Structure of Three-Dimensional Supersonic and Hypersonic Flow

T. H. CHONG AND LAWRENCE SIROVICH
Division of Applied Mathematics and Center for Fluid Mechanics
Brown University, Providence, Rhode Island 02912
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The Oseen equations in three dimensions for an arbitrary gas are considered. In particular, the cases of supersonic and hypersonic flow are studied. The asymptotic flow field displaying all the dissipative effects is found in terms of known functions. The explicit structure of the Mach cone and the wake is presented. The latter is shown to decouple, to lowest order, into a viscous and to a heat conducting wake, containing only entropy and vorticity perturbations, respectively. The above results are shown, by means of the nonlinear Navier–Stokes equations, to correctly describe the far field flow past a finite body.

I. INTRODUCTION

The present paper in the company of an earlier paper[1] completes a study of the compressible Oseen equations. In Ref. 1 the two-dimensional case is solved and in the present paper the solution of the three-dimensional problem is given. In all cases an explicit asymptotic form of the fundamental solution in terms of tabulated functions is now available. The uses of the fundamental solution as well as the nature of the asymptotic solution has been amply discussed in Ref. 1, and a short digest of this discussion appears in Sec. II of this paper. In this introduction we wish to discuss the use of the Oseen fundamental solution in describing the far field flow for steady motion past a body. In particular we wish to evaluate the present linear theory with respect to the nonlinear Navier–Stokes equations.

We start this discussion with the two-dimensional results from Ref. 1. We denote the total drag (per unit width) on a body by $-\hat{D}$. To illustrate the flow at large distances we consider the fluid velocity perturbation parallel to the upstream velocity U and for simplicity we ignore the effects of lift and energy flux. It then follows from Ref. 1 that to lowest order

$$u_1 \sim \frac{-\hat{D}[\exp(-y^2U/4\zeta x)]}{(RT_0)^{1/2}\mu(4\pi\zeta Ux)^{1/2}} + \frac{\hat{D}}{\mu(RT_0)^{1/2}2U(M^2-1)(\pi\alpha x)^{1/2}} \cdot\left\{\exp\left[-\frac{[y-x/(M^2-1)^{1/2}]^2}{\alpha x}\right] + \exp\left[-\frac{[y+x/(M^2-1)^{1/2}]^2}{\alpha x}\right]\right\}.$$

(For the normalization see Sec. II.) The basic assumption characterizing the linearization of the nonlinear Navier–Stokes equations is that

$$u_1\frac{\partial}{\partial x}u_1 \ll \nabla^2 u_1. \qquad (1)$$

All quantities are dimensionless under a normalization which effectively leaves the Reynolds number $O(1)$. It suffices to consider condition (1) in the neighborhood of a Mach line, and we find

$$\frac{\hat{D}}{\mu(RT_0)^{1/2}} \ll 1. \qquad (2)$$

But since[2] $D/[\mu(RT)^{1/2}] = O(\mathrm{Re}\ M)$ (Re is the Reynolds number based on body size) condition (1) is violated unless the flow around the body is well into the Knudsen flow regime.

It is intuitively clear that the far field flow past a body is a continuum flow even though the body itself is in Knudsen flow. This has, in fact, been demonstrated directly via a kinetic theory approach.[3] Therefore according to (2), the two-dimensional far field flow past an object in Knudsen flow is self-consistently linear, and governed by the Oseen equations. Otherwise, unless there is some other special reason for a small value of $\hat{D}$ the linear theory does not apply there. [This does not imply that the linear theory is entirely invalid. For example, in considering flow past a thin body, an entirely different perturbation parameter, the thickness ratio, enters. Thus, although the far field is not linear unless (1) holds, there is an intermediate region of validity for the linear theory.]

It is interesting to note that according to the expression for u_1 above, the far field perturbations are small (provided that the body is finite) even where the linear theory is invalid. We have considered this point elsewhere and have demonstrated that for finite bodies independently of the size of the ratio $\hat{D}/\mu(RT_0)^{1/2}$, the far field is governed by a system of Burgers' equations, and the leading term for the far field flow may be given explicitly.

We next consider the three-dimensional case and again letting $-\hat{D}$ denote the drag we consider the u_1 perturbation at large distances and condition (1). This discussion is indicated in Sec. V and the three-dimensional counterpart to (2) is,

$$\frac{\rho_0 \hat{D}}{\mu^2(x_2^2 + x_3^2)^{3/8}} \ll 1. \tag{3}$$

Again, we have restricted our attention to the neighborhood of the Mach cone. It, therefore, follows that independently of the body (finite), at sufficiently large distances the flow field is self-consistently linear.

We mention in passing that when $(x_2^2 + x_3^2)^{1/2}$ is such that (3) is not valid, a nonlinear theory resembling that of the Burgers' equation has been derived and this "not too distant" far field may be analytically described.

To conclude this introduction, we comment on the relation of the results of this paper and its relation to other work. In Ref. 1 the two-dimensional problem was solved by using a boundary layer analysis on the wake and on the Mach cone, and the same method may be applied to the problem studied here. (In fact, Salathe[4] in an independent investigation, uses the formulation and method of Ref. 1 to study the present problem.) We will, however, use a different and much simpler approach, based on the reduction of the problem, by Fourier transforms, to the evaluation of integrals. The integrals are then shown to fall into a class which have already been asymptotically investigated.[5,6] From this analysis it is found that as in the two-dimensional case,[1] the wake decouples into two different wakes. One is structured by the viscosity coefficient and contains only vorticity disturbances. The other is structured by the heat conductivity and contains only entropy perturbations. (It should be noted that the wake alone was studied earlier by Ryzhov.[7]) The analytical forms describing these wakes involve only elementary functions and the dissipative structure of the Mach cone is also explicitly found in closed form.

It should be noted that Ref. 1 and the present work are greatly different than the pioneering contributions in dissipative gas dynamics by Lagerstom, Cole, and Trilling[8] and Cole and Wu.[9] In addition to their restriction to two dimensions most of their work is of a qualitative value (except in Ref. 9 where a special Prandtl number and the ideal gas yields a solution) since a number of assumptions were made in order to achieve specific results. Also, though their modal decomposition gives an interesting view of a flow, we do not find it necessary here. This modal decomposition may, however, be easily derived from our final results.

Finally, although we do not consider nonlinear theory here, we mention the important contribution by Chu and Kovaznasy[10] in this direction. They consider nonlinear theory by following the above-mentioned modes in their interaction. With the results in Ref. 1 and here, this may be carried out in more specific detail.

II. FORMULATION OF PROBLEM

For completeness in this section we briefly review the formulation given in Ref. 1.

