Approximate Method in Kinetic Theory. Couette Flow and the Kramers Problem

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An approximate method especially suited to near-continuum flows is developed. The method is applied to the case of Couette flow with arbitrary specular and diffuse reflection. The result is in very close agreement with all known limits. In the case of purely diffuse reflection the approximate theory leads to a slip coefficient which is within $1\frac{1}{2}\%$ of the exact value. The analytical behavior in the neighborhood of the walls is exactly described. The limit of purely specular reflection is exactly desciibed by the approximation method. The method only involves elementary calculations.

I. INTRODUCTION

One of the most important applications of kinetic theory is to near-continuum boundary-value problems. The approach taken in such problems eventually reduces to the use of the Navier–Stokes equations (or some other system of macroscopic equations) with slip boundary conditions. The slip boundary condition itself leads to a canonical boundary-value problem in kinetic theory (Couette flow or the Kramers problem for velocity slip and heat flow between parallel plates for the temperature slip). These canonical problems themselves present great difficulties and only in certain cases have they met with success. Complete results have been obtained for the Couette flow and Kramers problems in the limit of purely diffuse reflections.[1–5] Certain partial and limiting results have been obtained for the Kramers problem with partly specular reflections.[4] Even in the successful cases the solutions which are attained involve analytical forms of great complexity and are almost useless except in qualitative discussions; e.g., even the determination of the slip coefficient involves a numerical calculation which is far from trivial.[1,6]

A number of moment approximations for such problems have been developed.[7–9] When comparison with an exact calculation is possible, these are found to furnish fairly accurate values of the slip coefficients. A general feature which greatly mars the moment methods is their failure to give a correct analytical description of the kinetic layer; e.g., velocity and temperature derivatives should have logrithmic infinities at a boundary, but the moment methods lead to finite values of these slopes. Another difficulty lies in the choice of moment equations. Although intuition provides some information, there is a great deal of arbitrariness in the selection of moments of the Boltzmann equation to govern the moments.

In this paper we consider a new method of finding approximate solutions. The method is relatively simple and free of arbitrariness, and has so far provided exceedingly good results. A description of the method under general circumstances is given in Sec. II. In brief the method attempts a uniform description of a flow field in terms of hydrodynamical-type variables. For regions bounded away from a wall the description reduces to the Chapman–Enskog equations (in the cases considered here only the Navier–Stokes equations are recovered). And in the neighborhood of a wall the method furnishes a description of the kinetic layer. As with the moment method the approximate method is in general nonlinear; however, none of the difficulties of the former method mentioned in the previous paragraph appear in the latter method.

In Sec. III the approximate method is applied to the case of Couette flow with arbitrary specular and diffuse reflection. The expression for the slip flow coefficient is found as a function of the degree of specular and diffuse reflection. In the limit of purely diffuse reflections this result is within 1.5% of the exact answer,[1,6] and in the limit of purely specular reflections the exact answer[4] is obtained. In addition, the entire flow field is obtained in terms of known functions. Using the calculations of Pao[10]

[1] D. R. Willis, Phys. Fluids 5, 127 (1962).
[2] C. Cercignani, Ann. Phys. (N. Y.) 20, 219 (1962).
[3] A. Leonard and T. W. Mullikan, Ann. Phys. (N. Y.) 30, 235 (1964).
[4] C. Cercignani, J. Math. Anal. Appl. 10, 568 (1965).
[5] C. Cercignani, J. Math. Anal. Appl. 11, 93 (1965).
[6] S. Albertoni, C. Cercignani, and L. Gotusso, Phys. Fluids 6, 993 (1963).
[7] E. P. Gross, E. A. Jackson, and S. Ziering, Ann. Phys. (N. Y.) 1, 141 (1957).
[8] L. Lees, J. Soc. Indust. Appl. Math. 13, 278 (1965).
[9] S. F. Shen, in *Rarefied Gas Dynamics*, J. A. Laurmann, Ed. (Academic Press Inc., New York, 1963), Vol. II, p. 112.
[10] Y. P. Pao, Phys. Fluids 9, 409 (1966).

the exact behavior of the velocity in the neighborhood of the wall is obtained in Appendix A. On comparison with the approximate results in this regime both have the same analytical form with coefficients that are about 2% different. The closely related Kramers problem is also solved in Appendix B.

