Evolution of finite disturbances in dissipative gasdynamics

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The initial-value problem for small amplitude disturbances governed by the Navier–Stokes equations is considered. This is shown to be governed by a system of Burgers and diffusion equations. The asymptotic solution of the finite amplitude initial-value problem is obtained by the introduction of asymptotic initial data. The analysis is also applied to the solution of a semi-infinite shock tube problem.

I. INTRODUCTION

The present investigation is in part aimed at describing the evolution of small but not infinitesimal disturbances in a gas governed by the Navier–Stokes equations. A study of the linearized problem appears in the early investigation of Lagerstrom *et al.*[1] In that investigation a study is made of the propagation of small disturbances in a compressible, viscous but nonheat conducting fluid. Wu[2] in a subsequent investigation was able to give a certain asymptotic description when the Prandtl number is $\frac{3}{4}$. A later investigation[3] contains the asymptotic solution in terms of elementary functions for a gas having arbitrary properties.

The need for a nonlinear description of gas dynamics goes back to Riemann (see Ref. 4 for a discussion outlining the breakdown of linear theory). The effect of nonlinearity in inviscid problems has been deeply investigated by Whitham.[5–7] He shows that linear theory becomes progressively worse as time becomes large, and he also furnishes a method for rendering the linear solution uniformly valid by a nonlinear transformation of the coordinate variables. The inclusion of both nonlinear and dissipative terms was carried out by Lighthill[8] in a study of propagating waves. (This was also done by Hayes,[9] although somewhat later.) Lighthill[8] shows that the long time behavior of a propagating wave is governed by a Burgers equation[10]—thereby also giving the Burgers equation a firm quantitative role in fluid mechanics. An earlier demonstration that the Burgers equation appears as an approximation to the Navier–Stokes equation is given in Appendix B of Lagerstrom *et al.*[1] This was also taken up by Cole.[11] (It is also of interest to point out as does Cole,[11] that the Burgers equation was considered by Bateman[12]—and also in a fluid dynamics context.) Subsequent investigations showing the role of the Burgers equation in wave propagation have been made by Moran and Shen[13] and Su and Gardner.[14] For applications the great value of the Burgers equation lies in the fact that Cole[11] and Hopf[15] found exact solutions to it. Other mathematical treatments of this equation have been given by Olejnik,[16] Ladyzhenskaya,[17] and Lax.[18]

Our investigation begins with a study of linearized theory, both inviscid and dissipative. We show, by computing second-order theory, that this breaks down due to both nonlinear and dissipative effects. This breakdown occurs in the form of secularities, and the method of multiple scales[19] is then used. This results in a description in terms of three modes: two propagating modes described by Burgers equations and a contact region described by a diffusion equation. All three modes are uncoupled. This description is also shown to be valid uniformly up to the initial instant. As a result, this description includes the linearized theory, and, in addition, it is demonstrated that the Whitham theory[5–7] follows in the inviscid limit. Further, the resolution of arbitrary initial data into the three modes is also obtained.

The inviscid limit naturally leads to unsteady discontinuous shock solutions. From the analysis of Sec. VI we obtain an unambiguous "center" of the shock which uniquely determines an internal structure. This may alternately be viewed as "shock slip" due to nonconstant up and down stream states. The inviscid limit is found to be nonuniform in time. For long times both nonlinear and dissipative effects must, in general, be included. The asymptotic solution in this limit takes on an especially simple form, and may be loosely interpreted as a sum of compression and N waves, each of which have been considered by Lighthill.[8] When the compression portion is small, the theory reduces to linear viscous theory. A description of the long time finite amplitude case is also obtained. This is based on the construction of asymptotic initial data for the problem. Finally, the analysis is applied to a shock-tube problem in the last section.

II. EQUATIONS OF MOTION

The Navier–Stokes equations for compressible gas dynamics are given by

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0,$$

$$\frac{\partial}{\partial t}(\rho\mathbf{u})+\nabla\cdot(\rho\mathbf{uu}+p-\mathbf{P})=0, \qquad (1)$$

$$\frac{\partial}{\partial t}[\rho(e+\tfrac{1}{2}u^2)]+\nabla\cdot[\rho\mathbf{u}(e+\tfrac{1}{2}u^2)+p\mathbf{u}-\mathbf{P}\cdot\mathbf{u}-\mathbf{Q}]=0,$$

where

$$P_{ij}=\mu(u_{i,j}+u_{j,i}-\tfrac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u})+\beta\nabla\cdot\mathbf{u}\delta_{ij} \tag{2}$$

and

$$Q=\kappa\nabla T. \tag{3}$$

In the linearized treatment,[3] no suppositions were made as to the nature of the gas either in its state equations or its various dimensionless dissipative parameters. The present derivation could again be carried out under the same general assumptions. However, to keep the notation simple, we assume that pressure p satisfies the perfect gas law

$$p=\rho RT \tag{4}$$

and the internal energy e satisfies

$$e=c_vT, \tag{5}$$

where c_v is a constant. In general, the dissipative parameters μ, β, and κ (viscosity, bulk viscosity, and heat conduction, respectively) are functions of temperature. Again for simplicity we take them constant, since in the following only their leading terms will enter.

Next, we introduce equilibrium quantities

$$\rho_0,\ T_0,\ a_0=\left[\left(\frac{\partial p_0}{\partial\rho_0}\right)_{T_0}\right]^{1/2},\qquad c_v=\left(\frac{\partial e_0}{\partial T_0}\right)_{\rho_0},$$

an unspecified length scale L, and the following normalized quantities:

$$\begin{aligned}\tilde{\mathbf{x}}&=\mathbf{x}/L, \quad \tilde{t}=a_0t/L, \quad \tilde{\rho}=(\rho-\rho_0)/\rho_0,\\ \tilde{\mathbf{u}}&=\mathbf{u}/a_0, \quad \tilde{T}=(c_v/a_0^2T_0)^{1/2}(T-T_0),\\ \tilde{P}_{ij}&=P_{ij}/\rho_0a_0^2, \quad \tilde{Q}_i=Q_i/\rho_0a_0^2(c_vT_0)^{1/2}.\end{aligned} \tag{6}$$

