# Kinetic theory of suction flow

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An exact solution is obtained for the suction flow problem. Hydrodynamical theory is shown to be valid only under a certain limit. The neighborhood of the boundary and the neighborhood of infinity are shown to always be kinetic layers. The exact solution is then used for a general determination of slip coefficients to be used as perturbation boundary conditions in the Chapman–Enskog–Hilbert method. One explicit result of this, is the resolution of the problem of slip velocity in a pressure gradient field.

## I. INTRODUCTION

A well-known solution of the incompressible Navier–Stokes equations is[1]

$$u/U=1-\exp[-(\rho V/\mu)y], \qquad v/V=-1.$$

This may be interpreted as describing the problem of suction flow in the half-space $y\geq 0$, in which the velocities normal and parallel to the wall at $y=0$ are given by $v$ and $u$, respectively. $V$ represents the constant suction velocity and $U$ the "free-stream" velocity parallel to the wall as $y\rightarrow\infty$. In order to view this problem as a kinetic theory flow, it is more convenient to make the variable $y$ dimensionless with respect to mean free path. On introducing the ratio $\epsilon=V/(2RT_0)^{1/2}$ which is proportional to the Mach number, this can be written as

$$u=U[1-\exp(-\epsilon\tilde{y})].$$

This form implies that hydrodynamical theory is obtained from kinetic theory under the outer limit $\epsilon\tilde{y}$ fixed, $\epsilon\rightarrow 0$. This is supported by our further work below.

In this paper, we find an exact kinetic theory solution to the suction flow problem. To our knowledge, the only other steady boundary value problem for which an exact kinetic theory solution exists is the Kramers problem.[2–4] Strictly speaking, the Kramers problem is not a self-consistent fluid mechanical solution in that it generates an infinite heat rate.

In itself, the suction flow problem is quite specialized—although in fluid mechanics it does provide a first step for a variety of other problems (e.g., in boundary layer and stability theory[1]). It might be of value, therefore, to point out some immediate consequences of this exact solution in kinetic theory. In contrast to the above-mentioned outer limit which leads to hydrodynamics, one can consider the formal inner limit $\tilde{y}$ fixed, $\epsilon\rightarrow 0$. Owing to linearity this generates an infinite number of exact solutions to the kinetic equation. Each of these is, in a sense, a canonical kinetic theory problem and, in fact, the first of these is precisely the Kramers problem.

Another point of importance lies in the observation that the outer flow, i.e., hydrodynamic flow, has nonvanishing velocity derivatives at the boundary. (By contrast, the Kramers problem has only a single nonvanishing velocity derivative at the wall.) This fact is instrumental in obtaining a large class, in fact infinite, of slip coefficients. These slip coefficients, once obtained, may be used in the Chapman–Enskog–Hilbert solution of an arbitrary boundary value problem.

Another application results from considering the analysis of the Poiseulle flow problem.[5] From this and our work we can separate the velocity slip contribution from a pressure gradient along a boundary. The explicit form of this is given in Sec. VII.

Another line of investigation opened to us by the exact solution, lies in examining the Chapman–Enskog–Hilbert procedure for the suction flow problem. More explicitly, if one considers the flow under the limit $z$ fixed, $\epsilon\rightarrow 0$, there is obtained ($z=\epsilon\tilde{y}$),

$$u_H=[1-\exp(-2z)]+\epsilon[2c_0\exp(-2z)]+\cdots,$$

the $c_i$ coefficients are related to successively higher-order slip coefficients. In fact, $c_0$ is the usual velocity slip coefficient.[2,6] These can be determined by the type of inner–outer analysis used by Darrozes[7] and Sone.[8] In our case, however, they are directly determined from the exact solution.

A point of some interest is that $u_H$ exhibits secularity, i.e., the expansion is not uniformly valid for $z\rightarrow\infty$. In Sec. V we demonstrate that the Chapman–Enskog–Hilbert procedure may be generalized by either the method of multiple scales or strained coordinates to render a uniformly valid hydrodynamical solution. However, if we examine the exact solution, we find that

$$u=u_H+O(\exp[-(\tilde{y})^{2/3}]).$$

Ironically, therefore, in spite of this improvement in the Chapman–Enskog–Hilbert procedure we see that the hydrodynamical theory is really only valid for $y=O(\epsilon^{-3})$. In other words, the Chapman–Enskog–Hilbert, or hydrodynamical theory, holds up to the region where secularity sets in, but beyond this, the flow requires a kinetic theory description. The region at infinity is a kinetic layer. Examples of remote kinetic layers have already been found in a variety of problems.[9–11]