II. A GENERAL APPROXIMATE PROCEDURE

We consider steady gasdynamic flows. (The unsteady case could also be considered but will be avoided for simplicity.) Provided that finite cross sections occur we may write the Boltzmann equation in the form[11]

$$\boldsymbol{\xi}\cdot\frac{\partial}{\partial \mathbf{x}} f + \nu f = Kf, \tag{1}$$

where ν is the collision frequency and, in general,

$$\nu = \nu(f) = \nu(\mathbf{x}, \boldsymbol{\xi}).$$

The operator K is, in general, also nonlinear. Equation (1) is to be solved subject to "outgoing" boundary conditions on the distribution function f. Denoting the outward normal at a boundary point by $\mathbf{n}$ and a boundary point by $\mathbf{x}_0$, we have

$$H(\boldsymbol{\xi}\cdot\mathbf{n})f(\mathbf{x}_0, \boldsymbol{\xi}) = H(\boldsymbol{\xi}\cdot\mathbf{n})F[\mathbf{x}_0, \boldsymbol{\xi}; f(\mathbf{x}_0, -\boldsymbol{\xi}\cdot\mathbf{n})]. \tag{2}$$

The last argument of F denotes the fact that F is usually a functional of the incoming distribution function. We will formally regard F as known.

The operator on the left-hand side of Eq. (1) is solvable in the sense that

$$\boldsymbol{\xi}\cdot\frac{\partial}{\partial \mathbf{x}} f_h = -\nu f_h$$

has the solution,

$$f_h(\mathbf{x}, \boldsymbol{\xi}) = H(k)f_h(\mathbf{x}_0 + k\boldsymbol{\xi}, \boldsymbol{\xi})$$
$$= H(k)f_h(\mathbf{x}_0, \boldsymbol{\xi}) \exp\left(-\int_0^k \nu(\mathbf{x}_0 + \boldsymbol{\xi}s, \boldsymbol{\xi})\, ds\right), \tag{3}$$

where again $\mathbf{x}_0$ is boundary point and the choice of k is clear from the arguments in Eq. (3).

We may also obtain a particular solution of f_p of Eq. (1). From the formal point of view at least, a solution of Eq. (1) is given by a solution of

$$f_p = \sum_{k=0}^{\infty}\left(-\frac{\boldsymbol{\xi}}{\nu}\cdot\frac{\partial}{\partial \mathbf{x}}\right)^k \frac{1}{\nu} K f_p. \tag{4}$$

Regarding K as $O(\nu)$ and expanding f_p in powers of $O(\nu^{-1})$, of course, leads to the Chapman–Enskog solution.

These remarks suggest solving Eq. (1) subject to the boundary conditions (2) in the following way. Write

$$f = f_p + f_h \tag{5}$$

and choose the decomposition so that f_p follows from Eq. (4) and f_h is of the form (3) with

$$H(\boldsymbol{\xi}\cdot\mathbf{n})f_h(\mathbf{x}_0) = H(\boldsymbol{\xi}\cdot\mathbf{n})\{F[\mathbf{x}_0, \boldsymbol{\xi}; f_h(-\boldsymbol{\xi}, \mathbf{n}) + f_p(-\boldsymbol{\xi}\cdot\mathbf{n})] - f_p(\mathbf{x}_0, \boldsymbol{\xi})\}. \tag{6}$$

The sum (5) will then constitute a formal solution if we can meet the following two relations:

$$\nu(f_p + f_h) = \nu(f_p), \qquad K(f_p + f_h) = K(f_p) \tag{7}$$

[i.e., substitution of Eq. (5) into Eq. (1) formally satisfies the problem].

For the moment we will assume that all this can be done and demonstrate the simplifications to which this leads. Since f_p is the Chapman–Enskog solution, it can be represented entirely in terms of the hydrodynamical moments. Then, from the form of f_h, it too can be represented in terms of only the hydrodynamical moments. Next, if this form of the solution is substituted into the conservation equations,

$$\frac{\partial}{\partial x_i}\int \xi_i(1, \boldsymbol{\xi}, \xi^2) f\, d\boldsymbol{\xi} = 0, \tag{8}$$

we obtain the solution to the Boltzmann equation by only solving the hydrodynamical equations. Moreover since f_h vanishes at several mean free paths from a boundary, Eq. (8) clearly goes over into the Chapman–Enskog equations in the main body of the flow.

