

# Kinetic Description of the Propagation of Plane Sound Waves in a Diatomic Gas

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Plane wave solutions for the propagation of sound in a polyatomic gas have been obtained by employing the method of Sirovich and Thurber for polyatomic kinetic models. An asymptotic analysis valid at small frequencies yields the correct continuum limits for the absorption and speed of propagation, and numerical calculations were made for a wide range of frequencies and at several values of the collision number for the relaxation of internal degrees of freedom. Calculations using values of model parameters suitable for nitrogen, oxygen, and air, over a large range of the rarefaction parameter, are in very good agreement with dispersion and absorption measured in these gases.

## I. INTRODUCTION

The kinetic theory of the dispersion and attenuation of sound produced by a sinusoidally oscillating piston has been extensively studied by numerous investigators in recent years. In particular,<sup>1-12</sup> effort has been directed toward theories that are capable of encompassing a broad frequency range, from the continuum (low-frequency) to the transition or near free molecule (high-frequency) regimes. With few exceptions, these studies have concerned themselves with monatomic gases, particularly since the Boltzmann equation, and appropriate models of it, are far better understood than its analog for internal degrees of freedom.

Recently, however, it has been possible to obtain higher-order kinetic models for the Boltzmann equation

with internal degrees of freedom.<sup>13</sup> This was done by employing in the Wang-Chang-Uhlenbeck equation<sup>14</sup> the technique first suggested by Gross and Jackson<sup>15</sup> for Maxwell molecules and later applied to arbitrary cross sections.<sup>16</sup> Thus, it was possible to extend a polyatomic kinetic model equation obtained previously<sup>17</sup> by an *ad hoc* procedure to encompass a correct description of an arbitrary (in principle) number of higher-order moments of the distribution function. This was necessary for an adequate description of sound relaxation processes in a diatomic gas.

Certain of the conceptual and mathematical difficulties associated with sound propagation in a monatomic gas arise in the polyatomic case as well, so perhaps it is pertinent to comment on these briefly, since there has been considerable discussion in the literature on the following points.

In the sound problem, the simplest method for obtaining closure of the moment equations is by truncating the  $(N + 1)$ -order moments appearing in the  $N$ th-order moment equations (hereafter, referred to as the Wang-Chang method).<sup>18,19</sup> The neglected terms stem from both the collision integral and the streaming terms. The extensive numerical

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<sup>1</sup> C. L. Pekeris, Z. Alterman, and L. Finkelstein, in *Symposium on the Numerical Treatment of Ordinary Differential, Integral and Integro-Differential Equations* (Birkhauser-Verlag, Basel, 1960), p. 388.

<sup>2</sup> C. L. Pekeris, A. Alterman, L. Finkelstein, and K. Frankowski, *Phys. Fluids* **5**, 1608 (1962).

<sup>3</sup> L. Sirovich and J. K. Thurber, in *Rarefied Gas Dynamics*, J. A. Laurmann, Ed. (Academic Press Inc., New York, 1963), Vol. 1, p. 159.

<sup>4</sup> L. Sirovich and J. K. Thurber, in *Rarefied Gas Dynamics*, J. H. de Leeuw, Ed. (Academic Press Inc., New York, 1965), p. 2.

<sup>5</sup> L. Sirovich and J. K. Thurber, *J. Acoust. Soc. Am.* **37**, 329 (1965).

<sup>6</sup> H. Weitzner, in *Rarefied Gas Dynamics*, J. H. de Leeuw, Ed. (Academic Press Inc., New York, 1965), Vol. 1, p. 1.

<sup>7</sup> R. J. Mason in Ref. 4, p. 48.

<sup>8</sup> D. Kahn and D. Mintzer, *Phys. Fluids* **8**, 1090 (1965).

<sup>9</sup> J. K. Buckner and J. H. Ferziger, *Phys. Fluids* **9**, 2309 (1966).

<sup>10</sup> J. K. Buckner and J. H. Ferziger, *Phys. Fluids* **9**, 2315 (1966).

<sup>11</sup> C. Cercignani and F. Sernagiotto, in *Rarefied Gas Dynamics*, C. L. Brundin, Ed. (Academic Press Inc., New York, 1967), Vol. 1, p. 381.

<sup>12</sup> R. J. Mason in Ref. 11, Vol. 1, p. 395.

<sup>13</sup> F. B. Hanson and T. F. Morse, *Phys. Fluids* **10**, 345 (1967).

<sup>14</sup> C. S. Wang-Chang and G. E. Uhlenbeck, Engineering Research Institute, University of Michigan, Report CM-681 (1951).

<sup>15</sup> E. P. Gross and E. A. Jackson, *Phys. Fluids* **2**, 432 (1959).

<sup>16</sup> L. Sirovich, *Phys. Fluids* **5**, 908 (1962).

<sup>17</sup> T. F. Morse, *Phys. Fluids* **7**, 159 (1964).

<sup>18</sup> C. S. Wang-Chang and G. E. Uhlenbeck, Engineering Research Institute, University of Michigan, Report NComr-23222 (1952).

<sup>19</sup> G. E. Uhlenbeck and G. W. Ford, *Lectures in Statistical Mechanics* (American Mathematical Society, Providence, Rhode Island, 1963), p. 90.

work of Pekeris *et al.*,<sup>1,2</sup> which is based on a modification of the Wang-Chang method, predicts that at high frequency the rate of decrease of the attenuation with frequency is much greater than for the attenuation measured in experiments. It has been shown<sup>20</sup> that the analytical behavior of the attenuation rate at high frequency predicted by the Wang-Chang method is a direct consequence of the truncation scheme, and results in an incorrect description of high-frequency sound propagation.

The use of model equations circumvents the difficulties associated with the simple truncation of the moment equations, in that only the collision integral is truncated. Further, a model equation of a given order retains an approximation to the expansion coefficients of the higher-order moments (eigenvalues in the case of a Maxwell force law). Thus, model equations retain the relaxation form of the collision integral while giving the correct representation of an arbitrary number of moments. The higher the order of the approximation, the more moments are represented exactly.

In the following analysis, employing appropriate model equations, a dispersion relation will be obtained for propagation of plane sound waves in a polyatomic gas. Therefore, we do not attempt solution of the full boundary-value problem but rather focus attention on the nature of plane waves which originate at an oscillating boundary. In a full discussion of the sound problem the contribution of the continuous, as well as the discrete spectra, must be considered.<sup>9,10,19</sup> It is almost a direct consequence of the continuous spectrum that beyond a critical frequency plane wave propagation ceases.<sup>2</sup> Both the continuous spectrum and the critical frequency are strongly dependent on the intermolecular force and, in the case of models, on the order of the model. In fact, the value of the critical frequency increases with the order of the model. Due to the form of the functions appearing in the dispersion relation it may be analytically continued beyond the critical frequency. The uncontinued sound mode for higher-order models then is essentially similar to the continued sound mode of lower-order models.<sup>5</sup> Therefore, the results of plane wave propagation employing model equations seem markedly insensitive to the continuous spectrum. This would seem to provide sufficient motivation for comparing the results of the model equation analysis (containing the analytical continuation) with high-frequency experimental results. These are arguments based

upon considerations from the theory of monatomic gases; however, they hold equally well for the dispersion and attenuation of sound in a polyatomic gas.

## II. CLASSICAL THEORY AND PHENOMENOLOGICAL EXTENSIONS

The classical theory originated over a century ago with the work of Stokes and Kirchhoff<sup>21</sup> and is essentially based on the Navier-Stokes equations. Only the dissipative mechanisms of viscosity and heat conduction are considered. Dissipation associated with volume dilation of the gas and relaxation effects due to internal degrees of freedom are absent. In this theory, the speed of propagation is the adiabatic speed,

$$c_0 = (\gamma R T_0)^{1/2}, \quad (1)$$

where  $\gamma$  is the ratio of specific heats,  $R$  is the gas constant, and  $T_0$  is the ambient temperature. The classical attenuation  $\alpha$  reflects both viscous and heat conduction effects:

$$\frac{\alpha c_0}{\omega} = \frac{1}{2\gamma r} \left( \frac{4}{3} + \frac{(\gamma - 1)^2 \lambda_{tr}}{\gamma R \mu} \right), \quad (2)$$

where, in standard notation,  $\omega$  is the frequency of vibration,  $r = p/\omega\mu$  is the rarefaction parameter,  $\mu$  is the coefficient of viscosity,  $p$  is the ambient pressure, and  $\lambda_{tr}$  is the coefficient of heat conduction associated with translational degrees of freedom. This theory is valid in regions of high pressure or very low frequency where the effect of the relaxation of internal degrees of freedom is not important.

