxford, 1962), Chap. 7.

[29] See, for example, A. Erdelyi, *Asymptotic Expansions* (Dover Publications, Inc., New York, 1956), Chap. 2.

[30] N. Bleistein, Commun. Pure Appl. Math. 19, 4 (1966).

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$$J \sim \frac{\beta(2\pi)^{\frac{1}{2}}}{\lambda^{\frac{1}{4}}} \left\{ \left[\frac{1}{\beta + 1} \left(\frac{2}{\beta}\right)^{\frac{1}{2}} \right]^{\frac{1}{2}} D_{-\frac{1}{2}}\left[(\beta - 1)\left(\frac{2\lambda}{\beta}\right)^{\frac{1}{2}} \right] + \frac{1}{2\lambda^{\frac{1}{2}}} \left(\frac{\beta}{2}\right)^{\frac{1}{4}} \frac{\{[1/(\beta + 1)]^{\frac{1}{2}} - 1/\sqrt{2}\}}{\beta - 1} D_{-\frac{3}{2}}\left[(\beta - 1)\left(\frac{2\lambda}{\beta}\right)^{\frac{1}{2}} \right] \right\}, \tag{6.19}$$

where D_ν is the parabolic cylinder function.[25]

The insertion of appropriate forms for J into Eq. (6.13) furnishes the asymptotic flow field for supersonic flow past a semi-infinite flat plate.[31] The first term of Eq. (6.13) is a boundary-layer term, and the remaining term is of higher order in the boundary layer. To see this, consider Eq. (6.16) for fixed y and $x \to \infty$; then J is $O(1/x)$, which is negligible compared with the first term. Along any ray from the origin in the region given by Eq. (6.16), J is $O(1/|y|^{\frac{1}{2}})$, whereas in the region (6.18), J is exponentially small. In fact, in the latter region the solution resembles the flow field of the fundamental solution. In the neighborhood of the Mach lines, Eq. (6.17), J is $O(1/|y|^{\frac{1}{4}})$. Here the perturbation is in its most concentrated form and falls off least rapidly. This effect is easily traceable to the singularity of Eq. (6.12) at the origin. From the point of fluid mechanics, this leading edge singularity seems justified.[11,32] Its true justification, however, must await a kinetic theory analysis.

7. AN EXTENSION OF THE OSEEN THEORY

An insight which the previous section provides is that the Oseen theory is incapable of describing all the essential aspects of a flow problem. If the Oseen equations are used in a self-consistant manner; i.e., if the exact boundary conditions are imposed, it is found that this leads to an incorrect value for skin friction and that the far field is incorrectly described. Only the boundary conditions, *ipso facto*, are correctly recovered in this approach. The often repeated remark that the Oseen theory is correct in the far field and qualitatively correct in the near field is, therefore, incorrect.[10]

Carrier and Lewis[11] in their study of the incompressible Oseen equations offer a modification of the Oseen theory, which overcomes some of the shortcomings of the self-consistent formulation. Instead of using the Oseen operator (3.9) directly, the free-stream velocity is replaced by cU with $0 < c < 1$ chosen to give the correct skin friction; other choices are also discussed there.[11] While this procedure does correctly give the skin friction and also returns the correct boundary conditions, it incorrectly describes the far field. This is quite striking in the supersonic case since the Carrier and Lewis[11] modification may fail to even predict Mach lines, and even if these occur they will be at the wrong inclination. The equations predict subsonic or supersonic flow depending on whether $cU < 1$ or $cU > 1$.

The flow field obtained at the close of the last section is at least formally correct at infinity since the correct skin friction is applied at the flat plate. This, however, does not furnish the correct boundary values at the plate. In fact, it is clear that Eq. (6.12) leads to a perturbed velocity at the wall given by

$$H(x)u(y = 0) = -\frac{C_f}{C_f^0}\, U = -0.597U$$

instead of the value $-U$. This then states that the dimensional, total velocity, V_1 at the wall has a slip at the wall given by

$$H(x)V, (y = 0) = 0.403U. \tag{7.1}$$

This is instead of the correct no-slip condition.