Attractive as the above procedure may be, the careful reader has already observed that it possesses a number of serious difficulties. First, we note that the series solution obtained from Eq. (4) may not exist for a number of reasons. It is known from associated problems[12–14] that the convergence of the Chapman–Enskog series is dependent on $\nu(\xi)$ and, in particular, on the limit $\xi \to \infty$. {If $\lim_{\xi\to\infty}[\nu(\xi)/\xi] \neq 0$, convergence is obtained. Otherwise, divergence occurs. Counter-examples to both cases occur under special conditions.} An even more serious objection

[11] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1953), Sec. 7.6.

[12] L. Sirovich and J. K. Thurber, J. Math. Phys. **8**, 888 (1967).

[13] L. Sirovich and J. K. Thurber, J. Math. Phys. **10**, 239 (1969).

[14] A. Ganz, M.S. thesis, Brown University (1966).

is the nonexistence of hydrodynamical derivatives at a boundary. Specifically, we have that [15]

$$f_p = f_0\Big[1 - \frac{\mu}{2pRT}(\xi - \mathbf{u})_i(\xi - \mathbf{u})_j \cdot(u_{i,j} + u_{j,i} - \tfrac{2}{3}\nabla\cdot\mathbf{u}\,\delta_{ij}) - \frac{\kappa}{pRT}(\xi - \mathbf{u})_i\Big(\frac{(\xi - \mathbf{u})^2}{5RT} - 1\Big)\frac{\partial T}{\partial x_i}\Big] + O\Big(\frac{1}{\nu^2}\Big) = f_{NS} + O\Big(\frac{1}{\nu^2}\Big)$$

with

$$f_0 = \frac{\rho}{(2\pi RT)^{3/2}}\exp\Big(-\frac{(\xi - \mathbf{u})^2}{2RT}\Big).$$

The tangential velocity u_t on approaching a boundary is known to behave as $\delta + x \ln x$ (see Appendix A), where δ is the microscopic slip and x the normal distance (suitably normalized). Hence, $\partial u_t/\partial x$ does not exist at a boundary. One method of overcoming this is by regarding f_p as an outer solution which is to be matched to an inner solution.[16] The inner solution is then found through a canonical problem. Our approach will be along different lines. Another serious question in connection with the above discussion involves the possibility of satisfying the conditions of Eq. (7). In fact a study of Eq. (7) indicates that it is highly unlikely that such a condition can ever be met.

In spite of the objections to the method (3)–(8) in solving Eqs. (1) and (2), it does serve as a guide for an approximate method of solution. Instead of dealing with the full solution of Eq. (4) we take

$$f_p = f_{NS},$$

where f_{NS} is given above. This from the formal viewpoint satisfies Eq. (4) to $O(1/\nu^2)$. (Actually any finite termination of the Chapman–Enskog expansion could be made. However, in our later work we will not go higher than f_{NS}.) Using this form for f_p we solve for f_h in the form Eq. (3) and with the boundary conditions (6). Since we are dealing with near-continuum flows

$$H(\xi\cdot\mathbf{n})f_h(\mathbf{x}_0, -\xi\cdot\mathbf{n}) \sim 0,$$

because this is the contribution from another wall or remote parts of the same wall; therefore, this term may be eliminated in Eq. (6). An important point now arises in trying to satisfy Eq. (7). These conditions will clearly be satisfied if $f_h = 0$. But from Eq. (3), $f_h \sim 0$ several mean free paths from a boundary. Therefore, we can expect to satisfy Eq. (7) to within good accuracy if we take

$$K[f_p(\mathbf{x}_0) + f_h(\mathbf{x}_0)] = K[f_p(\mathbf{x}_0)], \qquad \nu[f_p(\mathbf{x}_0) + f_h(\mathbf{x}_0)] = \nu[f_p(\mathbf{x}_0)], \tag{9}$$

where as before $\mathbf{x}_0$ represents a boundary point. Condition (9), as we shall see, furnishes needed boundary conditions for our macroscopic quantities. It should be noted that this is a condition in velocity space and hence in general leads to a number of conditions on the moments.