## II. FORMULATION OF THE KINETIC THEORY PROBLEM

We restrict ourselves to "incompressible" kinetic theory. In terms of the physical parameters this is obtained when the Mach number is small, and when the ratio of wall to free-stream temperature is sufficiently close to unity (the conditions are analogous to those under which a compressible material may be regarded as incompressible in fluid mechanics). The single relaxation equation[12] takes the form

$$\xi_2 \frac{\partial f}{\partial y} + f = \frac{\nu \rho_0}{(2\pi R T_0)^{3/2}} \exp\left(-\frac{(\xi_1-u)^2+(\xi_2+V)^2+\xi_3^2}{2RT_0}\right), \quad (1)$$

$$u = \rho_0^{-1} \int \xi_1 f \, d\xi.$$

The conditions of planar flow and constant suction velocity (a consequence of the continuity equation) have already been imposed on (1). As usual $f$ represents the distribution function and $\nu$ the (constant) collision frequency. $T_0$ and $\rho_0$ are the constant temperature and density, respectively. The far field condition on the velocity $u(y)$ is simply

$$\lim_{y\to\infty} u(y) = U. \quad (2)$$

The distribution function will be assumed to satisfy the diffuse boundary condition

$$H(\xi_2) f(y=0) = H(\xi_2) [\rho_0/(2\pi R T_0)^{3/2}] \times \exp[-\xi_1^2 + (\xi_2+V)^2 + \xi_3^2/2RT_0]. \quad (3)$$

(This is just the diffuse boundary condition in a frame of reference moving with vertical velocity $-V$. The effect of specular reflection will not be considered.)

Finally, since equilibrium is obtained as $y \to \infty$, Eqs. (1) and (2) imply

$$\lim_{y\to\infty} f = \frac{\rho_0}{(2\pi R T_0)^{3/2}} \exp\left(-\frac{(\xi_1-U)^2+(\xi_2+V)^2+\xi_3^2}{2RT_0}\right). \quad (4)$$

Multiplying (1) by $\xi_1$ we obtain the single nontrivial conservation equation

$$\frac{\partial}{\partial y} \int \xi_1 \xi_2 f \, d\boldsymbol{\xi} = 0. \quad (5)$$

Equations (5) and (4) then imply

$$\int f \xi_1 \xi_2 \, d\boldsymbol{\xi} = -\rho_0 U V. \quad (6)$$

The above problem may be transformed into a linear problem by introducing the reduced distribution function defined by

$$g(y, \xi_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_1 f \, d\xi_1 \, d\xi_3.$$

It has been pointed out that shear problems can, in general, be transformed to linear problems.[13] Also at this time we normalize the problem by writing

$$\hat{y} = [\nu/(2RT_0)^{1/2}] y, \quad \epsilon = V/(2RT_0)^{1/2}, \quad \xi = \xi_2/(2RT_0)^{1/2},$$
$$\phi = [(2RT_0)^{1/2}/\rho_0 U] \exp[-(\xi+\epsilon)^2]. \quad (7)$$

Imposing the above normalizations and dropping the circumflex on $\hat{y}$ we reduce our problem to the following:

$$\xi \frac{\partial \phi}{\partial y} + \phi = (\pi)^{-1/2} \int_{-\infty}^{\infty} \exp[-(\xi+\epsilon)^2] \phi \, d\xi = u(y), \quad (8)$$

$$u_\infty = \lim_{y\to\infty} u(y) = 1, \quad h(\xi)\phi_0 = H(\xi)\phi(y=0) = 0,$$
$$\phi_\infty = \lim_{y\to\infty} \phi = 1, \qquad \pi^{-1/2} \int \xi \exp[-(\xi+\epsilon)^2] \phi \, d\xi = -\epsilon. \quad (9)$$

A similar result can be obtained for Eq. (1), by linearizing for small $u/(2RT_0)^{1/2}$.

