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Effect of the Collision Frequency on Boundary Value Problems in Kinetic Theory

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An approximate method in kinetic theory is applied to a model equation with velocity-dependent collision frequency. The Kramers problem is solved and an expression for the slip coefficient in terms of the collision frequency is obtained. The behavior of the solution in the kinetic layer and the far field is examined. The analytical form of the solution in the kinetic layer is shown to be independent of the collision frequency. In the far field, the solution is seen to contain a nonhydrodynamical term with an analytical form which depends on the asymptotic behavior of the collision frequency for high-speed molecules. An expression for the decay rate of the nonhydrodynamical portion of the solution is given in terms of the collision frequency. This expression is seen to agree with other known results.

I. INTRODUCTION

The present investigation serves a twofold purpose. On the one hand, it extends an earlier investigation of the Kramers problem[1] to a gas with a velocity-dependent collision frequency. On the other hand, it yields, perhaps for the first time, specific quantitative information for problems involving model equations with a velocity-dependent collision frequency.

Only partial success has been achieved in applying the equations of kinetic theory to various problems. In almost all cases, the basic equation employed assumes a constant collision frequency for the molecules. One of the most disturbing and interesting results first emerged in the problem of steady-state sound propagation investigated by Kleitman and Ostrowski,[2] who found that even in the low-frequency limit, the dominant effect at large distances from the oscillating boundary is

$$O[\exp(-y^{2/3})] \qquad (y \to \infty). \tag{1}$$

This is in contrast to the classical result from fluid mechanics of exponential decay $O[\exp(-\lambda y)]$. Cercignani, in considering the steady-state shear wave[3] also found the same result, i.e., decay for large distances of the form (1). That this is not exclusively a linear effect is clear from the shock wave study of Lyubarskii[4] where it is found that the falloff at infinity is of the form (1), instead of the exponential decay of the disturbance predicted by the Navier–Stokes equations.

All of these results are disturbing since the slowly varying far field has always been thought to be the proper preserve for the hydrodynamical equations. The origin of this strange effect is not difficult to trace, and is basically due to the streaming term in a Boltzmann-type equation. In part, it is due to the form of this term for high-speed molecules. Stated in another way, in a gas with constant collision frequency, the high-speed molecules have extremely long free paths, and as a result carry nonhydrodynamical effects to the far field.

Although sometimes in a disguised form, the same falloff, (1), due to the streaming effect, is present in all steady or steady-state boundary value problems having a Boltzmann equation with constant collision frequency. The Kramers problem is an example. A gas is contained in a half-space bounded by a stationary wall with a given velocity gradient k at infinity. The velocity of the gas can be written as

$$u(y) = \Delta + ky + \hat{u}(y), \tag{2}$$

where Δ is the wall slip. For a constant collision frequency gas, it can be shown that[3]

$$\ln \hat{u}(y) = O(-y^{2/3}) \tag{3}$$

for large y. Here, the minus sign has been placed in the O-symbol to emphasize that this is a decay term.

It is clear that the hydrodynamical part of the solution (i.e., the linear term) shields the contribution of $\hat{u}(y)$. A similar discussion can be given for steady flow past a finite body.[5] There, the hydrodynamical part is algebraic and hence also shields the streaming effect term, (3).

Spectral studies[6] already indicate that the streaming effect term has to be modified for velocity-dependent collision frequencies. To discuss this, we need only classify the nature of the collisional frequency, $\nu(\xi)$, for high-speed molecules ($|\xi| \to \infty$). We write

$$\alpha = \lim_{\xi\to\infty} \frac{\ln \nu(\xi)}{\ln |\xi|}. \tag{4}$$

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For constant collision frequency molecules, $\alpha = 0$; at the other extreme, for rigid sphere molecules or potentials with radial cutoff, $\alpha = 1$.[7]

A spectral analysis shows that for $\alpha = 1$, the streaming effect is lost, and its contribution to the far field falls off exponentially. For $0 < \alpha < 1$, it has been conjectured[6] that this falloff is $O[\exp(-x^{\beta})]$, where $\frac{2}{3} < \beta < 1$, but no explicit form has yet been given. A major result of this paper is that the falloff is given by

$$O[\exp(-y^{2/(3-\alpha)})], \tag{5}$$

i.e.,

$$\beta = \frac{2}{3-\alpha}$$

with α defined by (4). This result is shown within the framework of the Kramers problem, i.e., from the form (2). The same result has also been found in the steady-state transverse[8] and longitudinal wave problems,[9] and it seems warranted to conclude the general result.