We consider a steady flow past a body whose surface s is denoted by $s(\mathbf{x}) = 0$. For convenience, we specify a steady internal flow inside the body and treat s as a singular surface. The discontinuous form of the Navier–Stokes equation is

$$\nabla\cdot(\rho\mathbf{u}) = 0, \tag{4}$$

$$\nabla\cdot(\rho\mathbf{u}\mathbf{u} + p\mathbf{1} - \mathbf{P}) = [p\mathbf{1} - \mathbf{P}]\cdot\mathbf{n}\delta(s), \tag{5}$$

$$\nabla\cdot\left[\rho\mathbf{u}\left(e_i + \frac{u^2}{2}\right) + p\mathbf{u} - \mathbf{P}\cdot\mathbf{u} + \mathbf{Q}\right] = [\mathbf{Q}]\cdot\mathbf{n}\delta(s), \tag{6}$$

$$\frac{P_{ij}}{\mu} = u_{i,j} + u_{j,i} + \left(\frac{\beta}{\mu} - \frac{2}{3}\right)\nabla\cdot\mathbf{u}\delta_{ij}, \tag{7}$$

$$\frac{\mathbf{Q}}{K} + \nabla T = [T]\mathbf{n}\delta(s). \tag{8}$$

No restriction will be placed on the dissipative coefficients μ, K, and β, and we write

$$p = p(\rho, T), \quad e = e(\rho, T), \quad p - \rho^2\frac{\partial e}{\partial \rho} = T\frac{\partial p}{\partial T}.$$

The last is the compatibility relation for the first two.

A. Oseen Approximation

To normalize our variables, we introduce the following upstream equilibrium quantities:

$$p_0,\ \rho_0,\ T_0,\ \mathbf{u}_0,\ a_0 = \left[\left(\frac{\partial p_0}{\partial \rho_0}\right)_{T_0}\right]^{1/2},\ c_v = \left(\frac{\partial e_0}{\partial T_0}\right)_{\rho_0},$$

$$c_0 = \left[\left(\frac{\partial p_0}{\partial \rho_0}\right)_{s_0}\right]^{1/2}. \tag{9}$$

Then, we define the following dimensionless quantities:

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{L}, \quad \tilde{\rho} = \frac{\rho - \rho_0}{\rho_0}, \quad \tilde{\mathbf{u}} = \frac{\mathbf{u} - \mathbf{u}_0}{a_0},$$

$$\tilde{T} = \left(\frac{c_v}{a_0^2 T_0}\right)^{1/2} (T - T_0), \quad \tilde{\mathbf{P}} = \frac{\mathbf{P}}{\rho_0 a_0^2}, \tag{10}$$

$$\tilde{\mathbf{Q}} = \frac{\mathbf{Q}}{\rho_0 a_0^2 (c_v T_0)^{1/2}}, \quad \mathbf{U} = \frac{\mathbf{u}_0}{a_0}, \quad \tilde{p} = \frac{p - p_0}{\rho_0 a_0^2}.$$

For the time being, the normalization scale L is left unspecified.

Substituting these into the governing equations (4)–(8), carrying only the first-order terms (Oseen approximation) and dropping the tildes, we get

$$\mathbf{U}\cdot\nabla\rho + \nabla\cdot\mathbf{u} = 0,$$

$$\mathbf{U}\cdot\nabla\mathbf{u} + \nabla\rho + \chi\nabla T - \zeta\nabla^2\mathbf{u} - \eta\nabla\nabla\cdot\mathbf{u} = \mathbf{n}\cdot(p\mathbf{1} - \mathbf{P})\delta(s), \tag{11}$$

$$\mathbf{U}\cdot\nabla T + \chi\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{Q} = (\mathbf{n}\cdot\mathbf{Q})\delta(s) - \chi\mathbf{U}\cdot(p\mathbf{n} - \mathbf{P}\cdot\mathbf{n})\delta(s),$$

$$\mathbf{Q} = -\xi\nabla T + \xi\mathbf{n}T\delta(s),$$

where

$$\eta = \frac{\beta + \frac{1}{3}\mu}{\rho_0 a_0 L}, \quad \zeta = \frac{\mu}{a_0\rho_0 L}, \quad \xi = \frac{K}{\rho_0 c_v L a_0},$$

$$\chi = (\gamma - 1)^{1/2}, \quad \gamma = \frac{c_p}{c_v} = \frac{c_0^2}{a_0^2}.$$

B. Reduced Problem

For a typical boundary value problem of uniform flow past a fixed body, let the solution of (11) be $\boldsymbol{\omega}$, the Oseen operator be $\mathbf{L}$, and the source terms be $\mathbf{F}$. Then, formally, we can write (11) as

$$\mathbf{L}\boldsymbol{\omega} = \mathbf{F}\ \delta(s). \tag{12}$$

The fundamental solution $\boldsymbol{\Omega}$ satisfies

$$\mathbf{L}\boldsymbol{\Omega} = 1\ \delta(s) \tag{13}$$

and from it, we can write the solution $\boldsymbol{\omega}$ in the form

$$\boldsymbol{\omega}(\mathbf{x}) = \int_s \boldsymbol{\Omega}(\mathbf{x} - \mathbf{y})\cdot\mathbf{F}(\mathbf{y})\ ds(\mathbf{y}). \tag{14}$$

Regarding $\boldsymbol{\Omega}$ as known and restricting $\mathbf{x}$ to be on the body in (14) we are led to an integral equation in the quantities $\mathbf{F}(\mathbf{y})$. Comparing (12) and (11) to identify the components of $\mathbf{F}(\mathbf{y})$ we see that this procedure results in the direct calculation of the forces and heat transfer at the body. That is, the flow field itself need not be found in order to find these quantities.