## III. SOLUTION OF THE SUCTION FLOW PROBLEM

The problem posed by (8) and (9) may be reformulated by direct integration. This gives

$$\phi = \left(\int_0^y H(\xi) + \int_y^\infty H(-\xi)\right) \times \exp[-|t-y|\,|\xi|^{-1}]\,|\xi|^{-1} u(t)\, dt. \quad (10)$$

On taking the velocity moment of this, we obtain the homogeneous integral equation,

$$u(y) = \int_0^\infty g(t-y) u(t) \, dt. \quad (11)$$

This conversion to an integral equation is similar to that used in the Milne problem.[12,14] $g(x)$ is given by

$$g(x) = \pi^{-1/2} \int_{-\infty}^{\infty} |\xi|^{-1} \exp\left(-(\xi+\epsilon)^2 - \frac{|x|}{|\xi|}\right) d\xi.$$

We will solve (11) by the Wiener–Hopf method. This requires the definition of the functions in (11) for all values of the independent variable. Defining

$$w(y) = H(y) u(y), \; n(y) = -H(-y) \int_{-\infty}^{\infty} g(y-t) u(t) \, dt,$$

instead of (11) we may write

$$w(y) = \int_{-\infty}^{\infty} g(y-t) w(t) \, dt + n(y). \quad (12)$$

Before resolving this equation by means of Fourier transforms we remark that since $u_\infty = 1$, the transform does not exist in the ordinary sense. However, in Pao[15] one can find a rigorous discussion verifying the use of distributions for the type of problem under study. With this in mind, we proceed formally. Using capital letters to denote a Fourier transform, e.g.,

$$W = \int_{-\infty}^{\infty} e^{i\omega y} w(y) \, dy \quad \left(= \int_0^\infty e^{i\omega y} w(y) \, dy\right),$$

Eq. (12) becomes

$$W = GW + N. \quad (13)$$

From their definitions, $W(\omega)$ and $N(\omega)$ are analytic in the upper and lower half of the complex $\omega$ plane, respectively. (In the following we use $\omega = t + is$, with

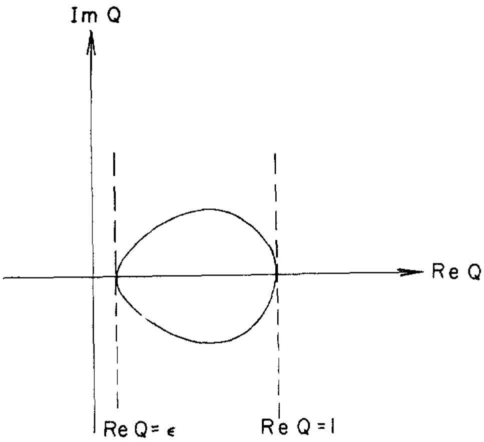


FIG. 1. Sketch of $Q(t)$ for $t$ real.

$t$ and $s$ real.) The $y$ integration in $G$ gives

$$1-G(\omega)=1-\pi^{-1/2}\int_{-\infty}^{\infty}\frac{\exp[-(\xi+\epsilon)^2]}{1-i\omega\xi}d\xi. \quad (14)$$

Defining

$$Q(\omega)=[(\omega-i)/\omega][1-G(\omega)]. \quad (15)$$

We rewrite Eq. (13) in the form

$$Q(\omega)\omega W(\omega)=(\omega-i)N(\omega). \quad (16)$$

The splitting of Eq. (16) is straightforward: It is clear from the definition (15), that $Q(\omega)$ defines two different analytic functions $Q^{\pm}(\omega)$ for $\mathrm{Re}\omega\gtrless 0$. In fact, the integral in (14) may be evaluated by standard methods in terms of the complex error function[16]

$$Q^{\pm}(\omega)=\frac{\omega-i}{\omega}\left[1-\frac{i\exp[-(\epsilon-i/\omega)^2]}{\omega}\times\left(-2\int_0^{-i/\omega}\exp(y^2)\,dy\mp i(\pi)^{1/2}\right)\right]. \quad (17)$$

In addition, we consider $Q(\omega)$ defined by (15) and (17) as a function of the real variable $t$. This may easily be sketched as $t$ varies on the real axis between $-\infty$ and $+\infty$, and its graph (in the complex $Q$ plane) is shown in Fig. 1.