In the same vein, we mention that, as in Ref. 1, the present work is an approximate treatment, and the result (5) must, therefore, be regarded as a conjecture. However, this result has been considered in relation to the steady-state oscillation problem and a rigorous proof in some generality has been given.[10]

II. A GENERAL APPROXIMATE PROCEDURE

In treating the Kramers problem we plan to use the approximate method given in Ref. 1. For completeness, we now review this method.

We consider steady flows, and assume that the Boltzmann equation may be written in the form

$$\boldsymbol{\xi}\cdot\frac{\partial}{\partial \mathbf{x}} f + \nu f = kf, \tag{6}$$

where ν is the collision frequency, and $\nu = \nu(f)$. The operator k is, in general, nonlinear. Equation (6) is subject to boundary conditions of the form

$$H(\boldsymbol{\xi}\cdot\mathbf{n})f(\mathbf{x}_0, \boldsymbol{\xi}) = H(\boldsymbol{\xi}\cdot n)F[\mathbf{x}_0, \boldsymbol{\xi}; f(\mathbf{x}_0, -\boldsymbol{\xi}\cdot\mathbf{n})], \tag{7}$$

where $\mathbf{x}_0$ is a point on the boundary, $\mathbf{n}$ is the outward normal, and F is a known functional.

Some observations regarding the solution of (6) may be made. The operator on the left-hand side of (6) may be solved in the sense that the solution to

$$(\boldsymbol{\xi}\cdot\nabla)f_h = -\nu f_h$$

can be written as

$$\begin{aligned} f(\mathbf{x}, \boldsymbol{\xi}) &= f(\mathbf{x}_0 + k\boldsymbol{\xi}, \boldsymbol{\xi}) \\ &= H(k)f_0(\mathbf{x}_0, \boldsymbol{\xi}) \exp\left[-\int_0^k \nu(\mathbf{x}_0 + \boldsymbol{\xi}s, \boldsymbol{\xi})\, ds\right]. \end{aligned} \tag{8}$$

The Chapman–Enskog series[11] may be regarded as a particular solution to (6), which we denote as f_p. It is clear, however, that difficulties arise if one attempts to write the solution to (6) as

$$f = f_p + f_h, \tag{9}$$

where f_h is chosen so that (9) satisfies the boundary conditions. In particular, two additional relations must be satisfied if (9) is to be a solution of (6):

$$\begin{aligned} \nu(f_p + f_h) &= \nu(f_p), \\ k(f_p + f_h) &= k(f_p). \end{aligned} \tag{10}$$

Even if this difficulty could be overcome, however, further complications arise. The function $(f_p + f_h)$ can be represented entirely in terms of hydrodynamical moments since by substituting (9) into the conservation equations a solution to the Boltzmann equation is obtained by solving hydrodynamical equation. Such a series solution may not exist, however, since convergence of the Chapman–Enskog series is certainly not guaranteed. A more serious objection is the nonexistence of hydrodynamical derivatives at the boundary, where the representation becomes meaningless.[12]

The above discussion may still serve as a guide if we abandon the search for an exact solution. The problem of convergence may be eliminated if the Chapman–Enskog series is truncated after a finite number of terms. It will also be convenient to regard the representation (9) as an operator on a set of unknown functions which have no direct connection with the hydrodynamical moments. The conservation equations

$$\frac{\partial}{\partial x_i}\int \xi_i[1, \boldsymbol{\xi}, \xi^2] f\, d\boldsymbol{\xi} = 0 \tag{11}$$

may be regarded as constraints on the unknown functions.