When the approximate form for $f = f_h + f_p$ found in this way is substituted into Eq. (7), we obtain five equations in the five unknowns $(\rho, \mathbf{u}, T)$. These are subject to the boundary conditions which follow from Eq. (9). An important subtlety now enters the method. The macroscopic quantities $(\rho, \mathbf{u}, T)$ which appear in the Chapman–Enskog form f_p (the macroscopic variables in terms of which we solve) can no longer be regarded as density, velocity, and temperature. For example, the true macroscopic velocity is

$$w = \frac{\int (f_p + f_h)\xi\, d\xi}{\int (f_p + f_h)\, d\xi} = \frac{\rho\mathbf{u} + \int \xi f_h\, d\xi}{\rho + \int f_h\, d\xi}.$$

Only at many mean free paths from a boundary do we have $\mathbf{w} \to \mathbf{u}$. This is quite important since it eliminates one of the aforementioned difficulties. As we shall see although $\mathbf{w}$ does not, in general, have derivatives at a boundary, $\mathbf{u}$ will always have a sufficient number of derivatives.

III. COUETTE FLOW

As an application of the approximate method outlined in the previous section, we consider Couette flow. In keeping with common practice we consider low-speed flow so that density and temperatures may be taken as constant. Also for simplicity we employ the BGK[17] equation. In this case the nonlinear Krook equation is

$$\Big(\xi_2\frac{\partial}{\partial y} + \nu\Big)f = \frac{\nu\rho_0\exp\left(-\{[\xi_1 - u(y)]^2 + \xi_2^2 + \xi_3^2\}/2RT_0\right)}{(2\pi RT_0)^{3/2}} = \nu f_0(u)$$

with

$$u(y) = \int \frac{\xi_1 f}{\rho_0}\, d\xi.$$

[15] Reference 11, Chap. 7.
[16] L. Trilling, Phys. Fluids 10, 1681 (1964).

[17] P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. 94, 511 (1954).

Allowing both specular and diffuse reflections the boundary conditions are given by

$$H(\mp\xi_2)f(\xi, \pm L) = H(\mp\xi_2)[\alpha f_0(\pm U) + (1 - \alpha)f(\pm L, -\xi_2)],$$

where $\pm U$ are the velocities of the upper and lower walls. (Our treatment allows different velocities and α's at the two walls, but in the interest of simplicity we avoid this generalization.) $H(\zeta)$ denotes the Heaviside function.

Next, we introduce the reduced distribution function

$$g = \int \xi_1 f \, d\xi_1 \, d\xi_3$$

and the following normalization:

$$z = \frac{y}{L}, \qquad \zeta = \frac{\xi_2}{(2RT)^{1/2}},$$
$$G = \frac{g}{\rho_0}, \qquad \omega = \frac{e^{-\zeta^2}}{(\pi)^{1/2}}, \tag{10}$$
$$\Theta = \frac{L\nu}{(2RT_0)^{1/2}}, \qquad V = \frac{U}{(2RT_0)^{1/2}},$$
$$v(z) = \frac{u}{(2RT_0)^{1/2}}.$$

The problem can now be stated as

$$\frac{\zeta}{\Theta}\frac{\partial}{\partial z}G + G = \omega v, \qquad v = \int G \, d\zeta, \tag{11}$$

$$H(\mp\zeta)G(\pm 1, \zeta) = H(\mp\zeta)[\pm\alpha\omega V + (1 - \alpha)G(\pm 1, -\zeta)]. \tag{12}$$

This formulation is the same as that given by a number of other authors.[1-3] The only difference lies in the fact that we have shown that the linear problem (11) and (12) solves the nonlinear problem formulated at the outset of this section.

According to the procedure outlined in the previous section the approximate Chapman–Enskog solution is taken as

$$G_p = \omega\left(v - \frac{\zeta}{\Theta}\frac{\partial}{\partial z}v\right).$$

A straightforward integration then yields,

$$G_h = H(\zeta)\exp\left(-\frac{\Theta(z+1)}{\zeta}\right)\Big[-\alpha\omega V + (1-\alpha)\cdot[G_p(-1, -\zeta) + G_h(-1, -\zeta)] - \omega\left(v_{-1} - \frac{\zeta}{\Theta}\frac{\partial}{\partial z}v_{-1}\right)\Big]$$
$$+ H(-\zeta)\exp\left(-\frac{\Theta(z-1)}{\zeta}\right)\Big[\alpha\omega V + (1-\alpha)\cdot[G_p(1, -\zeta) + G_h(1, -\zeta)] - \omega\left(v_1 - \frac{\zeta}{\Theta}\frac{\partial}{\partial z}v_1\right)\Big].$$

The subscripts ± 1 denote values at the upper and lower walls.