The splitting of $Q(\omega)$ is now accomplished by first defining

$$\eta^{\pm}(\omega)=(2\pi i)^{-1}\int_{-\infty}^{\infty}\frac{\ln Q(t)}{t-\omega}dt, \quad (18)$$

which as indicated defines two different analytic functions for $\mathrm{Im}\omega\gtrless 0$. The existence of $\eta$ is clear from Fig. 1 and the asymptotic properties of (17). The Plemelj formulas[17] then give

$$\eta(t_0^{\pm})=(2\pi i)^{-1}\int_{-\infty}^{\infty}\frac{\ln Q(t)}{t-t_0}dt\pm\tfrac{1}{2}\ln Q(t_0) \quad (19)$$

in which $t_0$ denotes a real number. $t_0^{\pm}$ is defined by

$$\lim_{\mathrm{Im}\omega\to 0^{\pm}}\omega=t_0^{\pm}.$$

Next, we observe that $\omega W(\omega)\exp[\eta^+(\omega)]$ represents a function which is analytic in the upper half-plane, and $(\omega-i)N(\omega)\exp[\eta^-(\omega)]$ a function analytic in the lower half-plane. Then, from the Plemelj formulas and from Eqs. (16) and (19) we conclude that

$$\lim_{\mathrm{Im}\omega\to 0^+}[\omega W\exp(\eta^+)]=\lim_{\mathrm{Im}\omega\to 0^-}[(\omega-i)N\exp(\eta^-)],$$

from which it follows

$$p(\omega)=\omega W\exp(\eta^+)=(\omega-i)N\exp(\eta^-),$$

with $p(\omega)$ a polynomial. It is obvious that $N(\omega)$ vanishes as $\omega\to\infty$ and from this we conclude that $p=C$ a constant, which for the moment is unknown. The solution for $W$ is therefore given by

$$W(\omega)=C\exp[-\eta^+(\omega)]/\omega.$$

The Fourier transform is inverted as

$$u(y)=(C/2\pi)\int_\Gamma\omega^{-1}\exp[-i\omega y-\eta^+(\omega)]\,d\omega, \quad (20)$$

where $\Gamma$ represents a path of integration parallel to and slightly above the real axis. (Note that $u=0$ for $y<0$.) Since

$$\int_\Gamma\omega^{-1}\exp(-i\omega y)\,d\omega=-2\pi iH(y),$$

we can write

$$u(y)=CiH(y)\exp[-\eta^+(0)]+\frac{C}{2\pi}\int_{-\infty}^{\infty}\exp(-ity)\times\left(\frac{\exp[-\eta^+(t)]-\exp[-\eta^+(0)]}{t}\right)dt, \quad (21)$$

where the integral may now be placed along the real axis as is indicated. The second term vanishes as $y\to\infty$ so we have from (21) that

$$C=-i\exp[\eta^+(0)]$$

and finally the solution to the problem as

$$u(y)=H(y)+\frac{i}{2\pi}\exp[\eta^+(0)]\int_{-\infty}^{\infty}\exp(-ity)\times\left(\frac{\exp[-\eta^+(t)]-\exp[-\eta^+(0)]}{t}\right)dt. \quad (22)$$

Although this completes the solution, the above form is not convenient for further calculations.

Since we are interested in the solution for $y>0$, it is natural to consider deforming the contour of integration into the lower half-plane. [For this purpose, it is more convenient for us to consider the solution in the form (20).] To accomplish this we first define the following function:

$$\begin{aligned} h(\omega)&=\exp[-\eta^+(\omega)], && \mathrm{Im}\omega>0,\\ &=\exp[-\eta^-(\omega)]/Q^-(\omega), && \mathrm{Im}\omega<0\quad \mathrm{Re}\omega<0,\\ &=\exp[-\eta^-(\omega)]/Q^+(\omega), && \mathrm{Im}\omega<0\quad \mathrm{Re}\omega>0. \end{aligned} \quad (23)$$

By straightforward application of the Plemelj formulas, we notice that $h(\omega)$ is analytic in the entire $\omega$ plane except for a branch cut on the negative imaginary axis. In view of the location of $\Gamma$ in the $\omega$ plane, instead of (20) we can write

$$u(y)=(C/2\pi)\int_\Gamma \omega^{-1}h(\omega)\exp(-i\omega y)\,d\omega.$$

We may deform the path of integration $\Gamma$ to $\hat{\Gamma}$ as shown in Fig. 2. The contribution of the integrand at $\infty$ vanishes sufficiently rapidly to allow this deformation. We can, therefore, write

$$u(y)=\lim_{\delta\to0}(C/2\pi)\int_{|\omega|=\delta}\omega^{-1}h(\omega)\exp(-i\omega y)\,d\omega$$
$$+(C/2\omega)\int_{\mathrm{Re}\,\omega=0,\,\mathrm{Im}\,\omega<0}\omega^{-1}[h(\omega)]\exp(-i\omega y)\,d\omega,$$

where the bracket in the second integral represents the jump in $h(\omega)$ across the branch cut. Using this value of $C$ obtained in (22) and the definition of $h(\omega)$ we find

$$u(y)=1+i\,\frac{\exp\eta^+(0)}{2\pi}\int_0^\infty s^{-1}\exp[-sy-\eta^-(-is)]$$
$$\times\{[Q^+(-is)]^{-1}-[Q^-(-is)]^{-1}\}\,ds.\quad(24)$$