Relation (10) poses an added problem; it is highly unlikely that it can ever be satisfied. One solution is obviously $f_h \equiv 0$. It may be noted from (8) that f_h will be small for distances greater than a few mean free paths from the boundary. We can thus expect to satisfy (10) with good accuracy by applying the condition only at the boundary, i.e.,

$$\begin{aligned} \nu[f_p(x_0) + f_h(x_0)] &= \nu[f(x_0)], \\ k[f_p(x_0) + f_h(x_0)] &= k[f_p(x_0)]. \end{aligned} \tag{12}$$

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These relations supply important conditions between the derivatives of the unknown quantities at the boundaries. By the same reasoning, if we deal with near-continuum flows,

$$H(-\boldsymbol{\xi}\cdot\mathbf{n})f_h(\mathbf{x}_0, \xi) \sim 0$$

since this represents the contribution from another wall or remote parts of the same wall.

These conditions are sufficient to determine the unknown quantities in representation (9); thus, the approximate distribution function is completely determined. The macroscopic quantities of interest may be obtained by taking the corresponding moment of the distribution function.

As in Ref. 1 we will truncate the Chapman–Enskog series after the first two terms. This corresponds to the Navier–Stokes level of approximation, and it may be noted that (11) reduces to the Navier–Stokes equations in the interior of the flow.

III. THE KRAMERS PROBLEM

We now apply this procedure to the Kramers problem.[3,13] A gas is contained in the half-space bounded by a stationary wall, $y = 0$. The velocity, $u'(y')$ has a constant gradient k' at infinity. A constant temperature and density is assumed throughout the flow field. The distribution function f is written as

$$f = f^\circ(1 + g)$$

and terms of $O(g^2)$ are neglected; i.e., the linearized equation will be considered.

As governing equation, we choose the following model due to Cercignani[13]:

$$\xi_2 \frac{\partial g}{\partial y} + \nu(\xi)g = \nu(\xi)\xi_1 \int \xi_1\nu(\xi)g\Omega\, d\boldsymbol{\xi}. \tag{13}$$

The following normalization has been introduced, denoting dimensional quantities by primes:

$$y = \frac{y'\nu_1}{(RT_0)^{1/2}}, \quad \xi = \frac{\xi'}{(RT_0)^{1/2}}, \quad \nu(\xi) = \frac{\nu'(\xi)}{\nu_1}, \quad k = \frac{k'}{\nu_1},$$

$$\Omega = \frac{\exp(-\xi^2/2)}{(2\pi)^{3/2}}, \quad \text{and} \quad \nu_1 = \int \xi_1^2\nu'(\xi)\Omega\, d\xi.$$

The function $\nu'(\xi)$, which we leave open for the present, represents the collision frequency. We assume for the boundary condition a combination of diffuse and specular reflection. For a stationary wall this is given by

$$H(\xi_2)g(y = 0, \xi_2) = H(\xi_2)(1 - \epsilon)g(y = 0, -\xi_2),$$

where the parameter ϵ represents the fraction of diffusely reflected molecules.

Using the method previously described, for the particular solution we write the first two terms of the Chapman–Enskog series

$$g_p = \xi_1\left[1 - \frac{\xi_2}{\nu(\xi)}\frac{\partial}{\partial y}\right]\hat{u}(y), \tag{14}$$

where $\hat{u}(y)$ denotes the velocity moment of g_p. We have not given the details of the Chapman–Enskog procedure leading to (14) since it is almost trivially executed. It is interesting to note that the function $A(\mathcal{C})$, in the notation of Chapman and Cowling,[14] is found exactly here (i.e., $A = \nu^{-1}$), whereas for the full Boltzmann equation it is never really found except for Maxwell molecules, where it is a constant.