C. Far Field Solution

Closely following the development in Ref. 1, we formally expand the source term

$$[\mathbf{F}](s) = \int_s \mathbf{F}(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')\ ds(\mathbf{x}') = \int_s \mathbf{F}(\mathbf{x}')\ ds(\mathbf{x}')\delta(\mathbf{x}) - \int_s \mathbf{F}(\mathbf{x}')\mathbf{x}'\ ds\cdot\nabla\delta(\mathbf{x}) + \cdots. \tag{15}$$

The leading term here is

$$\int_s [0, \mathbf{n}\cdot(p\mathbf{1} - \mathbf{P}), \chi\mathbf{n}\cdot(\mathbf{P} - p\mathbf{1})\cdot\mathbf{U} + \mathbf{n}\cdot\mathbf{Q}, \mathbf{n}T]\ ds\,\delta(\mathbf{x}) = [0, \mathfrak{F}, \mathfrak{H}, \mathfrak{Q}]\delta(\mathbf{x}) = \mathbf{G}_0\delta(\mathbf{x}). \tag{16}$$

In our subsequent discussion we will illustrate various aspects of our solutions by means of the far field flow. For this reason it is worthwhile noting that

$$\mathfrak{F} = \int_s \mathbf{n}\cdot(p\mathbf{1} - \mathbf{P})\ ds$$

represents the total force of the body on the fluid (this includes the lift, drag, and lateral forces). Also in

$$\mathfrak{H} = \int_s [\chi\mathbf{n}\cdot(\mathbf{P} - p\mathbf{1})\cdot\mathbf{U} + \mathbf{n}\cdot\mathbf{Q}]\ ds$$

the first term represents the rate of work done by the body on the fluid and the second the rate of heat flow from the body.

From (12)

$$\boldsymbol{\omega} \sim \boldsymbol{\Omega}(\mathbf{x})\cdot\mathbf{G}_0. \tag{17}$$

D. Fundamental Solution by Fourier Transform

To simplify the calculations, it is desirable to have a symmetric operator. Thus, if we substitute the heat conduction equation into the energy equation, then instead of $\mathbf{L}\boldsymbol{\omega}$, we get

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} \mathbf{U}\cdot\nabla & \nabla & 0 \\ \nabla & \mathbf{U}\cdot\nabla - \zeta\nabla^2 - \eta\nabla\nabla\cdot & \chi\nabla \\ 0 & \chi\nabla & \mathbf{U}\cdot\nabla - \xi\nabla^2 \end{bmatrix}\begin{bmatrix} \rho \\ \mathbf{u} \\ T \end{bmatrix}. \tag{18}$$

Denote by $\mathbf{V}(\mathbf{x})$ the fundamental solution which satisfies

$$\mathbf{A}\,\mathbf{V} = \mathbf{1}\,\delta(\mathbf{x}). \tag{19}$$

Then, for example, from (17), the lowest order far field solution is

$$\mathbf{v} \sim \mathbf{v}_f = \mathbf{V}(\mathbf{x}) \begin{bmatrix} 0 \\ \mathfrak{F} \\ \mathcal{H} \end{bmatrix} + \mathcal{Q}\cdot\nabla\mathbf{V}(\mathbf{x}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{20}$$

Introducing the Fourier transform into (19) yields

$$\mathbf{A}(i\mathbf{k})\,\mathbf{V}(i\mathbf{k}) = \mathbf{1},$$

where $\mathbf{k} = (k_1, k_2, k_3)$ is the transform variable.

Therefore,

$$\mathbf{V}(i\mathbf{k}) = \mathbf{A}^{-1}(i\mathbf{k})$$

and

$$\mathbf{V}(\mathbf{x}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \exp(i\mathbf{k}\cdot\mathbf{x})\mathbf{A}^{-1}(i\mathbf{k})\, dk_1\, dk_2\, dk_3$$

$$= \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \exp(i\mathbf{k}\cdot\mathbf{x}) \frac{\mathbf{C}(i\mathbf{k})}{|\mathbf{A}|}\, dk_1\, dk_2\, dk_3, \tag{21}$$

where $\mathbf{C}$ is the classical adjoint of $\mathbf{A}$ and $|\mathbf{A}|$ is the determinant of $\mathbf{A}$.

III. ASYMPTOTIC EVALUATION OF THE FUNDAMENTAL SOLUTION

We now fix the unspecified scale L in (11) by the condition that

$$\max(\xi, \eta, \zeta) = O(1).$$

L, therefore, is a scale length based on the dissipation. (From the point of view of kinetic theory the scale L becomes a mean free path.) In this section we search for the solution to $\mathbf{V}(\mathbf{x})$, (21), for $|\mathbf{x}|$ large, i.e., in the limit of dimensional distance large compared with the dissipative scale L. (Again, from the viewpoint of kinetic theory this is the limit of dimensional scales large compared to mean free path—which is, in fact, the limit under which the Navier–Stokes equations are derived from the Boltzmann equation.)

To begin the calculation we consider the explicit form of the 5×5 matrix $\mathbf{A}(i\mathbf{k})$,

$$\mathbf{A}(i\mathbf{k}) = \begin{bmatrix} ik_1U & i\mathbf{k} & 0 \\ i\mathbf{k} & (ik_1U + \zeta k^2)\mathbf{1} + \eta\mathbf{k}\mathbf{k} & i\chi\mathbf{k} \\ 0 & i\chi\mathbf{k} & ik_1U + \xi k^2 \end{bmatrix}, \tag{22}$$

where a combined matrix dyadic notation is employed. It will also be convenient to split the matrix into its dissipative and nondissipative parts.

$$\mathbf{A}(i\mathbf{k}) = i\mathbf{A}_\epsilon + \mathbf{A}_D, \tag{23}$$

where the matrix $i\mathbf{A}_\epsilon$ refers to the Euler operator given by

$$\mathbf{A}_\epsilon(i\mathbf{k}) = \begin{bmatrix} k_1U & \mathbf{k} & 0 \\ \mathbf{k} & k_1U\mathbf{1} & \chi\mathbf{k} \\ 0 & \chi\mathbf{k} & k_1U \end{bmatrix}.$$

The determinant of $\mathbf{A}$ is of ninth degree in k_1, and we write

$$\det \mathbf{A} = C \prod_{j=1}^{9} [k_1 - r_j(k_\perp)], \tag{24}$$

where

$$k_\perp = (k_2^2 + k_3^2)^{1/2}$$

and

$$C = iU\xi\zeta^2(\zeta + \eta).$$

That $r_j = r_j(k_\perp)$ follows from symmetry.

We now demonstrate that the imaginary part of r_j cannot change sign and also that r_j can only be zero when $k_\perp = 0$. Suppose that the imaginary part of r_j does become zero for $k_\perp \neq 0$, then there exists a k_1 real (k_2 and k_3 are, of course, real) such that $\det \mathbf{A} = 0$. But this implies the existence of a vector (complex, in general) $\mathbf{v}$ such that

$$\mathbf{A}(i\mathbf{k})\mathbf{v} = 0. \tag{25}$$

On taking the inner product and using (20) we have

$$-i(\mathbf{v}, \mathbf{A}_\epsilon\mathbf{v}) + (\mathbf{v}, \mathbf{A}_D\mathbf{v}) = 0.$$

But both $\mathbf{A}_\epsilon$ and $\mathbf{A}_D$ are real symmetric and hence this implies that

$$(\mathbf{v}, \mathbf{A}_D\mathbf{v}) = 0.$$

But if $\mathbf{k} \neq 0$, this can only be true if

$$\mathbf{v} = (1, 0, 0, 0, 0)$$

and on substituting this in (25) and taking note of the form of $\mathbf{A}$ this is impossible. Next assuming $k_1 = 0$ and $k_\perp \neq 0$ and repeating the argument again leads to a contradiction.