From the explicit form for $Q^\pm(\omega)$ given by (17) we obtain

$$Q^-(-is)-Q^+(-is)$$
$$=[2(1+s)/s^2]i\pi^{1/2}\exp[-(\epsilon+s^{-1})^2],$$
$$Q^+(-is)Q^-(-is)=[(s+1)/s]^2F(\epsilon,s),$$

where

$$F(\epsilon,s)=\left(1-\pi^{-1/2}\int_{-\infty}^{\infty}\frac{\exp[-(\xi+\epsilon)^2]}{1-s\xi}\,d\xi\right)^2+\frac{\pi\exp[-2(\epsilon+s^{-1})^2]}{s^2}$$
$$=\left(1-\frac{2\exp[-(\epsilon+s^{-1})^2]}{s}\int_0^{\epsilon+(1/s)}\exp(t^2)\,dt\right)^2+\frac{\pi\exp[-2(\epsilon+s^{-1})^2]}{s^2}.\quad(25)$$

On inserting this into (24),

$$u(y)=1-(\epsilon/\pi^{1/2})\times\int_0^\infty\frac{\exp[-sy-\eta^-(-is)+\eta^-(0)-(\epsilon+s^{-1})^2]}{s(s+1)F(\epsilon,s)}\,ds,\quad(26)$$

where we have used

$$\eta^+(0)-\eta^-(0)=\ln Q(0)=\ln\epsilon$$

which follows from the Plemelj formulas and (19).

Note that (20), (22), and (26) all represent exact solutions to the suction flow problem.

## IV. THE OUTER SOLUTION

As already mentioned in the introduction, hydrodynamical theory can be expected to emerge under the limit

$$\epsilon\to0,\qquad z=\epsilon y\quad\text{fixed}.$$

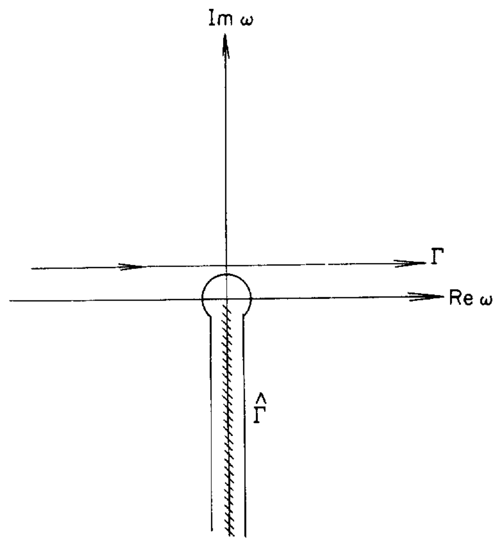


FIG. 2. Paths of integration $\Gamma$ and $\hat{\Gamma}$.

This is the outer limit in the sense mentioned by Van Dyke,[18] and we now consider this limit.

Introducing the outer variable into (26) we obtain

$$u(z,\epsilon)=1-(\epsilon/\pi^{1/2})\times\int_0^\infty\frac{\exp\{-\sigma z-[\epsilon+(\epsilon\sigma)^{-1}]^2-\eta^-(-i\epsilon\sigma)+\eta^-(0)\}}{\sigma(1+\epsilon\sigma)F(\epsilon,\epsilon\sigma)}\,d\sigma.\quad(27)$$

The limit above on $u(z,\epsilon)$ is straightforward since $F(\epsilon,\epsilon\sigma)$ vanishes in the neighborhood of $\sigma=2$, when $\epsilon\to0$. Since $F(\epsilon,\epsilon\sigma)$ is analytic in $\sigma$ real, we define $F^*(\epsilon,\epsilon\sigma)$ (analytic) for $\sigma$ complex by

$$F^*(\epsilon,\epsilon\sigma)=\left(1-\pi^{-1/2}\int_{-\infty}^{\infty}\frac{\exp[-(\eta+\epsilon)^2]}{1-\epsilon\sigma\eta}\,d\eta\right)\times\left(1-\pi^{-1/2}\int_{-\infty}^{\infty}\frac{\exp[-(\eta+\epsilon)^2]}{1-\epsilon\sigma\eta}\,d\eta-\frac{2i(\pi)^{1/2}\exp\{-[\epsilon+(\epsilon\sigma)^{-1}]^2\}}{\epsilon\sigma}\right)$$
$$=f(\sigma)[f(\sigma)+j(\sigma)]$$

and notice that

$$\lim_{\mathrm{Im}\,\sigma\to0^+}F^*(\epsilon,\epsilon\sigma)=F(\epsilon,\epsilon\sigma),\qquad\sigma\in R.$$