Another interesting feature along the same lines is that the viscosity may be exactly computed, and, in fact, is

$$\mu = \int \frac{\Omega\xi_1^2\xi_2^2}{\nu(\xi)}\, d\boldsymbol{\xi}. \tag{15}$$

It will be recalled that in the Chapman–Enskog procedure, this cannot be exactly computed except for Maxwell molecules.

The choice for the homogeneous solution is obviously

$$g_h = H(\xi_2)\left[-\epsilon\xi_1\hat{u}(0) + (2 - \epsilon)\frac{\xi_1\xi_2}{\nu(\xi)}\frac{\partial\hat{u}}{\partial y}\bigg|_0\right] \cdot \exp\left(-\frac{\nu(\xi)y}{\xi_2}\right).$$

Our approximate solution is taken as

$$g \approx g_p + g_h. \tag{16}$$

Once again we note that $\hat{u}(y)$ is not the macroscopic fluid velocity, although it tends toward it in the interior flow [see (18) below].

The condition (12) immediately provides the microscopic slip condition at the wall:

$$\frac{\partial\hat{u}}{\partial y}\bigg|_0 = \left(\frac{\pi}{2}\right)^{1/2}\frac{\epsilon}{(2 - \epsilon)}\hat{u}(0). \tag{17}$$

This relation should not be confused with the macroscopic slip condition which will be obtained directly from the solution.

The representation (16) is then substituted into the momentum equation

$$p_{12} = -\int \xi_1\xi_2 g\Omega\, d\boldsymbol{\xi} = \text{const}$$

to obtain a differential equation for $u(y)$. This equation may be solved completely using (17) and the observation that $\partial u/\partial y \to \partial\hat{u}/\partial y$ as $y \to \infty$.

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Representation (16) is thus completely determined, and the hydrodynamical quantity of interest, in this case the velocity, is obtained by taking the corresponding moment of the approximated distribution function.

The solution is given by

$$u(y) = k\left[y + \frac{\beta(2\pi)^{1/2}}{c}\left(1 - \frac{\epsilon}{2}\right) + \frac{\epsilon}{24c}\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty \frac{r^7}{\nu^2(r)}\exp\left(-\frac{r^2}{2}\right)dr + \frac{\epsilon}{2\pi c}\int_{\xi_2>0} J(\xi)\exp\left(-\frac{\nu(\xi)y}{\xi_2} - \frac{\xi^2}{2}\right)d\xi\right], \quad (18)$$

where

$$\beta = \frac{2}{15(2\pi)^{1/2}}\int_0^\infty \frac{r^6}{\nu(r)}\exp\left(-\frac{r^2}{2}\right)dr,$$

$$c = \epsilon\left(\frac{\epsilon\pi\beta}{2(2-\epsilon)} + 1\right),$$

and

$$J(\xi) = \xi_1^2\left[1 - \frac{\xi_2}{\nu(\xi)}\left(\frac{\pi}{2}\right)^{1/2}\right]\left[\frac{\xi_2^2}{\nu(\xi)} - \beta\right].$$

The parameter β is seen to be exactly the normalized viscosity, (15), as computed from the Chapman–Enskog expansion.

In the Appendix we show that the integral term of (18) falls off like $\exp(-y^{2/(3-\alpha)})$ for y large, where $0 \leq \alpha \leq 1$ depending on the behavior of the collision frequency for large molecular speeds. It is clear, therefore, that the slip coefficient is given by

$$\lambda = \frac{\beta(2\pi)^{1/2}}{c}\left(1 - \frac{\epsilon}{2}\right) + \frac{\epsilon}{24c}\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty \frac{r^7}{\nu^2(r)}\exp\left(-\frac{r^2}{2}\right)dr. \quad (19)$$