We restrict attention to near-continuum flows

$$\Theta \gg 1. \tag{13}$$

From this

$$H(-\zeta)G_h(+1, -\zeta) \approx 0,$$
$$H(\zeta)G_h(-1, -\zeta) \approx 0,$$

and we can write

$$G = G_p + G_h = \omega\left(1 - \frac{\zeta}{\Theta}\frac{\partial}{\partial z}\right)v + H(\zeta)\omega\exp\left(-\frac{\Theta(z+1)}{\zeta}\right)\cdot\left(-\alpha(V + v_{-1}) + \frac{2-\alpha}{\Theta}\zeta\frac{\partial}{\partial z}v_{-1}\right)$$
$$+ H(-\zeta)\omega\exp\left(-\frac{\Theta(z-1)}{\zeta}\right)\cdot\left(\alpha(V - v_{+1}) + \frac{(2-\alpha)}{\Theta}\zeta\frac{\partial}{\partial z}v_{+1}\right). \tag{14}$$

Next, in keeping with the approximate procedure we impose (9) at $z = \pm 1$. Since ν is constant, the second part of Eq. (6) is immediate. Our condition is, therefore,

$$K(G_h + G_p)|_{z=\pm 1} = KG_p|_{z=\pm 1}$$

but in our case

$$K(G_p + G_h) = \omega \int (G_p + G_h)\, d\zeta$$

and Eq. (9) is satisfied if

$$\int G_h(\pm 1, \zeta)\, d\zeta = 0.$$

On imposing this we easily obtain,

$$V \mp v_{\pm 1} = +\frac{\epsilon}{\Theta(\pi)^{1/2}}\frac{\partial v_{\pm 1}}{\partial z}, \tag{15}$$

where

$$\epsilon = \frac{2-\alpha}{\alpha}. \tag{16}$$

Equation (15) should not be interpreted as giving the slip velocity; as we shall see it yields the microscopic slip relation.

The conservation equations are trivial with the exception of the momentum equation

$$\frac{\partial}{\partial z}\int \zeta G\, d\zeta = 0.$$

Integrating we obtain

$$\int \zeta G\, d\zeta = C, \tag{17}$$

where C, a constant, is the dimensionless shear stress. Substituting Eq. (14) into Eq. (17) we obtain

$$C = -\frac{1}{2\Theta}\frac{\partial v}{\partial z} - \alpha(V + v_{-1})F_1(\Theta[1+z]) + \frac{2-\alpha}{\Theta}F_2(\Theta[1+z])\frac{\partial v_{-1}}{\partial z} - \alpha(V + v_1)F_1(\Theta[1-z]) + \frac{2-\alpha}{\Theta}F_2(\Theta[1-z])\frac{\partial v_1}{\partial z}, \tag{18}$$

where

$$F_n(x) = \frac{1}{(\pi)^{1/2}}\int_0^\infty \exp\left(-\zeta^2 - \frac{x}{\zeta}\right)\zeta^n\, d\zeta. \tag{19}$$

Both tables and properties of $F_n(x)$ are available.[18] In particular, we note.[18]

$$F_n(x) = \frac{1}{\sqrt{3}}\left(\tfrac{1}{2}x\right)^{n/3}\exp\left[-3\left(\tfrac{1}{2}x\right)^{2/3}\right]\cdot\left[1 + O\left(\frac{1}{x^{2/3}}\right)\right] \tag{20}$$

and

$$F_0(x) = \frac{1}{(\pi)^{1/2}}\left[\tfrac{1}{2}(\pi)^{1/2} + x\ln x + O(x)\right], \tag{21}$$

$$F_1(x) = \frac{1}{(\pi)^{1/2}}\left[\tfrac{1}{2} - \tfrac{1}{2}(\pi)^{1/2}x + O(x^2\ln x)\right]. \tag{22}$$

Therefore, under the limit (13) we have from Eq. (20) that

$$C = -\frac{1}{2\Theta}\frac{\partial v_0}{\partial z} = -\frac{s}{2\Theta}, \tag{23}$$

where a zero subscript denotes $z = 0$. s is, therefore, the dimentionless (constant) slope in the region bounded away from the walls.