Herzfeld and Rice,<sup>22</sup> from the thermodynamics of irreversible processes, and Kneser,<sup>23</sup> from statistical mechanics, developed similar theories to explain the attenuation in polyatomic gases for the case in which the relaxation of internal energy is important. They tacitly assumed that the relaxation processes of translational and internal degrees of freedom are sufficiently decoupled and the excess attenuation above the classical value results solely from the relaxation of the internal degrees of freedom. For a single internal degree of freedom, the relaxation equation is

$$\frac{dE_{int}}{dt} = -\frac{[E_{int} - E_{int}(T_{tr})]}{\tau}, \quad (3)$$

where  $\tau$  is the relaxation time and  $E_{int}(T_{tr})$  is

<sup>21</sup> K. F. Herzfeld and T. A. Litovitz, *Absorption and Dispersion of Ultrasonic Waves* (Academic Press Inc., New York, 1959), Sec. 7.

<sup>22</sup> K. F. Herzfeld and F. O. Rice, *Phys. Rev.* **31**, 691 (1928).

<sup>23</sup> H. O. Kneser, *Ann. Physik* **16**, 337 (1933).

<sup>20</sup> L. Sirovich and J. K. Thurber, *J. Math. Phys.* **8**, 888 (1967).

the internal energy that would exist if the temperature associated with internal energy were the same as the translational temperature  $T_{tr}$ . When the relaxation mechanisms of translational and internal degrees are strongly coupled, the assumption that these effects are additive fails.

Greenspan<sup>24, 25</sup> developed an *ad hoc* procedure that combined the classical theory (both Navier-Stokes and Burnett) and the relaxation theories of Herzfeld, Rice, and Kneser for the case in which the relaxation mechanisms of translational and internal energy are coupled. The same combinatorial rules for translational and internal relaxation as in the Becker gas, whose Prandtl number is 0.75, were assumed. These combinatorial rules replace simple addition in the Herzfeld, Rice, and Kneser theories. This method is mentioned since Greenspan used it to explain his experiments in nitrogen, oxygen, and air. However, it is apparent that a self-consistent theory is desirable, one in which the combination of translation and internal relaxation follows from the theory rather than a combination assumed beforehand. Such a theory (for the dispersion relation) will be developed below.

### III. PREVIOUS KINETIC STUDIES OF POLYATOMIC SOUND PROPAGATION

In their pioneering report on polyatomic gases, Wang-Chang and Uhlenbeck<sup>14</sup> briefly considered sound propagation. They gave the "easy transfer limit" (in the extended first Chapman-Enskog approximation) of the speed of propagation and the attenuation:

$$c \sim (\gamma RT_0)^{1/2} = c_0, \quad (4)$$

$$\frac{\alpha c_0}{\omega} \sim \frac{1}{2\gamma r} \left( \frac{4}{3} + \frac{\kappa}{\mu} + \frac{\gamma - 1}{\gamma} f \right). \quad (5)$$

A generalized Eucken factor is given by  $f \equiv (\gamma - 1)\lambda/\mu R$  where  $\lambda = \lambda_{tr} + \lambda_{int}$  is the total heat conductivity associated with the average flux of total energy. The difference between this attenuation and the classical attenuation is due to the appearance of a bulk viscosity term ( $\kappa/\mu$ ) associated with relaxation effects, and the inclusion of a heat-transfer coefficient that is the sum of the heat-transfer coefficients for translational and internal modes. This result for the speed and the attenuation is limited to small frequencies and is essentially in agreement with the Herzfeld, Rice, and Kneser theories.

More recently, Monchick,<sup>26</sup> following the success of Mason and Monchick<sup>27</sup> in determining polyatomic transport coefficients, utilized the Wang-Chang and Uhlenbeck equation to obtain a kinetic description of the propagation of sound in a polyatomic gas. Monchick modified the iteration of Maxwell, Truesdell, and Ikenberry,<sup>28</sup> in such a way as to retain the feature of nonnormal solutions, i.e., solutions not restricted to spatial and temporal dependence solely through the number density, mean velocity, and mean energy. The distribution function was expanded about a local Maxwellian in which the temperature characterized the total energy. The main advantage of such a solution is to obtain effective transport coefficients which are frequency dependent, as in the relaxation theories. The iteration involves the truncation of an infinite matrix, as in the Wang-Chang method. Monchick calculated sound dispersion and attenuation in nitrogen and oxygen for both the extended Wang-Chang method and the iterative scheme. These were carried out to a relatively high order. The comparison with experiment showed that Monchick's iterative theory did not agree with experiment to as high frequencies as the modified Wang-Chang theory,<sup>29</sup> but, qualitatively, they exhibit the same behavior.

Brau<sup>30</sup> extended the Krook<sup>31</sup> model to internal degrees by expanding the collision cross section rather than the entire collision operator. The linearized version of this model is similar to the  $L^{(2)}$  model of Ref. 13. The important restriction common to these two models is that the generalized Eucken factor  $f$  is equal to  $\gamma^{-1}$ . Brau also studied plane wave solutions using the model equation method for collision numbers  $Z$  (the ratio of the relaxation time for internal degrees of freedom,  $\tau_R$ , to the total collision frequency  $\tau_{coll}$ ) of order 10 and  $10^4$  and for rarefaction  $r > 1$ . Brau investigated the limit  $Z \rightarrow \infty$  in which the effects of the internal degrees of freedom are frozen as long as  $Z \rightarrow \infty$  faster than  $r \rightarrow \infty$  or as long as  $\tau_R \rightarrow \infty$  for fixed  $\tau_{coll}$ . In this limit, the polyatomic gas behaves as if its specific heat were that of a monatomic gas. Comparison with experiment is now being made.<sup>32</sup>

<sup>26</sup> L. Monchick, Phys. Fluids 7, 882 (1964).

<sup>27</sup> E. A. Mason and L. Monchick, J. Chem. Phys. 36, 1622 (1962).

<sup>28</sup> E. Ikenberry and C. Truesdell, J. Ratl. Mech. Anal. 5, 1 (1956).

<sup>29</sup> L. Monchick, in *Molecular Relaxation Processes* (Academic Press Inc., New York, 1966), p. 257.

<sup>30</sup> C. Brau, Ph.D. thesis, Harvard University (1965).

<sup>31</sup> P. A. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. 94, 511 (1954); P. Welander, Arkiv Fysik 7, 507 (1954).

<sup>32</sup> C. Brau (private communication).

<sup>24</sup> M. Greenspan, J. Acoust. Soc. Am. 26, 70 (1953).

<sup>25</sup> M. Greenspan, J. Acoust. Soc. Am. 37, 155 (1959).