To facilitate comparison with other results we introduce the dimensional slip coefficient

$$\lambda' = \left(\frac{2}{\pi}\right)^{1/2}\lambda\ell = z\ell,$$

where ℓ is the mean-free-path

$$\ell = \frac{1}{\nu_1}\left(\frac{\pi R T_0}{2}\right)^{1/2}.$$

In the limit of purely diffuse reflections and constant collision frequencies, we obtain, as in Ref. 1

$$\lambda' = 1.1667\ell.$$

This was shown to be within 1.5% of the exact calculation of the slip coefficient for the constant collision frequency model. In the limit of purely specular we find

$$\lambda' = \frac{2\beta}{\epsilon}\ell.$$

For a constant collision frequency, this reduces to the exact result found by Cercignani.[15]

We note briefly that the asymptotic solution to the Couette flow problem (i.e., a few mean-free paths from the plates) may be obtained directly from the Kramers problem. The wall slip is given by

$$\frac{zU\ell}{(L + z\ell)},$$

where $2L$ is the plate separation and U, $-U$ are the velocities of the top and bottom plates.

An expansion of the solution in the kinetic layer indicates that the analytical form of the solution in this region is not affected by the collision frequency. From the Appendix, the velocity for y small is given by the expansion

$$u(y) = \frac{k\beta(2\pi)^{1/2}}{c} - \left[\frac{\beta\epsilon k}{2c}\int_0^\infty r^3\nu(r) \cdot \exp\left(-\frac{r^2}{2}\right)dr\right]y\ln y + O(y).$$

IV. CONCLUDING REMARKS

An approximate method has been used to solve the Kramers problem with velocity dependent collision frequency. The solution is seen to be of the form

$$u = ky + \Delta + \tilde{u}(y),$$

where $\tilde{u}(y) \to 0$ at infinity. The kinetic layer and the hydrodynamical portion of the solution have analytical forms which are independent of the collision frequency. The behavior of $\tilde{u}(y)$ in the far field depends on the form of the collision frequency for large molecular velocities, and is given by

$$\ln \tilde{u}(y) = O(-y^{2/(3-\alpha)}), \quad (20)$$

where $0 \leq \alpha \leq 1$. Evidence indicates that such a term is always present in steady or steady-state boundary value problems except, perhaps, for certain special boundary conditions. However, its effect is shielded in problems where the hydrodynamical portion of the solution decays at a slower rate.

In problems such as steady-state shear wave propagation, a nonhydrodynamical term of the form (20) is seen to dominate in the far field. In such cases, the approximate method can be used to determine the region in which the flow is hydro-

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dynamical, and under what limits the solution approaches that predicted by the Navier–Stokes equation. This will be described in a following paper in which a study of time-dependent boundary value problems will be presented. For unsteady motions, the behavior of the far field will be seen to have an analytical form quite different from that of the steady or steady-state case.

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APPENDIX A. LARGE ARGUMENT ASYMPTOTICS

Consider the integral

$$I_{nm}(y) = \int_{\xi_2>0} \xi_1^{2n}\xi_2^{m} \exp\left(-\frac{\nu(\xi)y}{\xi_2} - \frac{\xi^2}{2}\right) d\xi$$

for $y \to \infty$. Simple considerations show that in the region of integration the exponential function

$$f(\xi, y) = \left[\frac{\nu(\xi)}{\xi_2}\, y + \frac{\xi^2}{2}\right]$$

has a single global maximum given by

$$\xi_1 = 0, \qquad \xi_3 = 0, \qquad \xi_2 = \zeta(y),$$

where $\zeta(y)$ is determined from

$$\zeta + \frac{\nu'(\zeta)y}{\zeta} - \frac{\nu(\zeta)y}{\zeta^2} = 0.$$

The asymptotic evaluation of I_{nm}, therefore, follows from the multidimensional use of Laplace's method. To obtain this we first write

$$\beta_{ij} = \frac{\partial^2}{\partial\xi_i\,\partial\xi_j} f(\xi_1 = 0, \xi_2 = \zeta, \xi_3 = 0; y),$$

$$\gamma = f(\xi_1 = 0, \xi_2 = \zeta, \xi_3 = 0; y)$$

and introduce the variables

$$\mathbf{n} = (\eta_1, \eta_2, \eta_3) = [\xi_1, \xi_2 - \zeta(y), \xi_3].$$

The lowest order term is seen to be

$$I_{nm}(y) \sim [\zeta(y)]^m \exp[-\gamma(y)] \cdot \int_{-\infty}^{\infty} \eta_1^{2n} \exp\left(-\frac{\beta_{ij}\eta_i\eta_j}{2}\right) d\mathbf{n}.$$