Speeds are normalized with respect to the isothermal speed of sound a_0, since this gives the system of equations a symmetry which proves useful. Restricting (1) to one space dimension and imposing the above normalization, we find

$$\begin{aligned}&\frac{\partial\rho}{\partial t}+\frac{\partial u}{\partial x}+\rho\frac{\partial u}{\partial x}+u\frac{\partial\rho}{\partial x}=0,\\ &\frac{\partial u}{\partial t}+\rho\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\rho u\frac{\partial u}{\partial x}+\frac{\partial\rho}{\partial x}+\chi\frac{\partial T}{\partial x}\\ &\qquad+\chi T\frac{\partial\rho}{\partial x}+\chi\rho\frac{\partial T}{\partial x}=\zeta\frac{\partial^2u}{\partial x^2},\\ &\frac{\partial T}{\partial t}+\rho\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+\rho u\frac{\partial T}{\partial x}+\chi\frac{\partial u}{\partial x}+\chi\rho\frac{\partial u}{\partial x}\\ &\qquad+\chi^2\rho T\frac{\partial u}{\partial x}+\chi^2T\frac{\partial u}{\partial x}=\chi\zeta\left(\frac{\partial u}{\partial x}\right)^2+\xi\frac{\partial^2T}{\partial x^2}\end{aligned} \tag{7}$$

with

$$\begin{aligned}\zeta&=(\beta+\tfrac{4}{3}\mu)/\rho_0a_0L, \quad \xi=\kappa/\rho_0c_va_0L,\\ \chi&=(\gamma-1)^{1/2}, \quad \gamma=c_p/c_v=c_0^2/a_0^2.\end{aligned} \tag{8}$$

c_0 and a_0 are the adiabatic and isothermal speeds of sound, respectively. In Eq. (7) tildes have been dropped with the understanding that all quantities are now dimensionless. This normalization has been chosen to agree with the linearized theory where it proves convenient. We leave the length scale L unspecified for the moment. In the course of our analysis it proves useful to change the choice of L.

In vector notation, we write

$$\mathbf{V}=\mathbf{v}_0+\mathbf{v},$$

where

$$\begin{aligned}\mathbf{V}&=[\rho/\rho_0,\ u/a_0,\ T/\chi T_0],\\ \mathbf{v}_0&=[1,0,\chi^{-1}],\end{aligned} \tag{9}$$

and

$$\mathbf{v}=(\tilde{\rho},\tilde{u},\tilde{T}).$$

Again, we drop tildes in $\mathbf{v}$ with the understanding that quantities are dimensionless.

If we, for example, assume that $\mathbf{v}$ is of compact support, integrating (7) over all space yields

$$\int_{-\infty}^{\infty}\rho\,dx=m,\qquad\int_{-\infty}^{\infty}(u+\rho u)\,dx=P,$$

$$\int_{-\infty}^{\infty}[T+\rho T+\tfrac{1}{2}\chi u(u+\rho u)]\,dx=E \tag{10}$$

with m, P, E constant. The above results just tell us that total mass, momentum, and energy are conserved for all time. Similar expressions hold in two and three dimensions.

III. BREAKDOWN OF LINEAR THEORY

In this section we consider the description furnished by small disturbance theory. The perturbed initial data is taken to be small, say $O(\epsilon)$, with ϵ a measure of this data. (The restriction to small initial data will be lifted in Sec. VII.) Assuming that small data give rise to small disturbances we obtain to the lowest order,

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\u\\T\end{bmatrix}+\begin{bmatrix}0&1&0\\1&0&\chi\\0&\chi&0\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\rho\\u\\T\end{bmatrix}=\begin{bmatrix}0&0&0\\0&\zeta&0\\0&0&\xi\end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{bmatrix}\rho\\u\\T\end{bmatrix} \tag{11}$$

or in symbolic form

$$\frac{\partial}{\partial t}\mathsf{v}+\mathsf{A}\frac{\partial}{\partial x}\mathsf{v}=\mathsf{D}\frac{\partial^2}{\partial x^2}\mathsf{v}. \tag{12}$$

Writing

$$\mathbf{w}^1=(\chi,0,-1),\quad \mathbf{w}^2=(1,\gamma^{1/2},\chi),\quad \mathbf{w}^3=(-1,\gamma^{1/2},-\chi) \tag{13}$$

and

$$\sigma=\{\gamma\zeta+(\gamma-1)\xi\}/2\gamma,\qquad \nu=\xi/\gamma$$

the approximate solution to (11) subject to

$$\mathbf{v}(t=0)=\mathbf{v}^0(x) \tag{14}$$

is given by the space convolution[3]

$$\mathbf{v}\sim V_{\mathrm{NS}}*\mathbf{v}^0 \tag{15}$$

with

$$\mathbf{V}_{\mathrm{NS}}=\frac{\exp(-x^2/4\nu t)}{\gamma(4\pi\nu t)^{1/2}}\,\mathbf{w}^1\mathbf{w}^1+\frac{\exp[-(x-\gamma^{1/2}t)^2/4\sigma t]}{2\gamma(4\pi\sigma t)^{1/2}}\mathbf{w}^2\mathbf{w}^2+\frac{\exp[-(x+\gamma^{1/2}t)^2/4\sigma t]}{2\gamma(4\pi\sigma t)^{1/2}}\mathbf{w}^3\mathbf{w}^3. \tag{16}$$

This is composed of diffuse waves traveling to the right and left with the adiabatic speed $\gamma^{1/2}$ and a contact or purely diffuse mode at the origin.