## IV. THE MODEL EQUATIONS

The equation for which we wish to employ a suitable model representation is the Wang-Chang Uhlenbeck kinetic equation for gases with internal degrees of freedom.<sup>14</sup> This equation may be written as

$$\frac{\partial f_l}{\partial t} + \mathbf{v} \cdot \frac{\partial f_l}{\partial \mathbf{x}} = \frac{\delta f_l}{\delta t},$$

where  $\delta f_l/\delta t$  symbolizes the collision integral and  $f_l$  is the distribution function for particles in the  $l$ th quantum state. The internal degrees of freedom are treated quantum mechanically, and the translational degrees of freedom classically. The lowest moments of the distribution function, appropriately summed over internal states are given (in standard notation) by the following:

$$\left\{ \begin{array}{l} n \\ n\mathbf{U} \\ \frac{3}{2}nkT_{tr} \\ n(E) \\ P_{ij} \\ \mathbf{q}_{tr} \\ \mathbf{q}_{int} \end{array} \right\} = \sum_l \int \left\{ \begin{array}{l} 1 \\ \mathbf{v} \\ \frac{mc^2}{2} \\ E_l \\ mc_i c_j \\ \frac{mc^2}{2} \mathbf{c} \\ E_l \mathbf{c} \end{array} \right\} f_l d\mathbf{v}, \quad (6)$$

where  $\mathbf{v}$  is the molecular velocity,  $\mathbf{c}$  is the peculiar velocity, and  $E_l$  is the energy of the  $l$ th state. The summational invariants are 1,  $\mathbf{v}$ , and  $\frac{1}{2}mc^2 + E_l$ . For a gas with internal degrees of freedom, the total energy (as opposed to the translational energy) is a summational invariant. Kinetic models of this equation will be used in studying the propagation of sound. Due to the inherent linearity of this problem, an expansion of the distribution function about an absolute Maxwellian,  $f_l = f_l^0(1 + g_l)$ , will be more appropriate than the local expansion used in Ref. 13,

$$f_l = f_{0l}(1 + \phi_l), \quad (7)$$

where

$$f_l^0 = n_0 Q^{-1}(T_0) \left( \frac{m}{2\pi k T_0} \right)^{3/2} \exp \left( -\frac{mv^2}{2kT_0} - \frac{E_l}{kT_0} \right),$$

$$f_{0l} = nQ^{-1}(T) \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -\frac{mc^2}{2kT} - \frac{E_l}{kT} \right),$$

and

$$\begin{aligned} \phi_l \approx & g_l - \rho - \xi_l u_l \\ & - \frac{k}{c_s} \left[ \frac{1}{2}(\xi^2 - 3) + G(\epsilon_l - \bar{\epsilon}) \right] \left( \frac{3}{2}\tau_{tr} + G(\epsilon) \right). \end{aligned} \quad (8)$$

The notation is standard;  $Q(T) = \sum_l \exp(E_l/kT)$  is the partition function and  $T$  is the total temperature. The dimensionless macroscopic perturbation variables and parameters appearing here are given in Appendix A. The general form of the  $n$ th-order model equation is

$$\left( \frac{\partial}{\partial t} + \xi_l \frac{\partial}{\partial x_l} \right) g_l = \frac{n_0 L^{(N)}}{(\pi^3 Q_0 \nu_N)}, \quad (9)$$

where  $L^{(N)}$  is the approximate representation of the collision integral  $\delta f_l/\delta t$ . These models have been discussed in detail elsewhere,<sup>13</sup> and here we will be concerned only with the linearizations and normalizations appropriate to the sound problem. The  $N = 2$  and  $N = 3$  models are given below:

$$\begin{aligned} & n_0 L^{(2)} / (\pi^3 Q_0 \nu_2) \\ & = [\rho + \xi_l u_l - \frac{1}{2}(3 - \xi^2)\tau_{tr} + G(\epsilon_l - \bar{\epsilon})(\epsilon) - g_l] \\ & \quad + \nu_{200}^0 [\rho_l - \rho - G(\epsilon_l - \bar{\epsilon})(\epsilon)] \\ & \quad - \nu_{010} [\frac{1}{2}(3 - \xi^2) + \frac{3}{2}(\epsilon_l - \bar{\epsilon})][(\epsilon) - \tau_{tr}], \end{aligned} \quad (10)$$

$$\begin{aligned} & n_0 L^{(3)} / (\pi^3 Q_0 \nu_3) \\ & = [\rho + \xi_l u_l - \frac{1}{2}(3 - \xi^2)\tau_{tr} + G(\epsilon_l - \bar{\epsilon})(\epsilon) \\ & \quad + \frac{1}{2}(\xi_1^2 - \frac{1}{3}\xi^2)p_{11} - \frac{1}{5}(5 - \xi^2)\xi_1 S_{tr} \\ & \quad + (\epsilon_l - \bar{\epsilon})\xi_1 S_{int} - g_l] \\ & \quad + \nu_{300}^0 [\rho_l - \rho - G(\epsilon_l - \bar{\epsilon})(\epsilon)] \\ & \quad - \nu_{010} [(\epsilon) - \tau_{tr}][\frac{1}{2}(3 - \xi^2) + \frac{3}{2}(\epsilon_l - \bar{\epsilon})] \\ & \quad - \nu_{200} [\frac{1}{2}(\xi_1^2 - \frac{1}{3}\xi^2)p_{11}] \\ & \quad + (\frac{3}{2}\nu_{200} + \frac{5}{6}\nu_{010})[\frac{1}{5}(5 - \xi^2)\xi_1 S_{tr}] \\ & \quad - \frac{1}{2}\nu_{010} [-(\epsilon_l - \bar{\epsilon})\xi_1 S_{tr} + \frac{1}{5}(5 - \xi^2)S_{int}] \\ & \quad - \nu_{101} [(\epsilon_l - \bar{\epsilon})S_{int}]. \end{aligned} \quad (11)$$

The  $L^{(2)}$  model correctly represents (in the sense of Chapman-Enskog) the first four macroscopic moments of the kinetic equation,  $\rho$ ,  $u$ ,  $\tau_{tr}$ , and  $(\epsilon)$ , and  $L^{(3)}$  gives, in addition to these, the correct representation for  $p_{11}$ ,  $S_{tr}$ , and  $S_{int}$ . These will be referred to as the four- and seven-moment models, respectively.

The dimensional cutoff coefficient

$$\nu_N \equiv -n_0 \beta_N (\pi^{3/2} Q_0)^{-1}$$

is chosen to normalize all other collision frequencies such that  $\nu_{rst} = -n_0 \beta_{rst} (\pi^3 Q_0 \nu_N)^{-1}$ . The  $\nu_{rst}$  in  $L^{(2)}$  and  $L^{(3)}$  are dimensionless, and  $\beta_{rst}$  are the bracket integrals of Ref. 13, which are related to the transport and relaxation coefficients. (See Appendix B.) For  $N = 2$ ,  $\nu_2 = -n_0 \beta_{200}^{200} (\pi^3 Q_0)^{-1}$  and for  $N = 3$ ,  $\nu_3 = -n_0 \beta_{300}^{300} (\pi^3 Q_0)^{-1}$ .

In the dimensionless form of the collision model just given, several collision frequencies have been eliminated in favor of a smaller set of parameters. In addition to  $\nu_{200}^{\circ}$  and  $\nu_{300}^{\circ}$ , whose coefficients vanish when moments in velocity and internal energy are taken,<sup>33</sup>  $\nu_3$  and  $\nu_{010}$ ,  $\nu_{200}$ , and  $\nu_{101}$  are the only independent frequencies in the seven-moment model. For the sound problem, the viscosity, the relaxation time, the total heat conductivity, and the specific heat due to internal degrees of freedom are represented, respectively, by a more useful set of parameters: the rarefaction parameter,  $r = p\omega/\mu$ , the collision number,  $Z = \tau_R/\tau_{e011}$ , the total Eucken factor,  $f = (m/c_{v\mu})(\lambda_{tr} + \lambda_{int})$ , and the dimensionless internal specific heat,  $G = c_v^i/k$ . The collision number  $Z$  is defined as the ratio of the relaxation time for the adjustment of the energy of the internal degrees of freedom to the total collision frequency. This parameter eliminates the strong dependence of the relaxation time on the pressure. Therefore, it is more convenient to use the viscosity rather than the total collision frequency in calculating the collision number, since the viscosity is accessible to measurement. In general, collision cross sections, especially those for inelastic collisions, are not precisely known, so the relationship between the viscosity and collision frequency cannot be directly calculated. As an estimate, the relationship between the two is assumed to be the same as that for Maxwell molecules:  $\tau_{e011} = \frac{4}{3}[\mu]_1/p$ . The collision number can now be expressed as  $Z = \frac{5}{2}p[\tau_R/\mu]_1$ , where  $[\ ]_1$  signifies the first Chapman-Enskog approximation to the transport coefficients.