Next, a simple perturbation analysis can be performed to find the roots r_j for $k_\perp$ small. This leads to

$$r_j = -ia_j + O(k_\perp), \qquad j = 1, 4, \tag{26}$$

where the four constants a_j are positive and their specific form will be of no consequence. (They, of course, vanish with vanishing dissipation.) We can write the remaining five roots as

$$r_j = f_j(k_\perp) + O(k_\perp^3), \qquad j = 5, 9 \tag{27}$$

with

$$\begin{aligned} f_5 &= \frac{k_\perp}{(M^2-1)^{1/2}} + i\beta k_\perp^2, \\ f_6 &= -\frac{k_\perp}{(M^2-1)^{1/2}} + i\beta k_\perp^2, \\ f_7 &= \frac{i\xi}{\gamma U} k_\perp^2, \quad f_8 = f_9 = \frac{i\zeta}{U} k_\perp^2. \end{aligned} \tag{28}$$

The constant β is given by

$$\beta = \frac{U^3}{2\gamma(U^2-\gamma)} [\gamma(\zeta + \eta) + (\gamma - 1)\xi] \tag{29}$$

and $M = U/\gamma^{1/2}$ is the Mach number which we take greater than 1. Using the results of the previous paragraph we can conclude that r_j, $j = 1, 4$ always lie in the lower half of the complex k_1 plane and r_j, $j = 5, 9$ always lie in the upper half-plane.

Next turning to the matrix in the numerator of (21), $\mathbf{C}(i\mathbf{k})$, we notice that its entries are polynomials in $\mathbf{k}$. However, a direct calculation shows that it may be written as

$$\mathbf{C}(i\mathbf{k}) = (k_1 - r_9)\mathbf{B}(i\mathbf{k}), \tag{30}$$

where $\mathbf{B}$ too has only polynomial entries. The double root $r_8 = r_9$ may thus be eliminated and using (30) and (24) in (21) we can write

$$\mathbf{V}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \exp(i\mathbf{k}\cdot\mathbf{x}) \cdot \mathbf{B}(i\mathbf{k}) / \{C \prod_{j=1}^{8} [k_1 - r_j(k_\perp)]\}\, dk_1\, d\mathbf{k}_\perp. \tag{31}$$

We first perform the k_1 integration and note that for $x_1 > 0$ we may close the path of the integration in the upper half of the complex k_1 plane and for $x_1 < 0$ in the lower half-plane. Denoting the corresponding contours by $\mathcal{L}_+$ and $\mathcal{L}_-$, respectively, we may write

$$\mathbf{V}(\mathbf{x}) = \frac{1}{(2\pi)^3}\left[H(x_1)\int_{\mathcal{L}+} dk_1 + H(-x_1)\int_{\mathcal{L}-} dk_1\right] \cdot \int_{-\infty}^{\infty} d\mathbf{k}_\perp \exp(i\mathbf{k}\cdot\mathbf{x})\mathbf{B}/C \prod_{j=1}^{8}(k_1 - r_j), \tag{32}$$

where $H(x)$ represents the Heaviside function. But we have already shown that only r_j, $j = 1, 4$ lie in the lower half-plane. Since the a_j are all positive, continuity shows that the imaginary part of r_j, $j = 1, 4$ are bounded away from zero and hence $H(-x_1)\mathbf{V}(\mathbf{x})$ is exponentially small. Therefore, on performing the k_1 integration (32)

$$\mathbf{V}(\mathbf{x}) \sim \frac{iH(x_1)}{(2\pi)^2 C} \sum_{j=5}^{8} \iint_{-\infty}^{\infty} d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{R} + ir_jx_1) \cdot \mathbf{B}(r_j, k_2, k_3) \Big/ \prod_{\substack{l=1\\ l\neq j}}^{8} (r_j - r_l), \tag{33}$$

where $\mathbf{R} = (x_2, x_3)$. As we have already mentioned, the error estimate in (33) is exponentially small.

It is now convenient to split the integrands in (33) into two parts.

$$\mathbf{V}_{MC} = \frac{iH(x_1)}{(2\pi)^2 C} \sum_{j=5}^{6} \iint_{-\infty}^{\infty} d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{R} + ir_jx_1) \cdot \mathbf{B}(r_j, k_2, k_3) \Big/ \prod_{\substack{l=1\\ l\neq j}}^{8} (r_j - r_l), \tag{34}$$

$$\mathbf{V}_w = \frac{iH(x_1)}{(2\pi)^2 C} \sum_{j=7}^{8} \iint_{-\infty}^{\infty} d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{R} + ir_jx_1) \cdot \mathbf{B}(r_j, k_2, k_3) \Big/ \prod_{\substack{l=1\\ l\neq j}}^{8} (r_j - r_l). \tag{35}$$

As indicated in the expansions (25) and as will be clearer in the following, $\mathbf{V}_{MC}$ governs the structure of the flow in the neighborhood of the Mach cone and downstream of it, and $\mathbf{V}_w$ governs the structure of the wake.

The integrals in (34) and (35) are special cases of a general asymptotic analysis given elsewhere.[5,6] Under the properties already demonstrated for the r_j, it is proven that for x_1 large the asymptotic forms of such integrals are obtained by replacing r_j by f_j and further retaining the leading term in the expansion of

$$\mathbf{B} \Big/ \prod_{\substack{l=1\\ l\neq j}}^{8} (r_j - r_l)$$

for small $\mathbf{k}_\perp$.