Since v is odd in z, we may restrict attention to $-1 \le z \le 0$ and from Eq. (20) neglect the last two terms of Eq. (18). Henceforth, we consider

$$\frac{\partial v}{\partial z} = s - 2\Theta\alpha(V + v_{-1})F_1(x) + 2(2-\alpha)F_2(x)\frac{\partial v_{-1}}{\partial z} \tag{24}$$

with

$$x = \Theta(1 - z). \tag{25}$$

Integrating (24) from $z = -1$,

$$v = v_{-1} + s(z+1) + 2\alpha(V + v_{-1})\left[F_2(x) - \tfrac{1}{4}\right] + \frac{2(2-\alpha)}{\Theta}\frac{\partial v_{-1}}{\partial z}\left(\frac{1}{2(\pi)^{1/2}} - F_3(x)\right). \tag{26}$$

Next, we determine the unknowns s, v_{-1}, $\partial v_{-1}/\partial z$. Since $v(0) = 0$, Eq. (24) at $z = 0$ gives

$$0 = v_{-1} + s - \frac{\alpha(V + v_{-1})}{2} + \frac{(2-\alpha)}{\Theta(\pi)^{1/2}}\frac{\partial v_{-1}}{\partial z}.$$

Equation (15) at the lower wall is

$$V + v_{-1} = \frac{\epsilon}{\Theta(\pi)^{1/2}}\frac{\partial v_{-1}}{\partial z}$$

and finally Eq. (24) at $z = -1$ gives

$$0 = s - \frac{\Theta\alpha}{(\pi)^{1/2}}(V + v_{-1}) - \frac{\alpha}{2}\frac{\partial v_{-1}}{\partial z}.$$

On solving these three relations we obtain

$$s = \frac{\Theta(2\epsilon + \pi)V}{(\pi)^{1/2}(\epsilon + 2)\epsilon + \Theta(2\epsilon + \pi)}, \tag{27}$$

$$\frac{\partial v_{-1}}{\partial z} = \frac{2\alpha^{-1}\Theta\pi V}{(\pi)^{1/2}(\epsilon + 2)\epsilon + \Theta(2\epsilon + \pi)}, \tag{28}$$

$$v_{-1} = -V + \frac{2\alpha^{-1}\epsilon V(\pi)^{1/2}}{(\pi)^{1/2}(\epsilon + 2)\epsilon + \Theta(2\epsilon + \pi)}. \tag{29}$$

On inserting these into Eq. (26) we have the solution v,

$$v(z) = -V + \frac{\Theta(2\epsilon + \pi)V(z+1)}{(\pi)^{1/2}(\epsilon + 2)\epsilon + \Theta(2\epsilon + \pi)} + \frac{2\epsilon V}{(\pi)^{1/2}(\epsilon + 2)\epsilon + \Theta(2\epsilon + \pi)}\cdot\left(\frac{(\pi)^{1/2}(\epsilon + 2)}{2} + 2(\pi)^{1/2}F_2(x) - 2\pi F_3(x)\right). \tag{30}$$

Although this is the solution for $v(z)$, it is not the macroscopic velocity. The latter is given by

$$\hat{w} = \int (G_p + G_h)\, d\zeta = v(z) + \frac{2\epsilon V(\pi)^{1/2}}{(\pi)^{1/2}(\epsilon + 2)\epsilon + \Theta(2\epsilon + \pi)}\cdot\left[(\pi)^{1/2}F_1(x) - F_0(x)\right]. \tag{31}$$

[18] M. Abramowitz and I. Stegun., Eds., *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series No. 55 (U.S. Government Printing Office, 1964), Sec. 27.5.

Therefore, we see that except for a narrow layer in the neighborhood of the wall, the velocity is just $v(z)$.

To facilitate comparison with other calculations we introduce the dimensional slip velocity

$$(2RT_0)^{1/2}(V - s) = \Delta = \frac{\epsilon(\epsilon + 2)(\pi)^{1/2}U}{\Theta(\pi + 2\epsilon) + \epsilon(\epsilon + 2)(\pi)^{1/2}}$$

$$= \frac{\epsilon(\epsilon + 2)2}{(\pi + 2\epsilon)}\, l\, \frac{dw_0}{dy}, \qquad (32)$$

where l is the mean free path

$$l = \frac{1}{\nu}\left(\frac{\pi R T_0}{2}\right)^{1/2} \qquad (33)$$

and dw_0/dy is the dimensional slope at $y = 0$. Also, we introduce the dimensional microscopic slip velocity,

$$(2RT_0)^{1/2}(V + v_{-1}) = \delta$$

$$= \frac{2\alpha^{-1}\epsilon U(\pi)^{1/2}}{(\pi)^{1/2}(\epsilon + 2)\epsilon + \Theta(2\epsilon + \pi)}$$

$$= \frac{4\alpha^{-1}\epsilon l}{(2\epsilon + \pi)}\frac{dw_0}{dy}. \qquad (34)$$