The collision frequencies of the  $L^{(2)}$  and  $L^{(3)}$  models may now be expressed in terms  $Z$ ,  $f$ ,  $G$ , and  $r$  by the following relations:

$$\begin{aligned} \nu_N &= p/([\mu]_1\nu_{200}) = \omega r/\nu_{200}, & \nu_{010} &= \frac{5}{8}G\nu_{200}/Z, \\ \nu_{101} &= (3 - 3\nu_{200}) \\ & \frac{\left[ \frac{8G}{27} + \left( \frac{5G}{9} + \frac{5}{3} \right) \nu_{010} + \frac{5}{8} \frac{G + \frac{3}{2}}{G} f(\nu_{010})^2 \right]}{\left[ \left( \frac{4}{3} + \frac{5}{8}\nu_{010} \right) (G + \frac{3}{2}) f - \frac{5}{3} \right]} \\ & + (3\nu_{200} - 2). \end{aligned} \quad (12)$$

For the four-moment model,  $\nu_{200} = 1$ , by definition, and it can be shown that  $\nu_{200} = \frac{2}{3}$  for the seven-moment model. For simplicity, the coefficients of the first approximation are set equal to the actual values of the transport coefficients. The approximate representation of the heat transfer moment in the

<sup>33</sup> The superscript  $\circ$  signifies the contribution to the cross section only from elastic collisions.

seven-moment model requires that  $f = \gamma$  or the total Prandtl number be equal to one. Thus, the four-moment model is determined by  $r$ ,  $Z$ , and  $G$ , and the seven-moment model by  $r$ ,  $Z$ ,  $G$ , and  $f$ .

## V. PLANE WAVE SOLUTIONS AND THE DISPERSION RELATION

We assume a plane wave form for the perturbed distribution function as follows:

$$g_i(\xi, x_1, t) = \tilde{g}(\xi, k, \omega) \exp [(i\omega t - kx_1)]. \quad (13)$$

The fixed frequency<sup>4</sup> of the boundary oscillating in the  $x_1$  direction is  $\omega$  and the propagation constant is denoted by  $k$ . Solutions of the dispersion relation  $k = k(\omega)$  are to be found for an induced oscillation in a half-space.

Substitution of the plane wave into the model of the kinetic equation with internal degrees of freedom yields

$$(i\omega - k\xi_1 + \nu_N)\tilde{g}_i = \nu_N\tilde{g}_i + n_0L^{(N)}/(\pi^{\frac{3}{2}}Q_0), \quad (14)$$

where the perturbation of the distribution function  $\tilde{g}_i$  has been added to both sides of the equation so that it will not appear explicitly on the right-hand side.

With these model equations, it is possible to obtain a closed system of equations for the macroscopic moments without further truncation of the moment equations. Dividing the above equations by  $(i\omega - k\xi_1 + \nu_N)$ , we obtain

$$\tilde{g}_i = \frac{(\nu_N\tilde{g}_i + n_0L^{(N)})/(\pi^{\frac{3}{2}}Q_0)}{(i\omega - k\xi_1 + \nu_N)}, \quad (15)$$

where  $L^{(N)}$  contains the perturbed moments and known functions of velocity and internal energy. Clearly, this division is permissible only if  $(i\omega - k\xi_1 + \nu_N) \neq 0$ . This equality gives rise to continuous or particle modes, as opposed to the discrete modes to which we restrict this study. (From the viewpoint of Laplace transforms, this problem does not arise.) To obtain a closed system of macroscopic moments, we must take those moments of  $g_i$  that appear in  $L^{(N)}$ . The moments taken for the four-moment model ( $L^{(2)}$ ) are  $1$ ,  $\xi_1$ ,  $\frac{1}{2}(\xi^2 - 3)$ ,  $\epsilon_i - \bar{\epsilon}$ . For the seven-moment model, we take  $\frac{1}{2}(\xi^2 - 5)\xi_1$ ,  $\xi_1^2 - \frac{1}{3}\xi^2$ , and  $(\epsilon_i - \bar{\epsilon})\xi_1$  as additional moments.

In general, the moment equations take the form

$$\sum_{ij}^N (c_{ij} - i\omega\delta_{ij})a_i = 0, \quad (16)$$

where, for example,

$$a_1 = \rho, \quad a_2 = u_1, \quad a_3 = \tau_{tr}, \quad a_4 = \langle \epsilon \rangle, \quad (17)$$

$$a_5 = p_{11}, \quad a_6 = S_{tr}, \quad \text{and} \quad a_7 = S_{int}.$$

The coefficients  $c_{ij}$  are too lengthy to be given here.<sup>34</sup> For the  $L^{(2)}$  model,  $N = 4$  and for the  $L^{(3)}$  model,  $N = 7$ . The condition for the existence of a plane wave is that the dispersion relation vanish, i.e.,  $D(\zeta, \omega) = \det(c - i\omega l) = 0$ , where  $\zeta \equiv (\nu_N + i\omega)/k$ . In obtaining the moment equations, integration over all velocity space and summation over all internal energy states gives rise, in the matrix elements, to singular integrals of the following type:

$$F(\zeta) = \int_{-\infty}^{\infty} \frac{\omega_1}{\xi_1 - \zeta} d\xi_1,$$

where  $\omega_1 = \exp(-\frac{1}{2}\xi_1^2)/(2\pi)^{\frac{1}{2}}$ . All such integrals of the form

$$\sum_l \int [\omega_l \xi_l^{2n} \xi_l^m / (\xi_l - \zeta)] d\xi_l$$

can be reduced to the sum of one polynomial in  $\zeta$  and the product of another polynomial in  $\zeta$  with  $F(\zeta)$ . This integral  $F(\zeta)$  is simply related to the plasma dispersion integral of Fried and Conte<sup>35</sup> and is discussed in Refs. 3 and 4.

Following the example used for a monatomic gas,<sup>5</sup> the dispersion relation is calculated for  $\text{Im}(\zeta) < 0$ , i.e., up to the cut of the function  $F(\zeta)$  on the real axis in the  $\zeta$  plane. This leads to a critical frequency beyond which the discrete sound mode ceases to exist for models similar to those used here.<sup>3,4,20</sup> The critical frequency, therefore, occurs when  $\text{Im}(\zeta) = 0$  or when  $\nu_N \text{Im}(k) = \omega_{crit} \text{Re}(k)$ . When  $\text{Im}(\zeta) > 0$ , it has been suggested that the analytic continuation of  $F(\zeta)$  be used in  $D^-(\zeta, \omega)$ . This leads to the analytic continuation of  $k(\omega)$  beyond the critical frequency. Specifically, one uses

$$F^-(\zeta) = -i(\frac{1}{2}\pi)^{\frac{1}{2}} \exp(-\frac{1}{2}\zeta^2) \cdot \left[ 1 - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{i\zeta/\sqrt{2}} \exp(-\frac{1}{2}t^2) dt \right] \quad (18)$$

for all  $\zeta$  instead of for just  $\text{Im}(\zeta) < 0$ . The integral defining  $F(\zeta)$  for  $\text{Im}(\zeta) > 0$  is related to  $F^-$  by

$$F^-(\zeta) = F(\zeta) - 2i(\frac{1}{2}\pi)^{\frac{1}{2}} \exp(-\frac{1}{2}\zeta^2). \quad (19)$$

This criterion for the critical frequency can be put into more familiar terms such as the velocity of propagation  $c$  and the attenuation  $\alpha$ . The velocity of propagation is defined as the velocity of planes

of constant phase. The phase of the plane wave is  $i\omega t - i \text{Im}(k)x_1$  and hence  $c = \omega/\text{Im}(k)$ , if  $k$  is independent of  $x_1$ . The reciprocal of the attenuation,  $1/\alpha$ , is the distance necessary to change the amplitude by  $1/e$  of its original value:  $\alpha = \text{Re}(k)$ . Thus, the criterion for the critical frequency can be stated as  $\alpha c = \nu_N$  or  $(\alpha c/\omega)_{crit} = \nu_{200}/r_{crit}$ . An explicit dependence of the critical frequency on the model appears through the parameter  $\nu_N$  which varies with the order of the model.

**VI. ASYMPTOTIC ROOTS OF THE FOUR MOMENT MODEL**

We first wish to examine the roots of the dispersion relation in the continuum limit. Thus, we consider the behavior of waves for the case in which  $\omega \rightarrow 0$ . We can then restrict attention to those roots for which  $k(\omega) \rightarrow 0$ , since these correspond to continuum modes. (Although roots where  $k(0) \neq 0$  do exist, they have no bearing on the sound problem.) These considerations lead to two pairs of asymptotic roots such that  $\omega/\nu_2 = O(\zeta^{-1})$  and  $\omega/\nu_2 = O(\zeta^{-2})$ .