We also note that[3]

$$\mathbf{V}_E=\lim_{\sigma,\nu\to 0}\mathbf{V}_{\mathrm{NS}}=[\delta(x)/\gamma]\mathbf{w}^1\mathbf{w}^1+[\delta(x-\gamma^{1/2}t)/2\gamma]\mathbf{w}^2\mathbf{w}^2+[\delta(x+\gamma^{1/2}t)/2\gamma]\mathbf{w}^3\mathbf{w}^3 \tag{17}$$

exactly satisfies the linearized inviscid equations, i.e., (12) with $\mathbf{D}=0$. These, subject to (14) are solved by

$$\mathbf{v}=\mathbf{V}_E*\mathbf{v}^0. \tag{18}$$

The purpose of this section is to examine in what way solutions to (11) approximate solutions to the nonlinear system (7). Barring exact or accurate estimates of solutions to (7) we must fall back on plausible criteria. Two such criteria will be considered. (i) If for a solution to (11) terms of (7) neglected in deriving (11) are small compared with those retained, we will state that the solution is self-consistent. Alternately, if on the basis of the linearized solution typical $O(\epsilon^2)$ terms are small compared with typical $O(\epsilon)$ terms, the linear solution is self-consistent. As a condition characterizing the linearization of (7) we may take

$$u\frac{\partial u}{\partial x}\ll\zeta\frac{\partial^2 u}{\partial x^2}. \tag{19}$$

(ii) A more severe condition, and hence one that is more reliable, is based on the computation or estimation of the second order solution. If the second-order solution as computed from the expansion of (7) is small compared with the linearized solution, the latter is then the leading term of a formal asymptotic expansion. When this is the case, the sense of the approximation is well known.

In both cases the conditions will depend on the region of x, t space. Also, neither criterion guarantees that the approximation is valid. Indeed, it is easy to contrive examples of self-consistent linearized solutions which are invalid (although there are no ready examples violating the second criterion).

First, we consider the linear approximation for a sharply varying initial disturbance. It will suffice to initially take

$$\rho^0(x)=\epsilon[H(x+l)-H(x)],\qquad u^0=T^0=0,$$

where $H(x)$ is the Heaviside function. Carrying out the calculation indicated by (15) and focusing on the right-traveling wave, we find

$$u(x,t)\sim\frac{\epsilon}{2(\gamma)^{1/2}}\times\left[\operatorname{erfc}\left(-\frac{x-\gamma^{1/2}t+l}{(4\sigma t)^{1/2}}\right)-\operatorname{erfc}\left(-\frac{x-\gamma^{1/2}t}{(4\sigma t)^{1/2}}\right)\right].$$

Using condition (19) the solution is self-consistently linear if $\epsilon l\ll 1$, and for $\epsilon l\geq O(1)$, a typical term in the ratio of inertial to diffuse terms is

$$\epsilon\left(\frac{\sigma t}{\gamma}\right)^{1/2}\frac{\exp(-X^2)\operatorname{erfc}(-X)}{X\exp(-X^2)-X_l\exp(-X_l^2)},$$

where

$$X=[x-\gamma^{1/2}t]/(4\sigma t)^{1/2}$$

and

$$X_l=[x-\gamma^{1/2}t+l]/(4\sigma t)^{1/2}.$$

We have made use of the freedom in the length scale L, (8), by taking $\zeta=O(1)$. From this it is seen that the nonlinear terms become as important as the diffusive terms for $t=O(\epsilon^{-2})$. If we denote dimensional time by t', then linear theory breaks down for

$$t'\approx\mu/\epsilon^2 p_0. \tag{20}$$

For air under normal conditions $\mu/p_0\approx 3\times 10^{-6}$ sec, so that this breakdown occurs at an early stage.

For this case the resolution of the initial discontinuity is first governed by diffusion. Then, in the time indicated in (20), diffusion has been too successful in spreading out the discontinuity and nonlinear steepening has become effective. In short, a nonlinear shock structure is required.

Next, we consider the case of smooth initial data

$$\rho^0=(\epsilon/\pi^{1/2})\exp(-x^2/l^2),\qquad u^0=T^0=0.$$

It is now convenient to choose the scale $L=l$ in (6). Then, considering the right-traveling wave, we find

$$u(x,t)\sim\frac{\epsilon\exp[-(x-\gamma^{1/2}t)^2/(1+4\sigma t)]}{2[\gamma\pi(1+4\sigma t)]^{1/2}}.$$

We note that this description is inviscid, i.e., $\sigma t\ll 1$, for dimensional time t' such that

$$\rho_0 l^2/\mu_0\gg t'. \tag{21}$$

For air under normal conditions and say $l\approx 1$ m, the ratio in (21) is $\approx 10^5$ sec; or, in other words as is well known for smooth phenomena, the description is inviscid (at least for "finite" time). We may, therefore, employ an inviscid description for smooth data.

Linearized inviscid theory predicts constant properties on characteristics and, therefore, leads to a self-consistent picture uniformly in $x-t$. We next examine this solution, (18), to see if it is the lead term of a formal asymptotic expansion, i.e., we compute the second order.

To compute the second order we write

$$\mathbf{v}=\mathbf{v}_0+\epsilon\mathbf{v}_1+\epsilon^2\mathbf{v}_2+\cdots \tag{22}$$

with $\mathbf{v}_0$ given by (9) and $\mathbf{v}_1$ satisfying (12) with $\mathsf{D}=0$. The solution for $\mathbf{v}_1$ subject to (14) is given by (18). $\mathbf{v}_2$ is governed by

$$\left(\frac{\partial}{\partial t}+\mathsf{A}\frac{\partial}{\partial x}\right)\mathbf{v}_2 = \mathsf{X} = -\begin{bmatrix} \rho_1\dfrac{\partial u_1}{\partial x}+u_1\dfrac{\partial}{\partial x}\rho_1 \\ \rho_1\dfrac{\partial u_1}{\partial t}+u_1\dfrac{\partial u_1}{\partial x}+\chi T_1\dfrac{\partial\rho_1}{\partial x}+\chi\rho_1\dfrac{\partial T_1}{\partial x} \\ \rho_1\dfrac{\partial T_1}{\partial t}+u_1\dfrac{\partial T_1}{\partial x}+\chi\rho_1\dfrac{\partial u_1}{\partial x}+\chi^2 T_1\dfrac{\partial u_1}{\partial x} \end{bmatrix}. \tag{23}$$

To solve this we note first that $\mathbf{w}^1$, $\mathbf{w}^2$, $\mathbf{w}^3$ are eigenvectors of A with respective eigenvalues 0, $\gamma^{1/2}$, $-\gamma^{1/2}$. Multiplying (23), for example, by $\mathbf{w}^2$ we obtain

$$\left(\frac{\partial}{\partial t}+\gamma^{1/2}\frac{\partial}{\partial x}\right)\mathbf{w}^2\cdot\mathbf{v}_2=\mathbf{w}^2\cdot\mathsf{X}. \tag{24}$$

In (24) the left-hand side is a derivative in the direction $(1, \gamma^{1/2})$ and the right-hand side is composed of functions of $x\pm\gamma^{1/2}t$, x multiplied two at a time. For initial data concentrated in the finite region the right-hand side of (24) tends to a constant as $t\rightarrow\infty$, with $x-\gamma^{1/2}t$ held fixed. Hence, the solution to (24) can be written as

$$\mathbf{w}^2\cdot\mathbf{v}_2=t\mathbf{w}^2\cdot\mathsf{X}+\Delta$$

with $\Delta=o(t)$ as $t\rightarrow\infty$, $x-\gamma^{1/2}t$ fixed.