In the limit of purely diffuse reflections ($\alpha = 1$, $\epsilon = 1$) we obtain

$$\Delta_\epsilon \approx 1.1667 l \frac{dw_0}{dy}, \qquad \delta_\epsilon \approx 0.7778 l \frac{dw_0}{dy}. \qquad (35)$$

The error in Δ in this limit is roughly the same as obtained in other approximate procedures.[7–9] In the limit of purely specular reflection we obtain

$$\Delta \sim \frac{2}{\alpha}\, l\, \frac{dw_0}{dy},$$

$$\delta \sim \frac{2}{\alpha}\, l\, \frac{dw_0}{dy}.$$

The first of these is the exact result as obtained by Cercignani.[4] The second is no doubt also the correct asymptotic result. It should be noted that the correct slip velocities in this limit are as given by the first relations in Eqs. (32) and (34). It should also be remarked that the correct slip condition in the limit of purely specular reflection is

$$\frac{dw}{dy} = 0.$$

(In curvilinear coordinates it would be $\mathbf{n}\cdot\mathbf{p} = 0$ at a boundary, where $\mathbf{p}$ is the Navier–Stokes stress.)

It is also of interest to consider the dimensional velocity $(2RT_0)^{1/2}\hat{w} = w$ in the neighborhood of the wall. Expanding (31) for small x we obtain

$$w = -U + \frac{2\epsilon U(\pi)^{1/2}\{[\frac{1}{2}(\epsilon + 1)] - [x \ln x/(\pi)^{1/2}] + O(x)\}}{(\pi)^{1/2}(\epsilon + 2)\epsilon + \Theta(2\epsilon + \pi)}. \qquad (36)$$

This clearly shows the nonexistence of the derivative of the velocity at the wall although v itself possesses a derivative at $x = 0$. For completely diffuse reflections we obtain

$$w = -U + \frac{U}{\Theta}[0.689 - 0.389\, x \ln x + O(x)]. \qquad (37)$$

In Appendix A it is shown that from the calculation of Pao[10] the exact expression is

$$w = -U + \frac{U}{\Theta}[0.707 - 0.399\, x \ln x + O(x)]. \qquad (38)$$

From comparison of Eqs. (37) and (38) we see that in addition to good numerical agreement there is also analytical agreement.

For completeness the closely related Kramers problem is discussed in Appendix B.

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APPENDIX A.

Using the work of Leonard and Mullikan,[3] Pao[10] has obtained a uniformly valid form for the macroscopic velocity in near-continuum Couette flow. In our notation this is given by

$$w(z) = \frac{1}{D_0(\Theta)}\left[zU - \frac{U}{\Theta(\pi)^{1/2}} \cdot \int_0^\infty \exp\left(-\sigma^2 - \frac{\Theta}{\sigma}(z + 1)\right)\frac{d\alpha}{F(\alpha)}\right],$$

where

$$D_0(\Theta) = 1 + \frac{1}{\Theta}\int_0^\infty [1 - \theta(t)]\, dt$$

and $F(\sigma)$ such that

$$F(0) = \sqrt{2}. \qquad (A1)$$

The functions $\theta(t)$ and $F(\sigma)$ are defined in Refs. 3 and 10.

For z bounded away from $z = -1$ and $\Theta \to \infty$, we obtain

$$w(z) \sim \frac{1}{D_0(\Theta)} Uz = w^0(z).$$

Comparing $U + w^0(-1)$ with the exact slip calculation, Eq. (35), we have that

$$K = \int_0^\infty [1 - \theta(t)]\, dt = 1.0162. \qquad \text{(A2)}$$

Another constant of interest may be found by considering the microscopic slip

$$U + w(-1) = \frac{U}{\Theta\, D_0(\Theta)} \left(\int_0^\infty [1 - \theta(t)]\, dt - \frac{1}{(\pi)^{1/2}} \int_0^\infty e^{-\sigma^2} \frac{d\sigma}{F(\sigma)} \right).$$

Comparing this with the exact calculation of microscopic slip (35), we find

$$I_0 = \frac{1}{(\pi)^{1/2}} \int_0^\infty \frac{e^{-\sigma^2}\, d\sigma}{F(\sigma)} = 0.3091. \qquad \text{(A3)}$$

Next consider

$$I(x) = \frac{1}{(\pi)^{1/2}} \int_0^\infty \exp\left(-\sigma^2 - \frac{x}{\sigma}\right) \frac{d\sigma}{F(\sigma)},$$

where x is given by Eq. (25).