A. Wake Calculation

Define the vector

$$\boldsymbol{\alpha} = (-\chi, 0, 0, 0, 1) \tag{36}$$

and the matrix

$$\mathbf{B} = \frac{1}{k_\perp^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_\perp^2 & 0 & 0 & 0 \\ 0 & 0 & k_3^2 & -k_2k_3 & 0 \\ 0 & 0 & -k_2k_3 & k_2^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

Then, carrying out the asymptotic approximation outlined in the previous paragraph from Ref. 5 we have

$$\mathbf{V}_w(\mathbf{x}) = \frac{H(x_1)}{(2\pi)^2 U\gamma} \cdot \left[\alpha\alpha \int_{-\infty}^{\infty} \exp\left(i\mathbf{k}_\perp\cdot\mathbf{R} - \frac{\xi}{\gamma U} k_\perp^2 x_1\right) d\mathbf{k}_\perp + \gamma \int_{-\infty}^{\infty} \exp\left(i\mathbf{k}_\perp\cdot\mathbf{R} - \frac{\zeta}{U} x_1 k_\perp^2\right) \mathbf{B}\, d\mathbf{k}_\perp\right] + O^*(x_1^{-3/2}),$$

where the symbol $O^*(x_1^{-p})$ represents a quantity which vanishes, for x_1 large, slightly less rapidly than x_1^{-p}. Each of the integrals is straightforward, and we find

$$[\mathbf{V}_w(x)]_{ij} = H(x_1)\left[\frac{\alpha_i\alpha_j}{4\pi\xi x_1}\exp\left(-\frac{\gamma U R^2}{(4\xi x_1)}\right) + \frac{\delta_{i2}\delta_{j2}}{4\pi\zeta x_1}\exp\left(-\frac{UR^2}{(4\zeta x_1)}\right) + W_{ij}\right] + O^*(x_1^{-3/2}) \quad (38)$$

with the matrix $\mathbf{W}$ having only four nonzero entries given by

$$\begin{aligned} W_{33} &= \frac{1}{2\pi R^2 U}\left\{\frac{x_3^2 U}{2x_1\zeta}\exp\left(-\frac{UR^2}{(4\zeta x_1)}\right) + \left(1 - \frac{2x_3^2}{R^2}\right)\left[1 - \exp\left(-\frac{UR^2}{(4\zeta x_1)}\right)\right]\right\}, \\ W_{44} &= \frac{1}{2\pi R^2 U}\left\{\frac{x_2^2 U}{2x_1\zeta}\exp\left(-\frac{UR^2}{(4\zeta x_1)}\right) + \left(1 - \frac{2x_2^2}{R^2}\right)\left[1 - \exp\left(-\frac{UR^2}{(4\zeta x_1)}\right)\right]\right\}, \\ W_{34} &= W_{43} \\ &= \frac{x_2 x_3}{\pi U R^4}\left[1 - \exp\left(-\frac{UR^2}{(4\zeta x_1)}\right)\left(1 + \frac{R^2 U}{4\zeta x_1}\right)\right]. \end{aligned} \quad (39)$$

Since $\xi/(\gamma U)$ and U/ζ are positive independently of the Mach number, the wake calculations are valid for all Mach numbers (i.e., at subsonic speeds also). This portion of the calculation is developed under the limit

$$\frac{x_1\zeta}{U} \gg 1, \qquad \frac{x_1\xi}{\gamma U} \gg 1.$$

Therefore, in the hypersonic limit $U \to \infty$, the development can only be presumed valid at relatively large distances.

We also point out that as in the two-dimensional case,[1] the viscous and heat conducting effects decouple to lowest order. From (38) and (39) we see that the wake structured by the heat conduction coefficient ξ only carries density and temperature variations, and that the pressure variation is zero so that this may be identified as an entropy wake. The wake structured by the viscosity coefficient ζ only carries velocity changes and since the divergence of velocity vanishes there, this is a vorticity wake.

B. Mach Cone Calculation

We define the vector

$$\mathbf{X} = \left(\frac{U}{(U^2-\gamma)^{1/2}},\ \frac{-\gamma}{(U^2-\gamma)^{1/2}},\ \frac{-\gamma^{1/2}k_2}{k_\perp},\ \frac{-\gamma^{1/2}k_3}{k_\perp},\ \frac{\chi U}{(U^2-\gamma)^{1/2}}\right). \quad (40)$$

Then, carrying out the asymptotic analysis discussed above, it follows that[5]

$$V_{MC} = \frac{H(x_1)}{(2\pi)^2 U\gamma} \cdot \mathrm{Re}\int_{-\infty}^{\infty}\exp\left(i\mathbf{k}_\perp\cdot\mathbf{R} + \frac{ik_\perp x_1}{(M^2-1)^{1/2}} - \beta k_\perp^2 x_1\right) \mathbf{XX}\, d\mathbf{k}_\perp + O^*\left(\frac{1}{x_1^{3/2}}\right), \quad (41)$$

where β is defined by (29). Unlike the wake calculation, the integrals involved in (41) are not straightforward. The following types of integrals (not unrelated) enter in the calculation:

$$I = \frac{1}{(2\pi)^2} \cdot \mathrm{Re}\int_{-\infty}^{\infty}\exp\left(i\mathbf{k}_\perp\cdot\mathbf{R} + \frac{ik_\perp x_1}{(M^2-1)^{1/2}} - \beta k_\perp^2 x_1\right) d\mathbf{k}_\perp,$$

$$I_i = \frac{1}{(2\pi)^2} \cdot \mathrm{Re}\int_{-\infty}^{\infty}\exp\left(i\mathbf{k}_\perp\cdot\mathbf{R} + \frac{ik_\perp x_1}{(M^2-1)^{1/2}} - \beta k_\perp^2 x_1\right) \cdot\frac{k_i}{k_\perp}\, d\mathbf{k}_\perp, \qquad i = 2, 3,$$

$$I_{ij} = \frac{1}{(2\pi)^2} \cdot \mathrm{Re}\int_{-\infty}^{\infty}\exp\left(i\mathbf{k}_\perp\cdot\mathbf{R} + \frac{ik_\perp x_1}{(M^2-1)^{1/2}} - \beta k_\perp^2 x_1\right) \cdot\frac{k_i k_j}{k_\perp^2}\, d\mathbf{k}_\perp, \qquad i, j = 2, 3. \quad (42)$$

Consistent with the error estimate in (41) a second asymptotic analysis may be performed.[5] This is carried out in the Appendix, and all the entries of

(42) are evaluated in terms of known functions to within the error estimate given in (41).

Analogous to (40) we define

$$\boldsymbol{\tau} = \left(\frac{U}{(U^2-\gamma)^{1/2}},\ \frac{-\gamma}{(U^2-\gamma)^{1/2}},\ \frac{\gamma^{1/2}x_2}{R},\ \frac{\gamma^{1/2}x_3}{R},\ \frac{\chi U}{(U^2-\gamma)^{1/2}}\right). \tag{43}$$

We then find

$$\mathbf{V}_{\mathrm{MC}} = \frac{H(x_1)}{U\gamma}\,\boldsymbol{\tau\tau}\,b(x_1, R) + O(x_1^{-3/2}). \tag{44}$$

The function $b(x_1, R)$ is given in terms of a parabolic cylinder function by (A8).