The above discussion demonstrates that linearized theory fails as the leading term of a formal asymptotic expansion when $t\sim\epsilon^{-1}$. This is clear from (22) with $\mathbf{v}_1$ and $\mathbf{v}_2$ substituted, for when $t=O(\epsilon^{-1})$, $\epsilon\mathbf{v}_1$, and $\epsilon^2\mathbf{v}_2$ become the same order. This particular difficulty with linearized inviscid theory has been resolved by Whitham.[5–7] An alternate treatment is implicitly given in the next section.

IV. METHOD OF MULTIPLE SCALES

The solution for $\mathbf{v}_2$, discussed at the close of the previous section indicates the presence of a slow variation in time, $O(\epsilon t)$, in the solution for $\mathbf{v}$. This in turn suggests that this secularity may be removed by the method of multiple scales,[19] i.e., by explicitly assuming a slow variation. We, therefore, write

$$\mathbf{v}=\mathbf{v}(x, t, t_1), \qquad t_1=\epsilon t. \tag{25}$$

Since neither the geometry nor the assumed initial data have a slow spatial variation, we suppress an x_1 dependence in (25).

Proceeding formally we introduce (22) and (25) into the governing equations (7) and obtain

$$\epsilon\left(\frac{\partial}{\partial t}+\mathsf{A}\frac{\partial}{\partial x}\right)\mathbf{v}_1+\epsilon^2\left(\frac{\partial}{\partial t}+\mathsf{A}\frac{\partial}{\partial x}\right)\mathbf{v}_2+\epsilon^2\frac{\partial}{\partial t_1}\mathbf{v}_1 -\epsilon^2\mathsf{X}-\epsilon\mathsf{D}\frac{\partial^2}{\partial x^2}\mathbf{v}_1=O(\epsilon^3, \epsilon^2\mathsf{D}).$$

D and X are defined by (12) and (23), respectively. The normalizing length L is now regarded as characterizing the scale variations in the flow. With the exception of shock waves—which we discuss momentarily—this makes D extremely small, much smaller than ϵ under normal circumstances.

To lowest order we have

$$\left(\frac{\partial}{\partial t}+\mathsf{A}\frac{\partial}{\partial x}\right)\mathbf{v}_1=0$$

which has (18) as a solution. More explicitly,

$$\mathbf{v}_1=\mathbf{w}^1g(x, t_1)+\mathbf{w}^2f(x-\gamma^{1/2}t, t_1)+\mathbf{w}^3h(x+\gamma^{1/2}t, t_1), \tag{26}$$

where the $\mathbf{w}^i$ are given by (13). g, f, h are scalar functions, which from their arguments are seen to represent the contact region, a wave to the right and a wave to the left, respectively.

Continuing to the next order we write

$$\left(\frac{\partial}{\partial t}+\mathsf{A}\frac{\partial}{\partial x}\right)\mathbf{v}_2=-\frac{\partial}{\partial t_1}\mathbf{v}_1+\mathsf{X}+\epsilon^{-1}\mathsf{D}\frac{\partial^2}{\partial x^2}\mathbf{v}_1=\mathsf{Y}. \tag{27}$$

In this we have retained the leading dissipative term. This may be justified by formally taking $\mathsf{D}=O(\epsilon)$, thereby permitting relatively sharp gradients in the theory. A more fundamental reason, however, is that the scaling which has been chosen is incorrect in shock waves. The rescaling which is correct in shock waves, as is well known, equates nonlinear and diffusive terms as done above. Outside the shock region the diffusive terms are small and we are then merely carrying negligible terms, i.e., in such case the theory is inviscid.

In its present form, solutions to (27) exhibit a secularity in the same way that was shown in Sec. III. To eliminate the secularity we can now impose the secularity condition

$$\lim_{t\to\infty}\mathbf{w}^i\cdot\mathsf{Y}=0, \qquad i=1, 2, 3.$$

Before going into this we will assume that our data are effectively concentrated in finite space e.g., it could be of compact support although such a severe restriction is not necessary. It then follows that as $t\to\infty$, the modes disengage, and, therefore, products of terms from different modes vanish in this limit.

Specifically considering

$$\lim_{t\to\infty} \mathbf{w}^2\cdot\mathbf{Y}=0$$

for $(x-\gamma^{1/2}t)<\infty$, we obtain the Burgers equation[10]

$$\frac{\partial}{\partial t_1}f+\tfrac{1}{2}[\gamma^{1/2}(\gamma+1)]f\frac{\partial}{\partial\alpha}f=\frac{\sigma}{\epsilon}\frac{\partial^2 f}{\partial\alpha^2} \quad (28)$$

with $\alpha=x-\gamma^{1/2}t$. This governs the right traveling mode, and in a similar way we obtain the Burgers equation

$$\frac{\partial h}{\partial t_1}-\tfrac{1}{2}[\gamma^{1/2}(\gamma+1)]h\frac{\partial h}{\partial\beta}=\frac{\sigma}{\epsilon}\frac{\partial^2 h}{\partial\beta^2} \quad (29)$$

with $\beta=x+\gamma^{1/2}t$ for the left traveling mode, and

$$\frac{\partial}{\partial t_1}g=\frac{\nu}{\epsilon}\frac{\partial^2}{\partial x^2}g \quad (30)$$

for the contact region. This last mode, therefore, remains linear. It is also interesting to note that secularity is solely due to dissipation in the last mode.