In addition, a simple calculation shows that

$$\exp[-\eta^-(-i\epsilon\sigma)+\eta^-(0)]$$
$$=(1+\epsilon\sigma)\exp\left[\frac{\sigma}{\pi}\int_0^\infty(1+\epsilon\sigma s)^{-1}\times\tan^{-1}\left(\pi^{1/2}s\Big/\left\{\exp[(s-\epsilon)^2]-2s\int_0^{s-\epsilon}\exp(y^2)\,dy\right\}\right)ds\right]$$
$$=(1+\epsilon\sigma)H(\epsilon\sigma),\quad(28)$$

(where $\tan^{-1}$ goes from $-\pi$ to 0 when $s$ passes from 0 to $+\infty$). In this notation (27) becomes

$$u(z)=1-\frac{\epsilon}{\pi^{1/2}}\int_0^\infty \frac{\exp\{-\sigma z-[\epsilon+(\sigma\epsilon)^{-1}]^2\}H(\epsilon\sigma)}{\sigma f(\sigma)[f(\sigma)+j(\sigma)]}\,d\sigma, \tag{29}$$

in which the integrand is analytic for $\mathrm{Im}\sigma>0$. $f(\sigma)$ has no zeros in the upper half-plane. Let $\sigma_c$ be such that[19]

$$f(\sigma_c)+j(\sigma_c)=0.$$

Then, a straightforward perturbation expansion yields

$$\sigma_c=2-4\epsilon^2+\cdots+i\,\frac{\pi^{1/2}\exp\{-[\epsilon+(2\epsilon)^{-1}]^2\}}{2\epsilon}\,(1+\cdots)$$

$$=\sigma_0+i(\pi^{1/2}/2\epsilon)\exp\{-[\epsilon+(2\epsilon)^{-1}]^2\}(1+\cdots). \tag{30}$$

By regarding $\epsilon$ as small, and rotating the path of integration to the ray $\theta=\theta_0$ (in the first quadrant), by Cauchy's theorem we obtain a contribution from the pole, plus a line integral:

$$u(z)=1-\frac{\epsilon^2H(\epsilon\sigma_c)\exp[-\sigma_c(\epsilon)z]}{f'(\sigma_c)+j'(\sigma_c)}+\frac{\epsilon}{\pi^{1/2}}$$

$$\times\int_{\theta=\theta_0}\frac{H(\epsilon\sigma)\exp\{-\sigma z-[\epsilon+(\epsilon\sigma)^{-1}]^2\}}{\sigma f(\sigma)[f(\sigma)+j(\sigma)]}\,d\sigma. \tag{31}$$

Although this is an apparent resolution into a point and continuous spectrum, one may show that there is no discrete spectra. Note however, that (31) is still an exact solution.

By construction the coefficient of the exponential in the above integral is bounded as $\epsilon\to0$ (since $H$ is uniformly bounded and the pole has been taken care of). Therefore, this integral can be evaluated in the limit $\epsilon\to0$ by a simple stationary point analysis and is $O\{\exp[-(z/\epsilon)^{2/3}]\}$ which is negligible in the outer limit. Therefore, defining

$$u_H(z)=1-\{\epsilon^2\sigma_0(\epsilon)/[\sigma_0(\epsilon)-2]\}H(\epsilon\sigma_0)\exp[-\sigma_0(\epsilon)z], \tag{32}$$

we have shown $u=u_H+O\{\exp[-(z/\epsilon)^{2/3}]\}$. This results in the expansion

$$u_H(z)=[1-\exp(-2z)]+\epsilon[2c_0\exp(-2z)]+\cdots, \tag{33}$$

$$c_N=-\pi^{-1}\int_0^\infty s^N$$

$$\times\tan^{-1}\left(\pi^{1/2}s\Big/\left\{\exp(s^2)-2s\int_0^s\exp(y^2)\,dy\right\}\right)ds.$$

This is the hydrodynamical solution as given in the introduction.