From the behavior of these roots, we see that  $\zeta \rightarrow \infty$  in the continuum limit, and we obtain the following asymptotic result for large  $\zeta$ :

$$F(\zeta) \sim \frac{1}{\zeta} \left( 1 + \frac{1}{\zeta^2} + \frac{3}{\zeta^4} + \frac{15}{\zeta^6} + \dots \right).$$

Although no fundamental difficulty exists in considering  $D_N(\zeta, \omega) = 0$  in the continuum limit for  $N > 4$ , for simplicity we shall consider the case for  $N = 4$ . After a straightforward calculation, the following asymptotic results for small  $\omega$  are obtained:

$$D_4(\zeta, \omega) = \left( i \frac{\omega}{\nu_2} \right)^4 + c_3 \left( i \frac{\omega}{\nu_2} \right)^3 + \left( i \frac{\omega}{\nu_2} \right)^2 c_2 + \left( i \frac{\omega}{\nu_2} \right) c_1 + c_0,$$

where

$$c_3 \sim W + \left( -\frac{22}{3} + \frac{13 - 6\gamma}{3} W \right) \zeta^{-2} + [-36 + (33 - 18\gamma)W] \zeta^{-4},$$

$$c_2 \sim -\left( \frac{5}{3} + \frac{9\gamma + 4}{3} W \right) \zeta^{-2} - [4 - (23\gamma + 1)W] \zeta^{-4}, \quad (20)$$

$$c_1 \sim -\gamma W \zeta^{-2} + \left( \frac{10}{3} - \frac{22 + 16\gamma}{3} W \right) \zeta^{-4},$$

$$c_0 \sim \gamma W \zeta^{-4}.$$

<sup>34</sup> F. Hanson, Ph.D. thesis, Brown University (1968).  
<sup>35</sup> B. Fried and S. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

Further,

$$W = \left(1 + \frac{3}{2G}\right) \nu_{010} = \frac{c_v}{c_s^2} \nu_{010} = \frac{5c_v}{6kZ},$$

$$\gamma = \frac{(1 + 5/2G)}{(1 + 3/2G)}.$$

We will first consider the case  $\omega/\nu_2 = O(\zeta^{-1})$  which corresponds to sound propagation. It is convenient to introduce the following notation:

$$\tilde{p} \equiv \frac{i\omega}{\nu_2}, \quad s \equiv \frac{k}{\nu_2}.$$

We now wish to find the expansion of  $s$  in terms of  $\omega$ . This proves difficult since  $W$  can vary from  $O(1)$  to an arbitrarily small value, corresponding, respectively, to the limits of very easy and very difficult transfer of energy. In order to find a description which is uniformly valid in  $W$  we write

$$s = b_1(\tilde{p}, W)\tilde{p} + b_2(\tilde{p}, W)\tilde{p}^2 + O(\tilde{p}^3).$$

We now seek to determine  $b_1$  and  $b_2$  in the limit  $\tilde{p} \rightarrow 0$  under the single assumption that  $b_1, b_2 = O(1)$  which will be verified *a posteriori*. Substituting this and the expansion

$$\zeta^{-1} = s(1 - \tilde{p} + \tilde{p}^2 \dots)$$

into (20) we obtain,

$$D_4 = \tilde{p}^3 \left\{ (1 - b_1^2 \frac{5}{3})\tilde{p} + W(1 - b_1^2\gamma) \right. \\ \left. + \tilde{p}^4 [(10b_1^4 - 10b_1b_2 - 12b_1^2) - \frac{\tilde{p}}{3} \right. \\ \left. + W[\gamma b_1^4 + 2\gamma b_1^2 - 2\gamma b_1b_2 - \frac{1}{3}(9\gamma + 4)b_1^2] \right\} \\ \left. + O(\tilde{p}^5 W) + O(\tilde{p}^6) = 0. \right. \quad (21)$$

Clearly, whatever the value of  $W$  the first term of Eq. (21) is of the lowest order and we set it equal to zero. Next, the second term is clearly of lower order than the neglected terms. Proceeding in this manner we obtain [to the order indicated in (21)]

$$b_1 = \pm \left( \frac{\tilde{p} + W}{\frac{5}{3}\tilde{p} + \gamma W} \right)^{\frac{1}{2}},$$

$$b_2 = b_1 \quad (22)$$

$$\frac{\{W[\gamma b_1^2 + 2\gamma - \frac{1}{3}(9\gamma + 4)] + \frac{1}{3}(\tilde{p})(10b_1^2 - 12)\}}{2\gamma W + 10\tilde{p}/3}.$$

For  $k(\omega)$  these results are uniformly in the limit  $\omega \rightarrow 0$ , independently of  $W$ . It should be noted that both  $b_1$  and  $b_2$  are complex. In order to find the speed and attenuation rate, the value of  $W$  is fixed and the coefficients  $b_1$  and  $b_2$  are then separated into real and imaginary parts. We now examine two limiting cases.

First consider the following limit:  $W \sim O(1)$ , ( $\omega/\nu_2 = r^{-1} \rightarrow 0$ ). This is the case of "easy" transfer of energy, and after some manipulation we find

$$\frac{k}{\nu_2} = \pm \frac{i}{(\gamma r)^{\frac{1}{2}}} \left[ 1 - \frac{1}{6\gamma r} \left( \frac{5 - 3\gamma}{W} + 3\gamma + 1 \right) \dots \right] + O\left(\frac{1}{r^{\frac{3}{2}}}\right). \quad (23)$$

In more common terms, the reciprocal speed ratio is  $c_0/c \sim \pm 1$  and the attenuation is  $\alpha c_0/\omega \sim (\pm 1/2\gamma r) \{ \gamma + \frac{1}{3} + [(5 - 3\gamma)/3W] \}$ . These represent two plane waves, one propagating in the plus  $x_1$  direction [ $\text{Im}(\zeta) < 0$ ] and another in the minus  $x_1$  direction [ $\text{Im}(\zeta) > 0$ ]. The sign of  $\text{Im}(\zeta)$  separates the asymptotic roots of the proper dispersion relation  $D(\zeta, \omega)$  according to the direction of propagation. For these roots, the sign of the attenuation is equal to the sign of the speed of propagation. The sound mode is immediately identified by the fact that the speed approaches the dimensionless adiabatic speed  $c_0 = (\gamma)^{\frac{1}{2}}$ . Since  $\kappa/\mu = (5 - 3\gamma)W$ , then  $\alpha c_0/\omega \sim (1/2\gamma r) [\gamma + \frac{1}{3} + (\kappa/\mu)]$ . This attenuation is the same as that obtained by Wang-Chang and Uhlenbeck,<sup>14</sup> when the total Eucken factor is taken as  $\gamma$ , which is the correct value for the four-moment model. Also, when  $W$  is very large, the attenuation for the classical Stokes-Kirchhoff gas with  $f = \gamma$  is obtained. Clearly, the limit of equilibrium between the degrees of freedom, i.e., the bulk viscosity  $\kappa$  tending to zero, corresponds to the Stokes-Kirchhoff gas.

Now consider the limit  $W \rightarrow 0$  first, and a fixed but small  $\omega$ . This corresponds to a "freezing" of the internal degrees of freedom, which occurs in the high-frequency limit for  $Z > O(1)$ . The internal modes fail to respond to the high-frequency oscillation. For this case

$$\frac{k}{\nu_2} = \pm \left(\frac{3}{5}\right)^{\frac{1}{2}} \frac{i}{r} \left(1 - \frac{3}{5} \frac{i}{r} \dots\right) \quad (24)$$

and

$$c_0/c \approx \pm(3\gamma/5)^{\frac{1}{2}},$$

$$\alpha c_0/\omega \approx \pm(3/5r)(3\gamma/5)^{\frac{1}{2}}.$$

In this limiting situation the gas behaves as if it were a monatomic gas with  $\gamma = \frac{5}{3}$ . This approach toward the "frozen limit" is actually unrealistic. For a gas in a particular experiment,  $W$  does not change significantly once the ambient temperature is fixed; the pressure can always be changed such that  $1/r \ll W$ .