At this point we find it convenient to set $\epsilon=1$, since this only played a formal role in our analysis. This replaces t_1 by t in (28), (29), and (30). Note that for example (28) may alternately be written as

$$\frac{\partial}{\partial t}f+\gamma^{1/2}\frac{\partial}{\partial x}f+\tfrac{1}{2}[\gamma^{1/2}(\gamma+1)]f\frac{\partial}{\partial x}f=\sigma\frac{\partial^2 f}{\partial x^2}.$$

An interesting feature now appears. Let us ignore the conditions under which (28)–(30) were derived and apply them to small initial data. Then, the nonlinear terms can be formally dropped. A simple substitution then shows that (16) satisfies the resulting equations. Hence, the derived system is valid up to the initial instant so that (28)–(30) are uniformly valid in their approach to the initial instant.

V. ASYMPTOTIC SOLUTION TO THE INITIAL-VALUE PROBLEM

To determine the solution of the flow field we solve (28)–(30) and assign appropriate data to these. Writing

$$\frac{\partial}{\partial t}F+aF\frac{\partial F}{\partial\alpha}=b\frac{\partial^2 F}{\partial\alpha^2}, \quad (31)$$

it has been shown that

$$F(\alpha,t;a,b,F_0)=\int_{-\infty}^{\infty}\left\{\frac{(\alpha-\beta)}{at},F_0(\beta)\right\}\exp\left[(2b)^{-1}\left(-a\int_0^\beta F_0(s)\,ds-\frac{(\alpha-\beta)^2}{2t}\right)\right]d\beta \times\left\{\int_{-\infty}^{\infty}\exp\left[(2b)^{-1}\left(-a\int_0^\beta F_0(s)\,ds-\frac{(\alpha-\beta)^2}{2t}\right)\right]d\beta\right\}^{-1} \quad (32)$$

is a solution.[11,15] The brace in the numerator gives two alternate forms for the coefficient of the exponential term. The equivalence of these two forms follows from integration by parts. Choosing the second form and multiplying numerator and denominator by $(4\pi bt)^{1/2}$, we obtain

$$\lim_{t\to0}F(\alpha,t;a,b,F_0)=F_0(\alpha).$$

This follows from the δ-function property of

$$(4\pi bt)^{-1/2}\exp(-x^2/4bt)$$

in the limit of $t\to0$ (or $b\to0$).

Another useful property of (32) applies to the case of small data, say $F_0=O(\epsilon)$. In this case, for fixed t, we have as $\epsilon\to0$,

$$F(\alpha,t;a,b,F_0)\sim\int_{-\infty}^{\infty}\frac{F_0(\beta)}{(4\pi bt)^{1/2}}\exp\left(-\frac{(\alpha-\beta)^2}{4bt}\right)d\beta.$$

Comparison with (16) shows this is just the linearized viscous solution. From these remarks the choice of initial data are clear. Noting the definition (26) and still employing $\mathbf{v}^0$ to denote the perturbed initial data, we define

$$g_0=\mathbf{w}^1\cdot\mathbf{v}^0/\gamma,\qquad f_0=\mathbf{w}^2\cdot\mathbf{v}^0/2\gamma,\qquad h_0=\mathbf{w}^3\cdot\mathbf{v}^0/2\gamma. \quad (33)$$

Then, the uniform asymptotic solution which we denote by $\mathbf{v}_B$, is given by,

$$\mathbf{v}\sim\mathbf{v}_B=\mathbf{w}^1F(x,t;0,\nu,g_0)+\mathbf{w}^2F[x-\gamma^{1/2}t,t;\gamma^{1/2}(\gamma+1)/2,\sigma,f_0]+\mathbf{w}^3F[x+\gamma^{1/2}t,t;-\gamma^{1/2}(\gamma+1)/2,\sigma,h_0]. \quad (34)$$

VI. LIMITING FORMS OF THE SOLUTION

We now turn to some limiting forms of solution (34). Certain of these have been already mentioned. From Sec. V we already know that (34) for t fixed and small initial data produce linearized theory. It then also follows that linear inviscid theory is valid except in regions of sharp gradients.

Inviscid Limit

Aspects of the inviscid limit of the Burgers equation are considered by a number of authors[8,15–18] as well as by Burgers[10] himself. To facilitate discussion we fix attention on the right traveling mode and, in particular, on the density perturbation of this wave. Denoting this by ρ^+ we have

$$\rho^+ = F[x-\gamma^{1/2}t, t; \gamma^{1/2}(\gamma+1)/2, \sigma, f_0]$$
$$= \left(\int_{-\infty}^{\infty} f_0(\beta) \exp[-(2\sigma)^{-1}\varphi(\alpha,\beta,t)]\, d\beta\right) \times \left(\int_{-\infty}^{\infty} \exp[-(2\sigma)^{-1}\varphi(\alpha,\beta,t)]\, d\beta\right)^{-1},$$

where

$$\varphi(\alpha,\beta,t) = \frac{(\alpha-\beta)^2}{2t} + \tfrac{1}{2}[\gamma^{1/2}(\gamma+1)] \int_{\beta_0}^{\beta} f_0(s)\, ds \tag{35}$$

and $\alpha = x - \gamma^{1/2}t$. The lower limit of integration β_0 can be arbitrary.

Under the inviscid limit, effectively $\sigma \to 0$, on (35) the main contributions to the integrals come from the minima of φ. These values of $\beta(\alpha, t)$ are such that

$$\tfrac{1}{2}[\gamma^{1/2}(\gamma+1)]f_0(\beta) + [(\beta-\alpha)/t] = 0,$$

where $\alpha = x - \gamma^{1/2}t$ and t are regarded as fixed. First, assuming that φ has a single global minimum b, this is determined implicitly by

$$b(\alpha,t) = \alpha - \tfrac{1}{2}[\gamma^{1/2}(\gamma+1)^{1/2}]f_0(b)t.$$

Then, from (35) the lead term is

$$\rho^+ \sim \rho_a = f_0(b) \tag{36}$$

and as may be verified directly

$$\frac{\partial}{\partial t}\rho_a + \tfrac{1}{2}[\gamma^{1/2}(\gamma+1)]\rho_a \frac{\partial \rho_a}{\partial \alpha} = 0 \tag{37}$$

which is just (28) with $\sigma = 0$.