IV. THE INVISCID LIMIT

It is of interest to consider the comparison of the asymptotic solution found in Sec. III with the analogous inviscid solution. If we denote the inviscid operator by

$$\mathbf{A}_\epsilon = \mathbf{A}(\zeta = 0,\quad \xi = 0,\quad \eta = 0), \tag{45}$$

the inviscid fundamental solution $\mathbf{V}^0$ is given by

$$\mathbf{V}^0 = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \exp(i\mathbf{k}\cdot\mathbf{x})\mathbf{A}_\epsilon^{-1}\,d\mathbf{k}. \tag{46}$$

Although not straightforward, we delete the calculations in obtaining $\mathbf{V}^0$. Again the solution can be described in terms of its behavior in the Mach cone $\mathbf{V}^0_{\mathrm{MC}}$ and in the wake, $\mathbf{V}^0_W$. That is, writing

$$\mathbf{V}^0 = \mathbf{V}^0_{\mathrm{MC}} + \mathbf{V}^0_W \tag{47}$$

we find

$$[\mathbf{V}^0_W(\mathbf{x})]_{ij} = H(x_1)\left(\frac{\alpha_i\alpha_j}{U\gamma}\,\delta(x_2)\delta(x_3) + \frac{\delta_{i2}\delta_{j2}}{U}\,\delta(x_2)\delta(x_3) + W^0_{ij}\right), \tag{48}$$

where

$$W^0_{33} = \frac{1}{2U}\,\delta(x_2)\delta(x_3) + \frac{1}{2U\pi R^2}\left(1 - \frac{2x_3^2}{R^2}\right),$$

$$W^0_{34} = W^0_{43} = \frac{x_2x_3}{\pi U R^4}, \tag{49}$$

$$W^0_{44} = \frac{1}{2U}\,\delta(x_2)\delta(x_3) + \frac{1}{2\pi R^2 U}\left(1 - \frac{2x_2^2}{R^2}\right)$$

with all other entries zero and

$$\mathbf{V}^0_{\mathrm{MC}}(\mathbf{x}) = \frac{H(x_1)}{U\gamma}\begin{bmatrix} \frac{U^2}{U^2-\gamma}I & -\frac{U\gamma}{U^2-\gamma}I & -\frac{U\gamma}{(U^2-\gamma)^{1/2}}I_2 & -\frac{U\gamma^{1/2}}{(U^2-\gamma)^{1/2}}I_3 & \frac{\chi U^2}{U^2-\gamma}I \\ & \frac{\gamma^2}{U^2-\gamma}I & \frac{\gamma^{3/2}}{(U^2-\gamma)^{1/2}}I_2 & \frac{\gamma^{3/2}}{(U^2-\gamma)^{1/2}}I_3 & -\frac{\chi\gamma U}{U^2-\gamma}I \\ & & \gamma I_{22} & \gamma I_{23} & -\frac{\chi\gamma U}{(U^2-\gamma)^{1/2}}I_2 \\ & & & \gamma I_{33} & -\frac{\chi\gamma U}{(U^2-\gamma)^{1/2}}I_3 \\ & & & & \frac{\chi^2U^2}{U^2-\gamma}I \end{bmatrix}, \tag{50}$$

where

$$I = \frac{H[x_1/(M^2-1)^{1/2} - R]\,x_1/(M^2-1)^{1/2}}{2\pi\,\{[x_1/(M^2-1)^{1/2}] - R\}^{3/2}\{[x_1/(M^2-1)^{1/2}] + R\}^{3/2}},$$

$$I_i = \frac{x_i}{R}\,A(x_1, R), \qquad i = 2, 3,$$

$$A(x_1, R) = \frac{H[x_1/(M^2-1)^{1/2} - R]}{2\pi}$$

$$\cdot\left(\frac{Rx_1/(M^2-1)^{1/2}}{\{[x_1/(M^2-1)^{1/2}] - R\}^{3/2}\{[x_1/(M^2-1)^{1/2}] + R\}^{3/2}([x_1/(M^2-1)^{1/2}] + \{[x_1^2/(M^2-1)] - R^2\}^{1/2})}\right.$$

$$\left. + \frac{1 + [x_1/(M^2-1)^{1/2}][x_1^2/(M^2-1)] - R^2\}^{-1/2}}{\{[x_1^2/(M^2-1)] - R^2\}^{3/2}([x_1/(M^2-1)^{1/2}] + \{[x_1^2/(M^2-1)] - R^2\}^{1/2})^2}\right),$$

$$I_{ij} = \frac{\delta_{ij}}{2}\,I + \left(\frac{x_ix_j}{R^2} - \frac{\delta_{ij}}{2}\right)B(x_1, R), \qquad i, j = 2, 3,$$

$$B(x_1, R) = -\frac{H[R_1 - x_1/(M^2-1)^{1/2}]}{\pi R^2} - \frac{H[x_1/(M^2-1)^{1/2} - R]}{2\pi}$$
$$\cdot\left(\frac{R^2x_1/(M^2-1)^{1/2}}{\{[x_1/(M^2-1)^{1/2}] - R\}^{3/2}\{[x_1/(M^2-1)^{1/2}] + R\}^{3/2}([x_1/(M^2-1)^{1/2}] + \{[x_1^2/(M^2-1)] - R^2\}^{1/2})}\right.$$
$$\left. + \frac{2(1 + [x_1/(M^2-1)^{1/2}]\{[x_1^2/(M^2-1)^{1/2}] - R^2\}^{-1/2})}{\{[x_1^2/(M^2-1)] - R^2\}^{1/2}([x_1/(M^2-1)^{1/2}] + \{[x_1^2/(M^2-1)] - R^2\}^{1/2})^4}\right).$$

When we take the inviscid limit of the viscous solution, we find that the leading terms of the wake solution (34) give the whole inviscid solution $\mathbf{V}_w^0$. On the other hand, the limit of the Mach cone solution (44) gives

$$\mathbf{V}_{MC}(\mathbf{x}) \sim \frac{-H(x)}{U\gamma}\,\boldsymbol{\tau\tau}$$
$$\cdot\frac{H[x_1/(M^2-1)^{1/2} - R]}{(32\pi^2R)^{1/2}\{[x_1/(M^2-1)^{1/2}] - R\}^{3/2}} \quad (51)$$

which is only the leading term of $\mathbf{V}_{MC}^0$ after we make the expansions in $\{[x_1/(M^2-1)^{1/2}] - R\} = O(R^{1/2})$. Thus, while the asymptotic solution $\mathbf{V}_w$ contains the whole inviscid solution $\mathbf{V}_w^0$, we have $\mathbf{V}_{MC}$ blending into $\mathbf{V}_{MC}^0$ at far field.