Now we briefly consider the roots that behave as  $\omega/\nu_2 = O(\zeta^{-2})$ . Expanding  $\zeta$  in half-integral powers of  $\omega$ , we obtain

$$c_0/c \sim \pm(\gamma r/2)^{\frac{1}{2}} \quad \text{and} \quad \alpha c_0/\omega \sim \pm(\alpha r/2)^{\frac{1}{2}}.$$

This asymptotic plane wave, with the real and imaginary parts of the propagation constant equal, is often called the diffusive wave because of its resemblance to the solution of the diffusion equation. For large  $r$  this mode is too heavily damped in comparison with the sound mode to dominate the solution.

Although we have focused attention on the case of forced oscillations the case of free waves can easily be treated. We recall that this involves finding complex  $\omega$  in terms of a  $k$  which is pure imaginary. By merely inverting the expansions found in this section the free wave results are obtained directly.

It is also possible to calculate the asymptotic limits of the analytically continued dispersion relation for large frequencies. These limits have not been computed for the four-moment model. However, the qualitative behavior of the roots for models of the type considered here should follow the results found in the case of simple gases.<sup>20</sup> For the speed of propagation,  $c_0/c \sim r(-\ln r^2)^{-\frac{1}{2}}$ , and for the attenuation  $\alpha c_0/\omega \sim (-\ln r^2)^{-\frac{1}{2}}$ , as  $\omega \rightarrow \infty$  (or  $r \rightarrow 0$ ). Both tend to zero as  $r \rightarrow 0$ . This behavior is markedly different from the behavior of solutions employing the Wang-Chang method.

## VII. NUMERICAL RESULTS

The sound mode of the analytically continued dispersion relation was calculated numerically for the four- and seven-moment models. The method of solution differs for two ranges of  $r$  values. For large  $r$ , the function  $F(\zeta)$  is calculated from asymptotic solutions which are obtained for the range  $4.0 \leq r \leq 120$ ; for the seven-moment model, asymptotic solutions are obtained for the range  $2.7 \leq r \leq 85$ . For  $10^{-3} \leq r \leq 2.7$ , the calculation is based on numerical integration of the differential equation of  $F(\zeta)$ .<sup>34</sup> In the small region where the numerical and asymptotic solutions overlap, the difference is negligible.

Above the critical frequency, an analytical continuation of the dispersion relation has been employed to extend the discrete solutions to positive  $\text{Im}(\zeta)$ . The critical frequency depends on the model equation and on the properties of the gas,  $Z_M$ ,  $f$ , and  $c_v^i/k$ . The values of the critical rarefaction parameter [ $r_{\text{crit}} \equiv (p/\omega\mu)_{\text{crit}}$ ] were calculated for different values of the collision number  $Z_M$  and

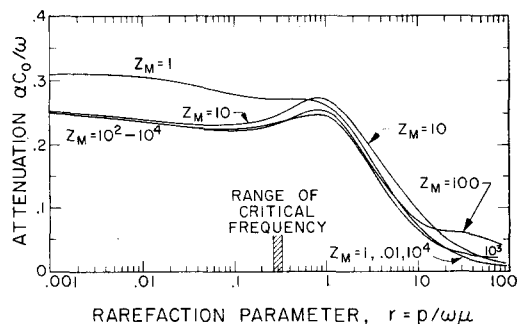


FIG. 1. Attenuation of sound according to the seven-moment model with  $Z_M$  as a parameter,  $c_v^i/k = 1.0$ ,  $f = 2.0$ ,  $\gamma = 1.4$ ,  $\text{Pr} = 0.7$ .

for the specific heat due to internal degrees of freedom  $c_v^i/k = 1.0$ . The collision number  $Z_M$  is based on the model collision frequency  $\nu_N$ . It is related to the collision number  $Z$  defined previously by

$$Z_M \equiv \nu_N \tau_R = \frac{4}{3} Z / \nu_{200}.$$

The critical frequency was higher ( $r_{\text{crit}}$  smaller) for the seven-moment model than for the four-moment model, and slightly dependent upon  $Z_M$ . Some characteristic values for the four-moment model ( $f = 1.4$ ) were  $Z_M = 1$ ,  $r_{\text{crit}} = 0.471$ ; and  $Z_M = 10^4$ ,  $r_{\text{crit}} = 0.468$ . For the seven-moment model ( $f = 2.0$ ),  $Z_M = 1$ ,  $r_{\text{crit}} = 0.321$ ; and  $Z_M = 10^4$ ,  $r_{\text{crit}} = 0.315$ . The critical frequency is rather insensitive to variation in  $Z_M$ .

There are numerous combinations of the many physical parameters, and it is, therefore, necessary to summarize our numerical results. In particular, the results of the seven-moment model will be studied. For both the four- and the seven-moment model, sound speed and attenuation depended only weakly upon the internal specific heat, and for the seven-moment model the dependence upon  $f$ , the Eucken factor, was negligibly small for  $f$  between 1.4 and 2.5. For the  $N = 4$  model,  $f$  is fixed at 1.4.

Figures 1 and 2 display the attenuation  $\alpha c_0/\omega$  and the reciprocal speed ratio  $c_0/c$  versus the rarefaction parameter  $r$  for the seven-moment model and for each of the model collision frequencies:  $Z_M = 1, 10, 100, 1000$ , and  $10\,000$ . The specific heat of the internal degrees of freedom is  $c_v^i/k = 1$  (i.e., an average of two internal degrees excited).

The curves for large collision numbers are noteworthy for  $r$  approximately 70. The attenuation for  $Z_M = 100$  exhibits a behavior typical of thermal relaxation phenomena. The speed for  $Z_M = 10^3$  and  $10^4$  is nearly constant at the frozen limit,  $c_0/c = (3\gamma/5)^{\frac{1}{2}}$ , while the speed for  $Z_M = 100$  is



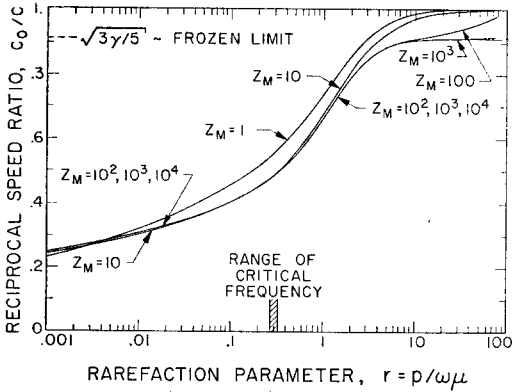


Fig. 2. Speed of propagation according to the seven-moment model with  $Z_M$  as a parameter,  $c_v^i/k = 1.0$ ,  $f = 1.4$ ,  $\gamma = 1.4$ ,  $Pr = 0.7$ .

between the frozen speed and the adiabatic speed,  $c_0/c = 1$ . The speeds for  $Z_M = 1$  and 10 are very close to the adiabatic speed or continuum limit.

For  $r$  near 10, the attenuation begins to rise steeply with decreasing  $r$  and the speed ratio begins to drop, indicating the onset of translational relaxation. For  $Z_M = 10$ , the attenuation is larger than that for other plotted values of  $Z_M$  and no higher point is apparent. For this value the internal and translation relaxations are coupled.

For smaller  $r$ , the attenuation reaches a maximum and slowly descends. The attenuation for  $Z_M = 1$  is much higher than for larger  $Z_M$  values which do not differ significantly in attenuation. This can be expected since a gas with a high collision number is less likely to follow the high-frequency sound vibrations. The speed ratio continues to decrease. At  $r = 0.001$ , the speed itself is about five times the adiabatic speed ( $c_0/c \sim 0.2$ ) for all  $Z_M$ .

The results of the four-moment model are qualitatively similar to the seven-moment model. The

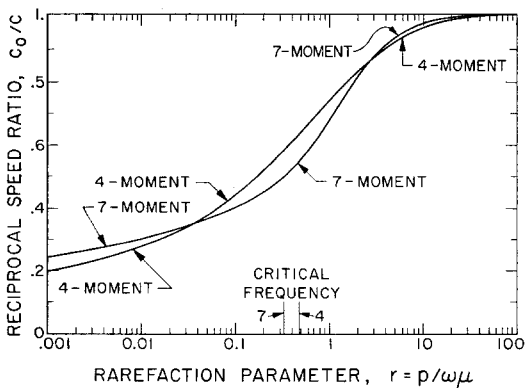


Fig. 3. Comparison of attenuation rates according to seven-moment and four-moment models with  $Z_M = 10$ ,  $c_v^i/k = 1.0$ ,  $f = 1.4$ ,  $\gamma = 1.4$ ,  $Pr = 1.0$ .