Next, suppose that φ has two global minima, b_1 and b_2. This signals the presence of a shock wave, say, at X_s. We then have $b_{1,2}(\alpha, t)$ determined implicitly through

$$\tfrac{1}{2}[\gamma^{1/2}(\gamma+1)]f_0(b_{1,2}) + (b_{1,2}-\alpha)/t = 0 \tag{38}$$

and, since $\varphi(b_1) = \varphi(b_2)$,

$$\tfrac{1}{2}[\gamma^{1/2}(\gamma+1)]\int_{b_1}^{b_2} f_0(s)\, ds = \frac{(\alpha-b_1)^2}{2t} - \frac{(\alpha-b_2)^2}{2t}. \tag{39}$$

Here, $\alpha = X_s - \gamma^{1/2}t$.

Solving (38) for $b_{1,2}(\alpha, t)$ and substituting into (39), gives the shock trajectory $X_s(t)$. To find the shock speed we differentiate (39) implicitly to find α_t. Using (38)

$$\alpha_t = \tfrac{1}{2}[\gamma^{1/2}(\gamma+1)]\tfrac{1}{2}[f_0(b_1)+f_0(b_2)]$$
$$= \tfrac{1}{2}[\gamma^{1/2}(\gamma+1)]\tfrac{1}{2}[\rho_a(b_1)+\rho_a(b_2)] = U_s - \gamma^{1/2}, \tag{40}$$

where we have used (36) to associate a density with conditions up- and downstream. Equation (40) gives the shock speed in a frame of reference moving with the adiabatic speed. As indicated to obtain the shock speed U_s in the rest frame, we add $\gamma^{1/2}$ to (40). The same result may also be obtained directly from the inviscid form of the Eq. (37).

A number of interesting geometrical constructions describing the above inviscid shock theory are given by Whitham[6] and Lighthill.[8] Their discussions also apply to a variety of more complicated phenomena, e.g., a shock overtaking another shock.

Another consequence of the above stationary point analysis is the calculation of density at the shockwave. Thus, we evaluate (35) by means of Laplace's formula, taking into account the presence of two global minima of φ. After some straightforward manipulation this gives

$$\rho^+ \sim \rho_s = \frac{f_0(b_1)\{1+\tfrac{1}{2}t[\gamma^{1/2}(\gamma+1)]f_x(b_2)\}^{1/2} + f_0(b_2)\{1+\tfrac{1}{2}t[\gamma^{1/2}(\gamma+1)]f_x(b_1)\}^{1/2}}{\{1+\tfrac{1}{2}t[\gamma^{1/2}(\gamma+1)]f_x(b_1)\}^{1/2} + \{1+\tfrac{1}{2}t[\gamma^{1/2}(\gamma+1)]f_x(b_2)\}^{1/2}},$$
$$= [\rho_a(b_1) + \rho_a(b_2)\Lambda]/(1+\Lambda) \tag{41}$$

where

$$\Lambda = \{[2+t\gamma^{1/2}(\gamma+1)\rho_{ax}(b_1)]/[2+t\gamma^{1/2}(\gamma+1)\rho_{ax}(b_2)]\}^{1/2}.$$

As before we have used (36) to associate up- and downstream densities. Thus, we have a density ρ_s to associate with the shock speed U_s—something which does not arise out of inviscid theory alone. One might envision using this result as follows. First, one constructs the locus of the shock $X_s(t)$ from inviscid theory. Next, one constructs a structured shock going from $\rho_a(b_1)$ to $\rho_a(b_2)$ at each instant, having a value for density given by (41) at $X_s(t)$. (Recall that the location of a structured shock requires an additional condition.) If the up- and downstream states are constant then $\rho_s = \tfrac{1}{2}[\rho(b_1)+\rho(b_2)]$. Thus (41) may be regarded as giving the leading term in the expansion of shock slip away from this case, when the up- and downstream states depart from this case.

The form of a structured shock in the present approximation also follows from (35). This calculation is made by Lighthill,[8] and in our notation has the form

$$\rho^+ \sim \frac{\rho_a(b_1) + \rho_a(b_2)\Lambda \exp\{+[\gamma^{1/2}(\gamma+1)\Delta\rho_s(x-X_s)/2\sigma]\}}{1+\Lambda \exp\{+[\gamma^{1/2}(\gamma+1)\Delta\rho_s(x-X_s)/2\sigma]\}}, \tag{42}$$

where

$$\Delta\rho_s=\rho_a(b_1)-\rho_a(b_2).$$

Notice that (42) checks with (41), in that it equals (41) at the shock locus $x=X_s$. In Ref. 8, Lighthill associates the mean value $\frac{1}{2}[\rho_a(b_1)+\rho_a(b_2)]$ with the inviscid shock. Using this it is concluded that the shock is incorrectly located, and Lighthill introduces the notion of "shock-wave displacement due to diffusion" in order to relocate the shock. As mentioned above the mean value is taken up at the shock wave only under special circumstances—and its use is really arbitrary. Also, since (41) does not involve any dissipative parameters, it would seem that the interpretation taken by us is more appropriate.

It is important to notice a nonuniformity in the inviscid limit. For (35) contains t in the denominator and, hence, although σ is small the inviscid limit becomes questionable for $t\to\infty$. We now treat this case.