## V. HYDRODYNAMICS AND THE CHAPMAN–ENSKOG–HILBERT METHOD

At this point we have an exact solution to the suction flow problem. In addition, we have also examined the solution in the outer limit, and, in fact, have the solution to all orders. It will now be illuminating to examine the Chapman–Enskog–Hilbert method for this problem in light of these exact results. Before going to this, we pause to review the general philosophy of this procedure.

Although the Chapman–Enskog–Hilbert development has been widely known for some time, it has not been generally applied in the study of boundary value problems. As has been pointed out,[20] the difficulty has centered on the proper specification of boundary conditions. At the Burnett approximation, for example, third derivatives appear in the momentum and energy equations (in contrast to second derivatives at the preceding, or Navier–Stokes step).

Additional boundary conditions are not known, nor are boundary conditions of any type a consequence of the Chapman–Enskog–Hilbert method. This difficulty was perhaps first considered by Schamberg,[21] and a more recent illustration of this point is to be found in the review by Cercignani.[22] In short, the problem of boundary conditions may be avoided by imposing on the Chapman–Enskog–Hilbert equations the same procedure used in developing them. For the procedure to be valid it is necessary that a small parameter, say $\epsilon$, be present. The assumptions of the theory then require that the dependent variables themselves be expanded in this $\epsilon$. This, of course, avoids the problem of boundary conditions—and is really the Hilbert expansion rather than the Chapman–Enskog expansion. For example, let us suppose that to the lowest order in $\epsilon$, the governing equations are the Navier–Stokes equations, e.g., we can define the "Navier–Stokes" equations for the problem at hand as being the governing equations when the expansion parameter $\epsilon$ is zero. Under this procedure, the third derivatives of the Burnett equations appear only as inhomogeneous terms; and in fact, at every order, the equations reduce to the Navier–Stokes operator with inhomogeneous terms involving only derivatives of the lower-order solutions. In this way there is no difficulty in specifying proper conditions—one datum each for the velocity and temperature.

Although this avoids one aspect of the difficulty of boundary conditions, the problem of furnishing the perturbation series for velocity and temperature at the boundary still remains. The specification of first-order temperature and velocity slip is well known, but our knowledge of higher-order effects is still incomplete. Some comments on this problem are to be found in the review by Grad.[23] Further light has been shed on the problem by the recent investigations of Darrozes[7] and Sone.[8] Their basic approach has been via the method of inner and outer expansions. We will illustrate these points in the framework of the suction flow problem. To solve the problem posed in (8) and (9) by the Chapman–Enskog–Hilbert method, one formally writes

$$G=G^{(0)}+\epsilon G^{(1)}+\epsilon^2G^{(2)}+\cdots,$$
$$u=u^{(0)}+\epsilon u^{(1)}+\epsilon^2u^{(2)}+\cdots. \tag{34}$$

The sole equation of motion for this simple problem becomes

$$\int_{-\infty}^{\infty}\xi G^{(j)}\,d\xi=-\delta_{j1}. \tag{35}$$

Substituting into (8) and equating orders of $\epsilon$ we find

$$G^{(0)}=\Omega_0[u^{(0)}(z)],$$
$$G^{(1)}=\Omega_0\left(u^{(1)}(z)-\xi\frac{\partial u^{(0)}}{\partial z}\,2\xi u^{(0)}\right),$$
$$\vdots$$
$$G^{(k)}=\Omega_0\left(u^{(k)}(z)-\xi\frac{\partial u^{(k-1)}}{\partial z}-2\xi u^{(k-1)}+\cdots\right),$$

where we have set

$$\Omega_0=\pi^{-1/2}\exp(-\xi^2).$$

On applying (35) we find a single first-order equation at each successive order. To solve, it is necessary to specify the velocity at the wall, $z=0$, in the form

$$u(0)=\alpha_0+\alpha_1\epsilon+\alpha_2\epsilon^2+\cdots. \tag{36}$$

The solution of the equations generated through (35) under the boundary conditions (36) is straightforward and we find

$$u(z)=[1+(\alpha_0-1)\exp(-2z)]+\alpha_1\epsilon\exp(-2z)+\epsilon^2(\alpha_2-4z)\exp(-2z)+O(\epsilon^3). \tag{37}$$

Before going to the determination of the $\alpha_i$ we make some comments on the form of the above solution.