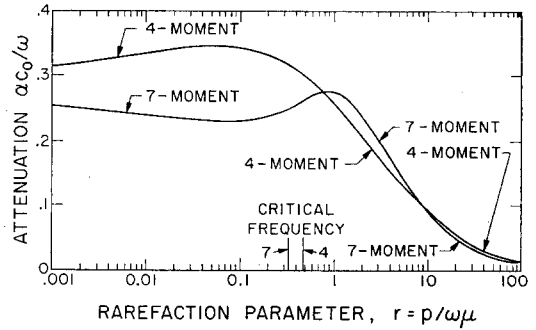


Fig. 4. Comparison of speeds of propagation according to seven-moment and four-moment models with  $Z_M = 10.0$ ,  $c_v^i/k = 1.0$ ,  $f = 1.4$ ,  $\gamma = 1.4$ ,  $Pr = 1.0$ .

prominent difference between the results for the models is that while the attenuation predicted by the four-moment model goes through a very broad maximum around  $r = 0.6$ , the seven-moment model predicts a relatively narrow maximum about  $r = 0.8$ . For very small  $r$ , the continued seven-moment dispersion relation predicts another maximum in the attenuation. A direct comparison of the two models is made in Figs. 3 and 4, with  $Z_M = 10$ ,  $c_v^i/k = 1.0$ , and  $f = 1.4$ . The four-moment model predicts larger reciprocal speed ratios in the transition regime (near  $r = 0.3$ ) and much higher attenuation for  $r < 0.6$  than the seven-moment model.

Comparison of the present seven-moment theory with high-frequency measurements of Greenspan in nitrogen, oxygen, and air at 300°K are shown in Figs. 5-7, respectively. This comparison is appropriate because the measurements exhibit one relaxation time. The present theory is not appropriate for gases with widely spaced internal relaxation times since it is tacitly assumed that the relaxation times of all the internal degrees of freedom can be represented by one relaxation time. Air is a

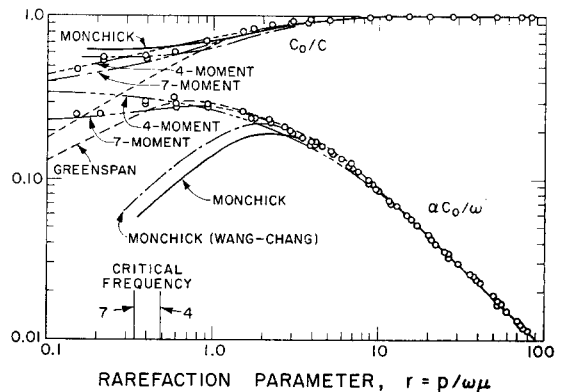


Fig. 5. Attenuation and speed of propagation in nitrogen at 300°K. Comparison of theories with experiment (O, Greenspan, Ref. 29).

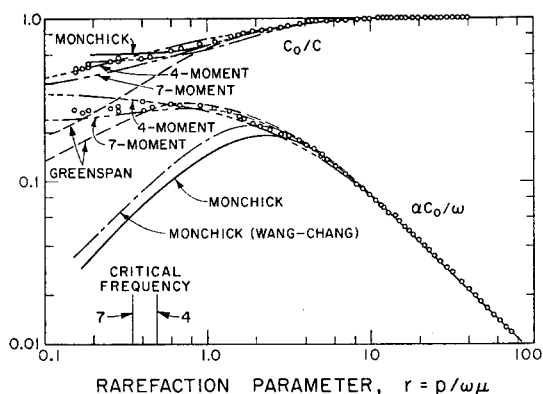


FIG. 6. Attenuation and speed of propagation in oxygen at 300°K. Comparison of theories with experiment (O, Greenspan, Ref. 29).

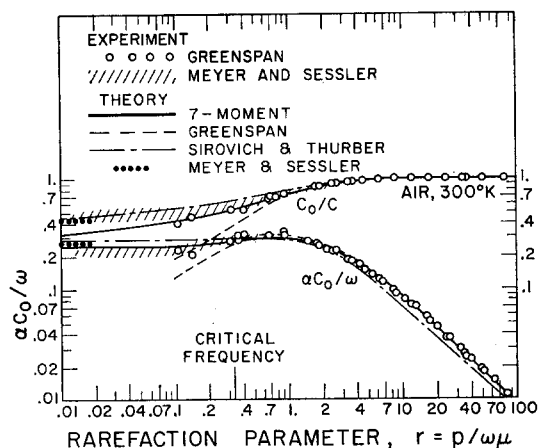


FIG. 7. Attenuation and speed of propagation in air at 300°K. Comparison of theories with experiment (O, Greenspan, Ref. 29; shaded, Meyer and Sessler, Ref. 38).

complicated mixture of polyatomic gases for which the present theory is not strictly applicable, but its behavior at 300°K does not differ significantly from the pure gases N<sub>2</sub> and O<sub>2</sub>.

The attenuation and reciprocal speed ratios are exhibited on a logarithmic scale. The experimental measurements were carried out in a finite geometry; a tube terminated at both ends by a transducer. When  $r$  is small (i.e., the collision frequency is small compared with the sound frequency), the available measurements are limited to receiver-transmitter separations which are small with respect to the mean free path. The width of the transition regime between the regime dominated by intermolecular collisions and the regime dominated by surface-molecule collisions (i.e., free-molecule flow regime) is uncertain. The experiments are not conclusively in favor of either the free-molecule theory or the theory employed here as long as the separation distance is not too small. Analytic continuation has been used here to extend the discrete sound mode to small values of  $r$  below  $r_{crit}$ . The motivation for this has been discussed previously.

The following values of gas properties were used with the seven-moment model:

Gas	$Z$	$f$	$c_v^i/k$
N <sub>2</sub>	5.08	1.96	1.0
O <sub>2</sub>	3.97	1.96	1.0
Air	4.62	1.98	1.01.

The properties  $f$  and  $c_v^i/k$  are taken from Prandtl numbers and specific heats in Ref. 36. The values of the collision number  $Z$  are not as accurately known

<sup>36</sup> J. Hilsenrath, C. W. Beckett, W. S. Benedict, L. Fano, H. J. Hoge, J. F. Masi, R. L. Nuttall, Y. S. Touloukian, and H. W. Woolley, National Bureau of Standards Circular 564 (1955).

as  $f$  and  $c_v^i/k$  and are not tabulated with other gas properties. For the present model equations, the collision number  $Z$  is determined in a manner similar to the way in which Greenspan determined  $Z$  from his own attenuation measurements. At a high value of  $r$  ( $r = 85$  for the seven-moment theory), the  $Z$  parameter is adjusted until the attenuation of the model is within the uncertainty of Greenspan's least-squares fit of his attenuation measurements. The final adjusted values of  $Z$ , tabulated above, are within or near the range of uncertainty in Greenspan's determination of  $Z$  based on experimental values of the Eucken factor and his *ad hoc* combination of continuum theories.

A similar fit using the four-moment model yields  $Z = 6.36$  for N<sub>2</sub> and  $Z = 5.18$  for O<sub>2</sub>. These higher  $Z$  values compensate for the Eucken factor  $f = 1.4$  specified by the four-moment model.

The critical frequencies are

Gas	$r_{crit}(7)$	$r_{crit}(4)$
N <sub>2</sub>	0.34	0.49
O <sub>2</sub>	0.35	0.49
Air	0.35	...

Below the critical frequency, the seven-moment results compare very well with the measurements. The four-moment theory did not compare as well with experimental measurements in N<sub>2</sub> and O<sub>2</sub>, and was not calculated for air.