Large Time Limit

For the purpose of performing the limit $t\to\infty$, it is convenient to assume that the initial data is of compact support. Specifically, we take g_0, f_0, h_0, of (33) to be zero outside the interval $(0, C)$. Considering the general solution (32), of the Burgers equation, we rewrite it as

$$F(\alpha,t,a,b,F_0)=\exp\left(-\frac{\alpha^2}{4bt}\right)\int_0^C F_0(\beta)\exp\left(-\frac{a}{2b}\int_0^\beta F_0(s)\,ds+\frac{\alpha\beta}{2bt}-\frac{\beta^2}{4bt}\right)d\beta$$
$$\times\left[\left(\int_{-\infty}^0+\exp\left(-\frac{a}{2b}M\right)\int_C^\infty\right)\exp\left(-\frac{(\alpha-\beta)^2}{4bt}\right)d\beta+\exp\left(-\frac{\alpha^2}{4bt}\right)\int_0^C\exp\left(-\frac{a}{2b}\int_0^\beta F_0(s)\,ds+\frac{2\alpha\beta-\beta^2}{4bt}\right)d\beta\right]^{-1} \tag{43}$$

where

$$M(F_0)=\int_0^C F_0(s)\,ds,\qquad N(F_0)=\int_0^C F_0(\beta)\beta\exp\left(-\frac{a}{2b}\int_0^\beta F_0(s)\,ds\right)d\beta. \tag{44}$$

The second constant N will be needed momentarily. We consider $t\to\infty$ in the sense; $C^2/\sigma t=O(1)$, $\alpha/bt=O(1)$. (Outside the latter region F is easily seen to be exponentially small.) A straightforward expansion next shows

$$F\sim R[\alpha,t,a,b,M(F_0),N(F_0)],$$

where

$$R[\alpha,t,a,b;A,B]$$
$$=\frac{\exp(-\alpha^2/4bt)\{(2b/a)[1-\exp(-aA/2b)]+(\alpha/2bt)B\}}{(\pi bt)^{1/2}\{2\exp(-aA/2b)+[1-\exp(-aA/2b)]\operatorname{erfc}[\alpha(4bt)^{-1/2}]+[aB\exp(-\alpha^2/4bt)/b(4b\pi t)^{1/2}]\}}. \tag{45}$$

Therefore, from (34) we have for $t\to\infty$,

$$\mathbf{v}_B\sim\mathbf{v}_A=\mathbf{w}^1R[x,t,0,\nu,M(g_0),N(g_0)]+\mathbf{w}^2R\,[x-\gamma^{1/2}t,t,\tfrac{1}{2}[\gamma^{1/2}(\gamma+1)],\sigma,M(f_0),N(f_0)]$$
$$+\mathbf{w}^3R\,[x+\gamma^{1/2}t,t,-\tfrac{1}{2}[\gamma^{1/2}(\gamma+1)],\sigma,M(h_0),N(h_0)]. \tag{46}$$

A point of interest is that the expression (45) itself is a solution of Burgers equation (31)—this may be verified by substitution. Another property of R is that

$$\int_{-\infty}^{\infty}R[\alpha,t,a,b,A,B]\,d\alpha=A. \tag{47}$$

The special solutions $R[\alpha,t,a,b,A,0]$ and $R[\alpha,t,a,b,0,B]$ have been considered by Lighthill.[8] The former is interpreted by him as arising from a pure compression pulse and the latter, referred to as a balanced N wave, as arising from a piston undergoing a single oscillation. We have, therefore, demonstrated that the development of an initial disturbance may asymptotically be resolved in terms of a compression and a balanced N wave. The compression wave in (45) is clearly the leading term, although this is not uniformly true for small M. In the latter case, if $M=O(1)$, the solution may be linearized, i.e., the flow is described by the linear dissipative equations.

VII. FINITE AMPLITUDE DISTURBANCES

To summarize our results we have formally shown that

$$\mathbf{v}\sim\mathbf{v}_B\sim\mathbf{v}_A,$$

where $\mathbf{v}_B$ is given by (34) and $\mathbf{v}_A$ by (46). The latter asymptotic form, of course, requires a longer elapsed time. Next, we notice from integrating the Burgers equation over the space variable that

$$\int_{-\infty}^{\infty} F(\alpha, t, a, b, F_0)\, d\alpha = \int_{-\infty}^{\infty} F_0\, d\alpha.$$

This requires mild restrictions on the data F_0. Then, since

$$(\mathbf{w}^1\mathbf{w}^1/\gamma)+(\mathbf{w}^2\mathbf{w}^2/2\gamma)+(\mathbf{w}^3\mathbf{w}^3/2\gamma)=\mathbf{1},$$

it also follows that

$$\int_{-\infty}^{\infty} \mathbf{v}_B\, dx = \int_{-\infty}^{\infty} \mathbf{v}_0\, dx.$$

But from (47) it also follows that $\mathbf{v}_A$ possesses the same property. We therefore, have that

$$\int_{-\infty}^{\infty} \mathbf{v}_B(x,t)\, dx = \int_{-\infty}^{\infty} \mathbf{v}_A(x,t)\, dx = \int_{-\infty}^{\infty} \mathbf{v}(x, t=0)\, dx. \tag{48}$$

Comparing this result with (10) we find a certain discrepancy. For in (10) quadratic and cubic terms enter while (48) is entirely linear in the perturbation. However, for the case of small initial disturbances (48) is seen to be a linearized expression of conservation. If the perturbed quantities are $O(\epsilon)$, the error in (48) is $O(\epsilon^2)$. A difficulty in this line of reasoning occurs if the integration itself is $O(\epsilon^{-1})$, then, e.g.,

$$\int_{-\infty}^{\infty} \rho u\, dx = O(\epsilon)$$

even though $\mathbf{v}=O(\epsilon)$. Therefore, if the support of the initial data is $O(\epsilon^{-1})$, some further investigation is required. We now take this up in the larger framework of finite amplitude initial data.

It is clearly plausible that finite amplitude initial disturbances after a sufficient time will decay and themselves become relatively small perturbations. Therefore, it is reasonable to assume that our analysis will also apply to finite amplitude initial disturbances after a certain time has elapsed in their evolution. One difficulty with this approach is signaled by the problem of conserving the global properties (10). Stated in other terms, our difficulty lies in choosing initial data which should be employed in solving the finite amplitude problem. We will refer to this as asymptotic initial data, and denote it by $\hat{\mathbf{v}}^o$.