Turning the above discussion around if the slip condition (7.1) is imposed as a boundary condition instead of no-slip, we obtain the correct skin friction and the formally correct far field flow. This suggests the following approximate procedure for the determination of skin friction for an arbitrary body.

If, for an arbitrary body, the local boundary-layer thickness is small compared with the bodies radius of curvature, we may expect the flow to be approximated by the flat-plate boundary layer (with a pressure gradient). The flat-plate case may then be regarded as a canonical problem for the local flow. Let ϕ denote the steady potential flow past an arbitrary body. Then letting t represent the tangential coordinate at the body, the tangential velocity at the body is denoted by $(\partial\phi/\partial t)_b$. It is proposed that in order to find the skin friction and far field of this body that the Oseen equations with the slip boundary condition u_s,

$$u_s = 0.403\left(\frac{\partial\phi}{\partial t}\right)_b, \tag{7.2}$$

be solved.

At present this is just a guess based on the semi-infinite flat-plate problem and no doubt the effect

[31] Although we only explicitly give u_1, the remaining flow variables can easily be represented in terms of J.

[32] G. F. Carrier and C. C. Lin, Quart. Appl. Math. 6 63 (1948).

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of a pressure gradient has to be considered. Several problems with simple geometrics are now under study using this approximation.

APPENDIX A. DERIVATION OF THE DISCONTINUOUS FORMS OF THE NAVIER–STOKES EQUATIONS

In a previous paper[8] the discontinuous forms of the Navier–Stokes equations were obtained. Here we present a simpler derivation. Let

$$S{:}S(\mathbf{x}, t) = 0 \tag{A1}$$

represent a smooth closed surface. We denote the finite volume it encloses by the superscript (i) and the exterior by (e). The normal directed into the exterior domain will be denoted by $\mathbf{n}$, and further (A1) is defined so that

$$\mathbf{n} = \frac{\nabla S}{|\nabla S|}.$$

As is well known, a normal velocity of the surface $\mathbf{V}\cdot\mathbf{n}$ is defined by

$$\mathbf{V}\cdot\mathbf{n} = \frac{-S_t}{|\nabla S|}.$$

Next we define the characteristic function $\theta_S(\mathbf{x}, t)$ so that $\theta_S = 1$, if $\mathbf{x}$ belongs to the interior, and zero otherwise.

The derivation given below depends on the following relations:

$$\frac{\partial \theta_S}{\partial t} = \mathbf{V}\cdot\mathbf{n}\ \delta(S), \tag{A2}$$

$$\nabla\theta_S = -\mathbf{n}\ \delta(S), \tag{A3}$$

where $\delta(S)$ is the "one-dimensional" distribution defined so that

$$\int \phi\ \delta(S)\ d\mathbf{x} = \int_S \phi\ dS, \tag{A4}$$

where ϕ is a testing function. We see that $\delta(S)$ transforms an integral over all space into a surface integral over S. To prove Eq. (A2), consider

$$A = \int \phi \frac{\partial \theta_S}{\partial t}\ d\mathbf{x},$$

where ϕ is a testing function. Then, by a parts integration,

$$A = \int \frac{\partial}{\partial t}(\phi\theta_S)\ d\mathbf{x} - \int \theta_S \frac{\partial \phi}{\partial t}\ d\mathbf{x}$$

$$= \frac{\partial}{\partial t}\int_S \phi\ d\mathbf{x} - \int_S \frac{\partial\phi}{\partial t}\ d\mathbf{x},$$

where the volume integrals in the last are over the interior of S. By a well-known result from calculus, the differentiation of the first integral may be carried out and we obtain

$$A = \int_S \mathbf{V}\cdot\mathbf{n}\phi\ dS$$

which proves Eq. (A2). The proof of Eq. (A3) follows in the same way; in fact,

$$\int \phi\nabla\theta_S\ d\mathbf{x} = \int \nabla(\theta_S\phi)\ d\mathbf{x} - \int_S \nabla\phi\ d\mathbf{x}$$

$$= -\int_S \mathbf{n}\phi\ dS,$$

since the first integral on the right-hand side vanishes by virtue of the vanishing of ϕ as $|x| \to \infty$.