V. THE FAR FIELD

We consider the far field solution of flow past a body. Denoting the drag by $-D$, the lift by $-L$, the lateral force by $-F$, and for simplicity, taking $\mathcal{Q} = 0$ in (20), we have

$$\mathbf{v}(\mathbf{x}) = \mathbf{V}(\mathbf{x})\begin{bmatrix}0\\ \mathfrak{F}\\ \mathcal{H}\end{bmatrix}, \quad (52)$$

where

$$\mathfrak{F} = (-D, -L, -F). \quad (53)$$

A typical term of the far field solution in the wake is then given by

$$u_1 \sim -\frac{H(x_1)D}{(2\pi)^3U\gamma\zeta x_1}\exp\left(-\frac{UR^2}{4\zeta x_1}\right) \quad (54)$$

and in the Mach cone,

$$u_1 \sim \frac{H(x_1)}{\gamma U}\left(-\frac{\gamma^2}{U^2-\gamma}D + \frac{\gamma^{3/2}x_2}{R(U^2-\gamma)^{1/2}}L\right.$$
$$\left. + \frac{\gamma^{3/2}x_3}{R(U^2-\gamma)^{1/2}}F - \frac{\chi U}{U^2-\gamma}\mathcal{H}\right)b(x_1, R). \quad (55)$$

For the purpose of illustration, we take $L = F = \mathcal{H} = 0$ for u_1 of the Mach cone solution, and the result is plotted in Fig. 1.

We also note that for our far field solution, we have, in the wake

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = O(x_1^{-5/2}),\quad \nabla^2\mathbf{u} = O(x_1^{-2}), \quad (56)$$

and in the Mach cone

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = O(x_1^{-3}),\quad \nabla^2\mathbf{u} = O(x_1^{-9/4}). \quad (57)$$

Therefore, our assumption that

$$(\mathbf{u}\cdot\nabla)\mathbf{u} \ll \nabla^2\mathbf{u} \quad (58)$$

is justified in the far field.

Note added in proof: After submission of this paper for publication, the work of Salathe[4] has appeared: E. P. Salathe, J. Fluid. Mech. **39**, 209 (1969). Although this is not a proper place for a relative discussion, several points merit immediate comment. Salathe introduces a cautionary remark concerning the use of the linearized theory at infinity. Our contention is that the linearized theory in three dimensions yields a self-consistently valid solution of the full Navier–Stokes equations. This is shown in Eq. (56)–(58). (As mentioned in our Introduction, this is not true in two dimensions.) Our subsequent investigation of the nonlinear theory supports this contention. Detailed comparison with the paper of Salathe is not possible since it contains an error. If this is accounted for, agreement between the two calculations results.

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APPENDIX

In each of the integrals (38), it is convenient to introduce $\mathbf{R}$ as the reference axis of integration

$$\mathbf{R} = R\mathbf{e}_1,$$

$$\mathbf{k}_\perp = k_\perp\cos\theta\,\mathbf{e}_1 + k_\perp\sin\theta\,\mathbf{e}_2.$$

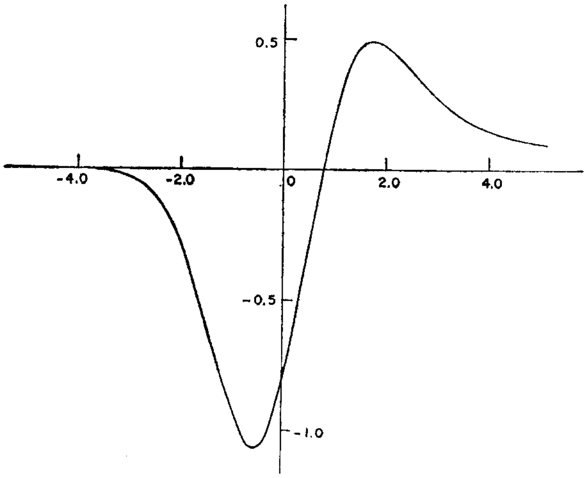

FIG. 1. Normalized Mach cone profile: x axis, $\{[x_1/(M^2-1)^{1/2}] - R\}(2\beta x_1)^{-1/2}$; y axis, $[H(x_1)U(U^2-\gamma)/\gamma D](2\beta x_1)^{3/4}\cdot(32\pi^2 R)^{1/2}u_1$.

Then, let us consider I in (38) first. We have directly

$$I = \frac{1}{2\pi}\operatorname{Re}\int_0^\infty k_\perp J_0(k_\perp R)\cdot\exp\left(\frac{ik_1x_1}{(M^2-1)^{1/2}} - \beta x_1 k_\perp^2\right)dk_\perp.$$

We first consider $R = o(x_1)$; the analysis is straightforward, and we find

$$I \sim -\frac{(M^2-1)}{2\pi x_1^2}$$

which is of a neglected order in (37). We, therefore, restrict attention to $R \geq O(x_1)$.

Let $p > 0$, and consider

$$\mathcal{J} = \frac{1}{2\pi}\operatorname{Re}\int_0^{O(x_1^{-p})} k_\perp J_0(k_\perp R)\cdot\exp\left(\frac{ik_1x_1}{(M^2-1)^{1/2}} - \beta x_1 k_\perp^2\right)dk_\perp.$$

Then

$$O(\mathcal{J}) \leq \int_0^{x_1^{-p}} k_\perp J_0(k_\perp R)\,dk_\perp.$$

The integral may be directly integrated and we find

$$O(\mathcal{J}) \leq \frac{J_1(R/x_1^p)}{Rx_1^p}.$$

Taking $p < 1$, then in view of $R \geq O(x_1)$, we can use the asymptotic approximation for J_1, and we find

$$O(\mathcal{J}) \leq \frac{1}{R^{3/2}x_1^{p/2}}$$

which is clearly of a neglected order. Next, we consider

$$I - \mathcal{J} = \frac{1}{2\pi}\operatorname{Re}\int_{O(x_1^{-p})}^\infty J_0(k_\perp R)k_\perp\cdot\exp\left(\frac{ik_1x_1}{(M^2-1)^{1/2}} - \beta x_1 k_\perp^2\right)dk_\perp.$$

The argument of the Bessel function is clearly large and we write

$$J_0(k_\perp R) = \left(\frac{2}{\pi k_\perp R}\right)^{1/2}\cos\left(k_\perp R - \frac{\pi}{4}\right) + O\left(\frac{1}{|k_\perp R|^{3/2}}\right).$$