Computation is simplified by the fact that β_{ij} is diagonal, and we note that

$$\beta_{11} = \beta_{33} = \frac{\nu(\zeta)y}{\zeta^3},$$

$$\beta_{22} = 3 + \frac{\nu''(\zeta)y}{\zeta}.$$

Therefore,

$$I_{nm} \sim (2\pi)^{3/2} \frac{(2n)!}{2^n n!} \frac{\zeta^{m+3n+3}}{[\nu(\zeta)y]^{n+1}} \cdot\left(\frac{\zeta}{3\zeta + \nu''(\zeta)y}\right)^{1/2} \exp[-\gamma(y)]. \qquad \text{(A1)}$$

The full development of I_{nm} may be found by formal expansion. Under mild assumptions on $\nu(\xi)$ this may be rigorously shown to give the asymptotic expansion of I_{nm}. We will, however, avoid these details.

The specific form of (A1) depends on the function $\nu(\xi)$. In general, one finds the following expansion for ν at infinity:

$$\nu = k|\xi|^\alpha + \alpha V + O(1),$$

where $0 \leq \alpha \leq 1$, and k and V are constants. The terms in the development of ζ are determined by

$$\zeta + \frac{(\alpha-1)ky}{\zeta^{2-\alpha}} - \frac{\alpha V y}{\zeta^2} = 0$$

and the leading terms in the asymptotic expansion of the exponential, $\exp[-\gamma(y)]$, are

$$\gamma = (1-\alpha)^{(\alpha-1)/(3-\alpha)}\left(\frac{3-\alpha}{2}\right)(ky)^{2/(3-\alpha)} + R(\alpha, y).$$

The first term is valid for $0 \leq \alpha \leq 1$. [Note $\lim_{\alpha\to 1}(1-\alpha)^{(\alpha-1)/(3-\alpha)} = 1$.] The second term is given by

$$R(\alpha, y) = \begin{cases} \dfrac{2\alpha V}{3-\alpha}[(1-\alpha)k]^{-1/(3-\alpha)} y^{(2-\alpha)/(3-\alpha)} + O(\alpha y^{(2-2\alpha)/(3-\alpha)}), & \alpha < 1 \\ \frac{3}{2} V^{2/3} y^{2/3}, & \alpha = 1. \end{cases}$$

We conclude that, for $0 \leq \alpha \leq 1$,

$$\lim_{y\to\infty} \ln I_{nm} = O(-y^{2/(3-\alpha)}),$$

where the minus sign has been placed in the O symbol to emphasize that this is a decay term.

In the cases $\alpha = 0$ and $\alpha = 1$, there is no difficulty in representing the asymptotic form of I_{nm} explicitly. In fact,

$$I_{nm}(y; \alpha = 0) \sim \frac{(2n)!}{n!\,2^n} \frac{(2\pi)^{3/2}}{\sqrt{3}} \cdot \exp[-\tfrac{3}{2}(ky)^{2/3}][(yk)^{m/3} + O(y^{(m-1)/3})]$$

which is well known in the literature,[16] and

$$I_{nm}(y; \alpha = 1) \sim \frac{(2\pi)^{3/2}(2n)!}{2^n n!\sqrt{3}} \frac{V^{(m+2n+2)/3}}{k^{n+1}} y^{(m-n-1)/3}[1 + O(y^{-1/3})] \cdot \exp[-ky - \tfrac{3}{2}(Vy)^{2/3}].$$

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APPENDIX B. SMALL ARGUMENT ASYMPTOTICS