Both the interior and the exterior of S are now regarded as being filled with fluids satisfying the Navier–Stokes equations. In general, the two fluid motions are independent. Denoting the interior and exterior dependent variables by the superscripts (i) and (e), respectively, we define

$$\rho = \theta_S\rho^{(i)} + (1 - \theta_S)\rho^{(e)},$$

$$\rho\mathbf{u} = \theta_S\rho^{(i)}\mathbf{u}^{(i)} + (1 - \theta_S)\rho^{(e)}\mathbf{u}^{(e)}.$$

Then, since $[\rho^{(e)}, \mathbf{u}^{(e)}]$ and $[\rho^{(i)}, \mathbf{u}^{(i)}]$ satisfy the continuity equation in the exterior and interior domains, respectively, we find

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial \mathbf{x}}\cdot\rho\mathbf{u} = [\rho(\mathbf{u} - \mathbf{V})]\cdot\mathbf{n}\ \delta(S). \tag{A5}$$

The bracket operator represents the jump across S,

$$[F] = F^{(e)} - F^{(i)}.$$

On applying the same formalism to the momentum and energy equations, it is clear that

$$\frac{\partial}{\partial t}\rho\mathbf{u} + \frac{\partial}{\partial \mathbf{x}}\cdot\rho\mathbf{u}\mathbf{u} + \frac{\partial}{\partial \mathbf{x}}p - \frac{\partial}{\partial \mathbf{x}}\cdot\mathbf{P} = [p\mathsf{I} - \mathbf{P}]\cdot\mathbf{n}\ \delta(S) + [\rho\mathbf{u}(\mathbf{u} - \mathbf{V})]\cdot\mathbf{n}\ \delta(S), \tag{A6}$$

$$\frac{\partial}{\partial t}\rho\left(e + \frac{u^2}{2}\right) + \frac{\partial}{\partial \mathbf{x}}\cdot\rho\mathbf{u}\left(e + \frac{u^2}{2}\right) + \frac{\partial}{\partial x}\cdot(p\mathbf{u} - \mathbf{P}\cdot\mathbf{u} + \mathbf{Q}) = \left[\rho\left(e + \frac{u^2}{2}\right)(\mathbf{u} - \mathbf{V}) + p\mathbf{u} - \mathbf{P}\cdot\mathbf{u} + \mathbf{Q}\right]\cdot\mathbf{n}\ \delta(S), \tag{A7}$$

where $\mathbf{P}$ and $\mathbf{Q}$ refer to the viscous stress tensor and the heat-conduction vector, respectively. The constitutive relations also have a discontinuous

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representation given by

$$\frac{P_{ij}}{\mu} - (u_{i,j} + u_{j,i} - \tfrac{2}{3}\,\delta_{ij}\nabla\cdot\mathbf{u}) - \frac{\beta}{\mu}\nabla\cdot\mathbf{u}\;\delta_{ij}$$
$$= (-[u_i]n_j - [u_j]n_i + \left(\frac{2}{3} - \frac{\beta}{\mu}\right)[\mathbf{u}]\cdot\mathbf{n}\;\delta_{ij})\;\delta(S) \quad \text{(A8)}$$

$$\frac{\mathbf{Q}}{\kappa} + \frac{\partial T}{\partial \mathbf{x}} = [T]\mathbf{n}\;\delta(S). \quad \text{(A9)}$$

The ratio of bulk to absolute viscosity β/μ must be taken as a constant in order for this to have a meaning on the right-hand side of Eq. (A8).

For a steady flow it is clear that

$$\mathbf{V}\cdot\mathbf{n} = 0.$$

At an impermeable surface

$$\mathbf{u}\cdot\mathbf{n} = 0,$$

and at a material surface, either for a viscous ($u = 0$) or inviscid ($\mathbf{P} = 0$) fluid

$$\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{u} = 0.$$

On imposing these conditions on the discontinuous Navier–Stokes equation, we obtain Eqs. (2.1)–(2.5).

The restriction of S to closed bounded regions may easily be lifted. Also, allowing a number of surfaces S adds no difficulty to the derivation. Finally, we point out that if S is chosen as an arbitrary control surface moving with the fluid, the use of Eqs. (A2) and (A3) leads to an economical derivation of the conservation equations.

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