For the case of finite amplitude disturbances we conjecture that the correct asymptotic behavior is obtained by taking the asymptotic initial data in (33) to be

$$\hat{\mathbf{v}}^o = \begin{bmatrix} \hat{\rho}^o \\ \hat{u}^o \\ \hat{T}^o \end{bmatrix} = \begin{bmatrix} \rho^o \\ u^o+\rho^o u^o \\ T^o+\rho^o T^o+\frac{1}{2}[\chi u^o(u^o+\rho^o u^o)] \end{bmatrix}. \tag{49}$$

Denoting the solution obtained with this data, by $\hat{\mathbf{v}}$, it then follows that

$$\int_{-\infty}^{\infty} \hat{\mathbf{v}}\, dx = \int_{-\infty}^{\infty} \hat{\mathbf{v}}^o\, dx$$

$$= \int_{-\infty}^{\infty} \begin{bmatrix} \rho^o \\ u^o+\rho^o u^o \\ T^o+\rho^o T^o+\frac{1}{2}(\chi u^o)(u^o+\rho^o u^o) \end{bmatrix} dx.$$

To complete the argument we observe that, for example,

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} (\hat{u}+\hat{\rho}\hat{u})\, dx = \int_{-\infty}^{\infty} \hat{u}\, dx$$

which follows from simple estimates. A similar limit applies to the remaining integrals of (10).

The solution obtained in this way will certainly result in an unreliable description for short times; but, of course, our analysis is not able to deal with the finite amplitude initial epoch. The main basis for the conjectured solution is: (a) the long time behavior should be governed by the weakly nonlinear equations which we have obtained, and (b) the asymptotic initial data (49) preserves the integrals given in (10).

VIII. A BOUNDARY PROBLEM

From the analysis carried out in the preceding parts of this paper, it follows that in the case of small amplitude initial disturbances, the problem may be treated as linear for $t \ll O(\epsilon^{-1})$. It has been shown[3] that a number of boundary value problems in linearized theory may be treated as equivalent initial value problems. We now apply our formulation to one such problem.

Consider the following mixed initial and boundary value problem

$$\mathbf{v}^o = \begin{bmatrix} \rho(t=0) \\ u(t=0) \\ T(t=0) \end{bmatrix} = \begin{bmatrix} \epsilon H(x)H(L-x) \\ 0 \\ 0 \end{bmatrix}$$

and

$$\frac{\partial T}{\partial x}(x=0,t) = u(x=0,t) = 0. \tag{50}$$

This describes the situation in a semi-infinite shock tube having an insulating wall.

In the linear theory this is solved by considering the pure initial value problem

$$\mathbf{v}^o = \begin{bmatrix} \rho(t=0) \\ u(t=0) \\ T(t=0) \end{bmatrix} = \begin{bmatrix} \epsilon H(x+L)H(L-x) \\ 0 \\ 0 \end{bmatrix}. \tag{51}$$

From the symmetry of the data, condition (50) is met for all time, and this initial value problem solves the problem posed above.

Denoting the linearized solution by the subscript l, it is given by[3]

$$\mathrm{v}_l=(\epsilon\chi/2\gamma)\mathrm{w}^1[\mathrm{erfc}(\hat{x}-\hat{L})-\mathrm{erfc}(\hat{x}+\hat{L})]$$
$$-(\epsilon/4\gamma)\mathrm{w}^2[\mathrm{erfc}(\bar{\alpha}-\bar{L})-\mathrm{erfc}(\bar{\alpha}+\bar{L})]$$
$$+(\epsilon/4\gamma)\mathrm{w}^3[\mathrm{erfc}(\bar{\beta}-\bar{L})-\mathrm{erfc}(\bar{\beta}+\bar{L})],$$

where

$$\hat{z}=z/(4\nu t)^{1/2},\qquad \bar{z}=z/(4\sigma t)^{1/2},$$

and $\alpha=x-\gamma^{1/2}t$ and $\beta=x+\gamma^{1/2}t$.

This solution is not valid for long times, the corrected form is given by (34) with the data (51) inserted and is

$$\mathrm{v}\sim\frac{\epsilon\chi}{2\gamma}\mathrm{w}^1[\mathrm{erfc}(\hat{x}-\hat{L})-\mathrm{erfc}(\hat{x}+\hat{L})]+\frac{\epsilon\mathrm{w}^2}{2\gamma}\left\{1+\frac{\mathrm{erfc}(\bar{\alpha}+\bar{L})+[2-\mathrm{erfc}(\bar{\alpha}-\bar{L})]\exp(-2\epsilon_0 L)}{\exp[-\epsilon_0(\alpha+L-\epsilon_0 t)]\{\mathrm{erfc}[\bar{\alpha}-\bar{L}-\epsilon_0(\sigma t)^{1/2}]-\mathrm{erfc}[\bar{\alpha}+\bar{L}-\epsilon_0(\sigma t)^{1/2}]\}}\right\}^{-1}$$
$$-\frac{\epsilon\mathrm{w}^3}{2\gamma}\left\{1+\frac{\mathrm{erfc}(\bar{\beta}+\bar{L})+[2-\mathrm{erfc}(\bar{\beta}-\bar{L})]\exp(2\epsilon_0 L)}{\exp[\epsilon_0(\beta+L+\epsilon_0 t)]\{\mathrm{erfc}[\bar{\beta}-\bar{L}+\epsilon_0(\sigma t)^{1/2}]-\mathrm{erfc}[\bar{\beta}+\bar{L}+\epsilon_0(\sigma t)^{1/2}]\}}\right\}^{-1}$$

with

$$\epsilon_0=\gamma^{1/2}(\gamma+1)\epsilon/8\sigma\gamma.$$

It is of interest to consider the asymptotic form for this problem. In particular, the pulse traveling to the right has the form

$$\begin{bmatrix}\rho\\ u\\ T\end{bmatrix}\sim\mathrm{w}^2R\,[x-\gamma^{1/2}t,\,t,\,\tfrac{1}{2}[\gamma^{1/2}(\gamma+1)],\,\sigma,\,M,\,N\,],$$

where R is defined by (45) with

$$M=2\epsilon_0 L$$

and

$$N=\epsilon_0^{-1}\{(1-\epsilon_0 L)-(1+\epsilon_0 L)\exp(-2\epsilon_0 L)\}$$

By contrast, the linear solution for pulse as $t\rightarrow\infty$ is

$$\begin{bmatrix}\rho_l\\ u_l\\ T_l\end{bmatrix}\sim\frac{\epsilon L}{\gamma}\begin{bmatrix}1\\ \gamma^{1/2}\\ \chi\end{bmatrix}\frac{\exp(-\alpha^2/4\sigma t)}{(4\pi\sigma t)^{1/2}}.$$

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