The graphs for N<sub>2</sub> and O<sub>2</sub> also show Monchick's theory<sup>26,29</sup> for local and absolute expansions truncated at nine polynomials. Also shown, for all three gases, is Greenspan's combined theory, and the Herzfeld, Rice, and Kneser, and quasi-Burnett theories.<sup>25</sup> Both calculations of Monchick<sup>29</sup> depart

from the  $N_2$  and  $O_2$  attenuation data near  $r = 2$  and rapidly tend to zero for small  $r$ . Comparison with sound speed measurements remains good. Greenspan's theory gives good agreement with his experiment for values of  $r$  near or greater than 0.5. The reason can be traced to the superiority of the Burnett theory (derived by the Chapman-Enskog method) in depicting sound attenuation in monatomic gases.<sup>37</sup> The attenuation from the Burnett theory is close to the experimental attenuation until  $r \sim 0.5$ , while the attenuations from the super-Burnett and the Navier-Stokes theories are close to the experimental attenuation only until  $r \sim 3$ . Monchick's theories, without considering the additional moments for internal degrees of freedom, are analogous to a "Navier-Stokes" theory derived by the Wang-Chang method. At the critical frequency, and certainly beyond it, only the present seven-moment theory seems to follow the trend of both the attenuation and speed ratio data. Even if our theory is not calculated beyond the critical frequency, it is the only self-consistent kinetic treatment that gives good agreement with experiment over the whole range in which it is proper to consider only the dispersion relationship.

Also shown in Fig. 7 for comparison are Meyer and Sessler's measurements in air at 20°C, an 11-moment theory for a monatomic gas of rigid spheres<sup>5</sup> and Meyer and Sessler's free-molecule theory.<sup>38</sup> Although not shown here, comparison, using the values of  $Z$  for Greenspan's data, with Sessler's measurements in nitrogen and oxygen at 20°C is also very good.

## VIII. CONCLUSION

The method of Sirovich and Thurber has been applied to obtain a dispersion relation for four- and seven-moment polyatomic kinetic model equations. The agreement with experimentally measured propagation and attenuation rates for oxygen and nitrogen is very good over a wide range of the rarefaction parameter. Asymptotic analysis of the four-moment model for small  $r$  yields qualitatively the same results for attenuation and propagation as found by Wang-Chang and Uhlenbeck, and phenomenological theories. These results furnished the starting iterates for the numerical calculation of the roots of the dispersion relation.

Numerical calculations show that the four-moment model, as compared with the seven-moment model,

<sup>37</sup> M. Greenspan, in *Dispersion and Absorption by Molecular Processes*, D. Sette, Ed. (Academic Press Inc., New York, 1962), Vol. 27, p. 73.

<sup>38</sup> E. Meyer and G. Sessler, *Ann. Physik* **149**, 151 (1957).

somewhat exaggerates the effects of  $c_i^j$  and  $f$ , although the variation of attenuation and propagation over the range of  $f$  and  $c_i^j$  for gases of interest is small. In the case where  $Z$  is very large, the polyatomic gas behaves as if it were a monatomic gas. The attenuation reaches a maximum value and the speed of propagation appears to approach the value for a monatomic gas. This case is often referred to as the "freezing of the internal degrees of freedom." This is similar to the results obtained by Brau.<sup>30</sup> As  $r$  grows unbounded for fixed  $Z$  the speed eventually approaches the equilibrium or adiabatic limit. In the case where the magnitude of  $Z$  is small or moderate and is of the same order as  $r$ , the maximum in the attenuation rate and the apparent approach to the "frozen speed" is not noticeable since the relaxation of the translational and internal degrees of freedom is strongly coupled.

As  $r$  decreases from unity, the attenuation rate would tend to zero more rapidly than these curves predict (e.g., as rapidly as the attenuation rates predicted by the Wang-Chang method), if the processes contributing to the attenuation and propagation of sound were merely the relaxation of the molecular degrees of freedom. While one must be careful in interpreting the physical meaning of the analytically continued sound mode, it seems plausible to attribute the slow decline of the attenuation rate to the streaming terms. The analytically continued portion of the sound mode for the polyatomic gas will furnish a good approximation to the experimentally measured sound mode even though the polyatomic kinetic model does fail to predict this mode above the critical frequency.

## ACKNOWLEDGMENTS

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## APPENDIX A

The dimensionless parameters introduced here are

$$\begin{aligned} \xi &= \mathbf{v}/RT_0, & \epsilon_i &= E_i/c_i^j T_0, \\ \omega_l &= (RT_0)^{3/2} f_l^0/n_0, & \rho_l &= \int \omega g_l d\xi, \\ \rho &= (n - n_0)/n_0, & u_1 &= U_1/(RT_0)^{1/2}, \end{aligned}$$

$$\begin{aligned} \tau_{tr} &= (T_{tr} - T_0)/T_0, & \bar{\epsilon} &= \sum_i E_i \exp\left(\frac{-E_i/kT_0}{c_v^i T_0}\right), & G &= c_v^i/k, & \nu_N &= -n_0 \beta_N / (\pi^{\frac{3}{2}} Q_0), \\ \langle \epsilon \rangle &= \frac{\langle E \rangle}{c_v^i T_0} - \bar{\epsilon}, & p_{11} &= (P_{11} - p)/p_0, & x &= \nu_N \hat{x} / (RT_0)^{\frac{1}{2}}, & t &= \nu_N \hat{t}, \\ S_{tr} &= (q_{tr})_1 / [p_0 (RT_0)^{\frac{1}{2}}], & S_{int} &= (q_{int})_1 / [p_0 G (RT_0)^{\frac{1}{2}}], & g_i &= (f_i - f_i^0) / f_i^0, & \phi_i &= (f_i - f_{0i}) / f_{0i}. \end{aligned}$$

APPENDIX B

Relation between relaxation parameters and transport coefficients

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$$\begin{aligned} -\nu_N \nu_{200} &= \frac{n \beta_{200}^{200}}{\pi^{\frac{3}{2}} Q} = -\frac{p}{[\mu]_1}, & -\nu_N \nu_{001} &= n \beta_{001}^{001} / \pi^{\frac{3}{2}} Q = -1 / [\tau_R]_1, \\ -\nu_N \nu_{010} &= \frac{n \beta_{010}^{010}}{\pi^{\frac{3}{2}} Q} = -\frac{2}{3} \frac{c_v^i}{k} / [\tau_R]_1, & -\nu_N \nu_{110} &= n (\frac{2}{3} \beta_{200}^{200} + \frac{5}{6} \beta_{010}^{010}) / \pi^{\frac{3}{2}} Q \\ &= -\frac{2}{3} \left(\frac{c_v^i}{c_v}\right)^2 p / [k]_1, & &= -\frac{\frac{2}{3} p}{[\mu]_1} - \frac{5}{9} \frac{c_v^i}{k} / [\tau_R]_1, \\ &\frac{n \beta_{010}^{001}}{\pi^{\frac{3}{2}} Q} = -\left(\frac{2c_v^i}{3k}\right)^{\frac{1}{2}} / [\tau_R]_1, & \frac{n \beta_{101}^{110}}{\pi^{\frac{3}{2}} Q} &= \frac{1}{2} \left(\frac{5k}{2c_v^i}\right)^{\frac{1}{2}} \frac{n \beta_{010}^{010}}{\pi^{\frac{3}{2}} Q}, \end{aligned}$$


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$$\begin{aligned} -\nu_N \nu_{101} &= \frac{n \beta_{101}^{101}}{\pi^{\frac{3}{2}} Q} = -\frac{2}{3} \frac{c_v^i}{k} \frac{1}{[\tau_R]_1} \\ &\cdot \left(1 + \frac{1}{3} \frac{c_v^i}{k} + \frac{2}{5} \frac{p[\tau_R]_1}{[\mu]_1} + \frac{1}{6} \frac{m[\lambda]_1}{(kp[\tau_R]_1)}\right) / \left(1 + \frac{4}{15} \frac{m[\lambda]_1}{(k[\mu]_1)} + \frac{2}{9} \frac{c_v^i}{k} \frac{m[\lambda]_1}{(kp[\tau_R]_1)}\right), \\ -\nu_N \nu_{300} &= -\frac{3}{2} \nu_N \nu_{200} = n \beta_{300}^{300} / \pi^{\frac{3}{2}} Q = \frac{3}{2} n \beta_{200}^{200} / \pi^{\frac{3}{2}} Q, \\ &= -\frac{3}{2} p / [\mu]_1. \end{aligned}$$