It is now required to find the expansion of I for small x. Writing

$$I(x) = I_0 + I_1 + \frac{1}{(\pi)^{1/2}} \int_0^\infty e^{-\sigma^2}(e^{-x/\sigma} - 1) \cdot \left(\frac{1}{F(\sigma)} - \frac{1}{F(0)} \right) d\sigma,$$

where

$$I_1 = \frac{1}{(\pi)^{1/2}} \int_0^\infty e^{-\sigma^2}(e^{-x/\sigma} - 1) \frac{d\sigma}{F(0)}.$$

The function $F(\sigma)$ is analytic at $\sigma = 0$. From this and straightforward estimates it follows that

$$I(x) - I_0 - I_1(x) = O(x).$$

The expansion of $I_1(x)$ for small x follows from Eqs. (21) and (22). Inserting the value of $F(0)$ from Eq. (A1) we get

$$I_1 = \frac{x \ln x}{(2\pi)^{1/2}} + O(x).$$

Therefore, the velocity field for small x is given by

$$\omega(z) = -U + \frac{U}{\Theta}(K - I_0) - \frac{U}{\Theta}\left(\frac{x \ln x}{(2\pi)^{1/2}} + O(x) \right). \qquad \text{(A4)}$$

Equation (A4) with numerical coefficients is given by Eq. (38).

APPENDIX B.

The Kramers problem[19] is closely related to the Couette problem and for completeness we give its solution by the approximate method. Using the normalization (10), with $\Theta = 1$, Kramer's problem is stated by

$$\zeta \frac{\partial G}{\partial z} + G = \omega v = \omega \int G\, d\zeta,$$
$$H(\zeta)G(0, \zeta) = H(\zeta)(1 - \alpha)G(0, -\zeta), \qquad \text{(B1)}$$

and

$$v(z \to \infty) \to \Delta + \hat{k}z, \qquad \text{(B2)}$$

where the slope $\hat{k}$ is given and Δ is to be determined.

Taking G_p such that

$$G_p = \omega\left(v - \zeta \frac{\partial v}{\partial z}\right)$$

we find

$$G = G_p + G_h = \omega\left(v - \zeta \frac{\partial}{\partial z} v\right) + H(\zeta)e^{-z/\zeta}\left(-\alpha v_0 + (2 - \alpha)\zeta \frac{\partial v_0}{\partial z}\right),$$

where the zero subscript denotes an evaluation $z = 0$.

Condition (9) now leads to

$$v_0 = \frac{\epsilon}{(\pi)^{1/2}} \frac{\partial v_0}{\partial z}, \qquad \text{(B3)}$$

where ϵ is given by Eq. (16). The momentum equation (17) is now

$$\frac{\partial v}{\partial z} = \hat{k} - 2\alpha v_0 F_1(z) + 2(2 - \alpha)F_2(z) \frac{\partial v_0}{\partial z}. \qquad \text{(B4)}$$

Evaluating this at $z = 0$

$$0 = \hat{k} - \frac{\alpha v_0}{(\pi)^{1/2}} - \frac{\alpha}{2} \frac{\partial v_0}{\partial z}. \qquad \text{(B5)}$$

Integrating Eq. (B5) from $z = 0$, gives

$$v = v_0 + \hat{k}z + 2\alpha v_0[F_2(z) - \tfrac{1}{4}] + 2(2 - \alpha) \frac{\partial v_0}{\partial z}\left(\frac{1}{2(\pi)^{1/2}} - F_3(z) \right). \qquad \text{(B6)}$$

Solving Eqs. (B3) and (B5) for v_0 and $\partial v_0/\partial z$ we get

$$v_0 = \frac{2\epsilon(\pi)^{1/2}\hat{k}}{\alpha(2\epsilon + \pi)},$$
$$\frac{\partial v_0}{\partial z} = \frac{2\pi\hat{k}}{(2\epsilon + \pi)\alpha}. \qquad \text{(B7)}$$

Substituting Eq. (B7) into Eq. (B6) solves the problem. The remaining calculations are as in Sec. III.

[19] H. A. Kramers, Nuovo Cimento Suppl. 6, 297 (1949).