In considering an expansion of $I_{nm}(y)$ for y small, for convenience, we confine our attention to functions $\nu(\xi)$ for which

$$\lim_{\xi_2\to 0}\frac{\nu(|\xi|)-\nu((\xi_1^2+\xi_3^2)^{1/2})}{\xi_2}$$

exists for all values of ξ_1 and ξ_3. We write

$$I_{nm}(y)=\int_{\xi_2>0}\xi_1^{2n}\xi_2^m\exp\left(-\frac{\nu(\hat{\xi})y}{\xi_2}-\frac{\xi^2}{2}-yF(\xi)\right)d\xi,$$

where

$$F(\xi)=\frac{\nu(|\xi|)-\nu(\hat{\xi})}{\xi_2}$$

and

$$\hat{\xi}=(\xi_1^2+\xi_3^2)^{1/2}.$$

Expanding the exponential involving $F(\xi)$ in a power series in y, we have

$$I_{nm}(y)=\int_{\xi_2>0}\sum_{j=0}^{\infty}y^j\frac{[-F(\xi)]^j}{j!}\xi_1^{2n}\xi_2^m\cdot\exp\left(-\frac{\nu(\hat{\xi})y}{\xi_2}-\frac{\xi^2}{2}\right). \quad \text{(B1)}$$

It is known that[16]

$$\int_0^\infty G(\zeta)\zeta^m\exp\left(-\frac{\lambda y}{\zeta}-\frac{\zeta^2}{2}\right)d\zeta=\sum_{k=0}^{m}\frac{(-\lambda y)^k}{k!}\int_0^\infty G(\zeta)\zeta^{m-k}\exp\left(-\frac{\zeta^2}{2}\right)d\zeta-\frac{G(0)(-\lambda y)^{m+1}}{(m+1)!}\ln y+O(y^{m+1}),$$

where λ is a constant.

Using this result in (B1), and noting that $\nu(\hat{\xi})$ is a constant with respect to ξ_2, after a regrouping of the expansion one obtains

$$I_{nm}(y)=\sum_{k=0}^{m}\frac{(-y)^k}{k!}\int_{\xi_2>0}\xi_1^{2n}\xi_2^{m-k}[\nu(|\xi|)]^k\exp\left(-\frac{\xi^2}{2}\right)d\xi-\frac{(-y)^{m+1}\ln y}{(m+1)!}\iint_{-\infty}^{\infty}\xi_1^{2n}[\nu(\hat{\xi})]^{m+1}\exp\left(-\frac{\hat{\xi}^2}{2}\right)d\xi_1\,d\xi_3+O(y^{m+1}).$$

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[1] L. Sirovich, Phys. Fluids **12**, 586 (1969).
[2] H. S. Ostrowski and D. J. Kleitman, Nuovo Cimento **44 B**, 49 (1966).
[3] C. Cercignani, Lecture Notes, Division of Applied Mathematics, Brown University (1965).
[4] G. Ya. Lyubarskii, Zh. Eksp. Teor. Fiz. **40**, 1050 (1961) [Sov. Phys. JETP **13**, 740 (1961)].
[5] L. Sirovich, in *Rarefied Gas Dynamics*, edited by L. Talbot (Academic, New York, 1961), p. 283.
[6] L. Sirovich and J. K. Thurber, J. Math. Phys. **10**, 239 (1969).
[7] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1953), Sec. 5.4.
[8] L. Sirovich and H. York (to be published).
[9] H. York, Ph.D. thesis, Brown University (1969).
[10] T. G. Richardson and L. Sirovich, J. Math. Phys. (to be published).
[11] Reference 7, Chap. 7.
[12] Y. Pao, Phys. Fluids **9**, 409 (1966).
[13] C. Cercignani, Ann. Phys. (N. Y.) **40**, 469 (1966).
[14] Reference 7, Sec. 7.31.
[15] C. Cercignani, J. Math. Anal. Appl. **10**, 568 (1965).
[16] M. Abramowitz and I. Stegun, Editors, *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series No. 55 (U. S. Government Printing Office, 1964), Sec. 27.5.

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