Consider

$$A = \frac{1}{2\pi}\operatorname{Re}\int_{O(x_1^{-p})}^\infty\left[J_0(k_\perp R) - \left(\frac{2}{\pi k_\perp R}\right)^{1/2}\cos\left(k_\perp R - \frac{\pi}{4}\right)\right]\cdot\exp\left(-\beta k_\perp^2 x_1 + \frac{ik_\perp x_1}{(M^2-1)^{1/2}}\right)dk_\perp.$$

Then, using straightforward estimates we find

$$O(A) \leq \frac{1}{Rx_1^{3(1-p)/2}}.$$

It only remains for us to consider

$$F = \frac{1}{2\pi}\operatorname{Re}\int_0^{O(x_1^{-p})} k_\perp\left(\frac{2}{\pi k_\perp R}\right)^{1/2}\cos\left(k_\perp R - \frac{\pi}{4}\right)\cdot\exp\left(-\beta k_\perp^2 x_1 + \frac{ik_\perp x_1}{(M^2-1)^{1/2}}\right)dk_\perp$$

for which we easily find

$$O(F) < \frac{1}{R^{1/2}x_1^{3p/2}}.$$

Therefore, for example, taking $p = \frac{2}{3}$ we have demonstrated

$$I = \left(\frac{1}{R\pi^3}\right)^{1/2}\operatorname{Re}\int_0^\infty k_\perp^{1/2}\cos\left(k_\perp R - \frac{\pi}{4}\right)\cdot\exp\left(-\beta k_\perp^2 x_1 + \frac{ik_\perp x_1}{(M^2-1)^{1/2}}\right)dk_\perp + O(x_1^{-3/2}). \tag{A1}$$

Next consider I_i of (38). A simple invariant argument states

$$I_i = \frac{x_i}{R}A(x_1, R), \qquad i = 2, 3. \tag{A2}$$

and we easily show

$$A(x_1, R) = \frac{1}{2\pi}\operatorname{Re}\, i\int_0^\infty \exp\left(\frac{ik_\perp x_1}{(M^2-1)^{1/2}} - \beta k_\perp^2 x_1\right)\cdot J_1(k_\perp R)k_\perp\,dk_\perp. \tag{A3}$$

The same arguments and estimates used in obtaining (A1) may again be applied to (A3). It suffices to say that we can replace $J_1(k_\perp R)$ by its asymptotic approximation and write

$$A(x_1, R) = \left(\frac{1}{R\pi^3}\right)^{1/2} \operatorname{Re} i \int_0^\infty k_\perp^{1/2} \cos\left(k_\perp R - \frac{3\pi}{4}\right) \cdot \exp\left(\frac{ik_\perp x_1}{(M^2 - 1)^{1/2}} - \beta k_\perp^2 x_1\right) dk_\perp + O(x_1^{-3/2}). \quad \text{(A4)}$$

Finally, consider I_{ij} in (38). Again using an invariant argument we can write

$$I_{ij} = \frac{\delta_{ij}}{2} I(x_1, R) + \left(\frac{x_i x_j}{R^2} - \frac{\delta_{ij}}{2}\right) B(x_1, R), \quad i, j = 2, 3, \quad \text{(A5)}$$

where $I(x_1, R)$ is given by (38) and

$$B(x_1, R) = -\frac{1}{2\pi} \operatorname{Re} \int_0^\infty k_\perp \cdot \exp\left(\frac{ix_1 k_\perp}{(M^2 - 1)^{1/2}} - \beta k_\perp^2 x_1\right) J_2(k_\perp R)\, dk_\perp \quad \text{(A6)}$$

and using previous arguments, (A6) becomes

$$B(x_1, R) = -\left(\frac{1}{2\pi^3 R}\right)^{1/2} \operatorname{Re} \int_0^\infty k^{1/2} \cos\left(kR - \frac{\pi}{4}\right) \cdot \exp\left(\frac{ix_1 k_\perp}{(M^2 - 1)^{1/2}} - \beta k_\perp^2 x_1\right) dk_\perp + O(x_1^{-3/2}). \quad \text{(A7)}$$

The integral occurring in (A7) is the same as the one appearing in (A1). Therefore, this and the integral in (A4) are the only integrals which need to be evaluated.

Both integrals may be evaluated using standard integral tables. Writing in an obvious notation

$$B(x_1, R) = \hat{b}(x_1, R) + O(x_1^{-3/2})$$

we find[3]

$$\hat{b}(x_1, R) = \frac{1}{(2\beta x_1)^{3/4}(32\pi^2 R)^{1/2}} \cdot \operatorname{Re}\left[\exp\left(\frac{i\pi}{4}\right) - \frac{[R - x_1/(M^2 - 1)^{1/2}]^2}{(8\beta x_1)}\right) \cdot D_{-3/2}\left(-\frac{i[x_1/(M^2 - 1)^{1/2} - R]}{(2\beta x_1)^{1/2}}\right) + \exp\left(-\frac{i\pi}{4}\right) - \frac{[R + x_1/(M^2 - 1)^{1/2}]^2}{(8\beta x_1)}\right) \cdot D_{-3/2}\left(\frac{i[R + x_1/(M^2 - 1)^{1/2}]}{(2\beta x_1)^{1/2}}\right)\Bigg],$$

where $D_p(z)$ is the parabolic cylinder function. The second term of the brackets is already of a neglected order [since $R + x_1/(M^2 - 1)^{1/2} \gg 1$]. Therefore, defining

$$b(x_1, R) = \frac{1}{(2\beta x_1)^{3/4}(32\pi^2 R)^{1/2}} \cdot \operatorname{Re}\left[\exp\left(\frac{i\pi}{4}\right) - \frac{[R - x_1/(M^2 - 1)^{1/2}]^2}{(8\beta x_1)}\right) \cdot D_{-3/2}\left(\frac{i[R - x_1/(M^2 - 1)^{1/2}]}{(2\beta x_1)^{1/2}}\right)\Bigg], \quad \text{(A8)}$$

we can write

$$B(x_1, R) = b(x_1, R) + O(x_1^{-3/2}).$$

A similar argument for $A(x_1, R)$, (A4), shows

$$A(x_1, R) = -b(x_1, R) + O(x_1^{-3/2}).$$

Therefore, we have shown

$$I = b(x_1, R) + O(x_1^{-3/2}),$$

$$I_i = -\frac{x_i}{R} b(x_1, R) + O(x_1^{-3/2}); \quad i = 2, 3, \quad \text{(A9)}$$

$$I_{ij} = \frac{x_i x_j}{R^2} b(x_1, R) + O(x_1^{-3/2}); \quad i, j = 2, 3.$$

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