Examining Eq. (37) we see that for $z$ sufficiently large, the second order is greater than the first two. This indicates a secularity and the Chapman–Enskog–Hilbert method must be modified for large $z$. The method of multiple scales is therefore suggested.[18] According to this method, one defines a sequence of scales,

$$z_k=\epsilon^kz \qquad k=0,1,\cdots,$$

and assumes that the expansions (34) contain these scales in the arguments. Equation (8) takes the form

$$\epsilon\xi\left(\frac{\partial}{\partial z_0}+\epsilon\frac{\partial}{\partial z_1}+\cdots\right)(G^{(0)}+\epsilon G^{(1)}+\cdots)+(G^{(0)}+\epsilon G^{(1)}+\cdots)=\Omega_0[1-2\xi\epsilon+\epsilon^2(2\xi^2-1)+\cdots][u^{(0)}+\epsilon u^{(1)}+\cdots].$$

Equating orders of $\epsilon$, and imposing condition (35), one can solve this perturbation expansion. For example at the lowest order,

$$u^{(0)}=1+A^{(0)}(z_1,z_2,\cdots)\exp(-2z). \tag{38}$$

As is well known, the elimination of secularities at successive steps determines the functional dependence of $A^{(0)}$ on the $z$'s. For example at the second step

$$u^{(0)}(z)=\tilde{A}(0)(z_3,z_4,\cdots)\exp(-2z_0+4z_2). \tag{39}$$

The comparison of this with (32) shows that we are generating $u_H$ by this process. As a final step in this type of analysis, one can return to Eq. (8) and seek a solution having a single strained variable,[18] i.e., we can seek a solution in the form

$$G=\Omega[1+f(\xi,\epsilon)\exp[\sigma(\epsilon)z]. \tag{40}$$

This assumes the single-strained variable $\sigma(\epsilon)z$. By a variety of arguments this can be shown to lead exactly to $u_H$.

To summarize, we have shown that the Chapman–Enskog–Hilbert theory can easily be solved to all orders for our problem. However, this procedure must be modified owing to nonuniformities—that is, as $z$ becomes large, the flow is no longer described by the Chapman–Enskog–Hilbert solutions. This led to the introduction of an infinite number of layers ($z_k=\epsilon^kz$) or alternatively a single-strained coordinate. By these devices, the modified procedure seems to be valid uniformly. Now, however, we notice a somewhat ironic point: Equation (32) gives $u_H$ with an explicit error estimate. From this we see that the error estimate given by the exact solution demonstrates that the multiscale method was to no avail since the hydrodynamical solution is only valid in the first layer; that is, for $z_k$ fixed, $\epsilon\rightarrow0$ the flow is nonhydrodynamical for all $k>0$. This is directly seen by noting that

$$O\{\exp[-(z/\epsilon)^{2/3}]\}=O[\exp(-\epsilon^{-2(k+1)/3})]$$

under this limit, and observing that only even $k$ occur.

It is important to also note that this discussion implies that the neighborhood of infinity is nonhydrodynamic and should be regarded as a kinetic layer. To complete the Chapman–Enskog–Hilbert theory, we must turn to the determination of the $\alpha$'s which give successive orders of velocity slip at the wall. The solution which we have determined, in any of the preceding three forms, is an outer solution, and is incorrect in the neighborhood of the wall $z=0$. The outer solution in this region, and hence the $\alpha_i$, must be determined in such a way as to lead to the correct solution for $z$ fixed, $\epsilon\rightarrow0$. It is necessary, therefore, to find an inner solution, which we denote by $G$, to connect to the outer solution. In this way the constants $\alpha_i$ will be determined.

Much progress on this connection problem has been achieved in recent years and in this section we will consider the formalism as restricted to the suction problem. We now obtain the inner limit of the outer solution[18]; that is, we re-express the outer solution in terms of the inner variable $y$, and expand in $\epsilon$. This yields

Inner limit

$$G=\Omega_0\{\alpha_0+\epsilon(-2y(\alpha_0-1)+\alpha_1-2\xi)+\epsilon^2[2y^2(\alpha_0-1)-2y\alpha_1+\alpha_2+\alpha_0(2\xi^2-1)]\}+O(\epsilon^3)$$

with

$$\Omega_0=\pi^{-1/2}\exp(-\xi^2).$$

We, therefore, seek an inner solution in the form

$$\mathcal{G}=\mathcal{G}^{(0)}+\epsilon\mathcal{G}^{(1)}+\epsilon^2\mathcal{G}^{(2)}+\cdots$$

to Eq. (8)

$$\xi\frac{\partial\mathcal{G}}{\partial y}+\mathcal{G}=\frac{\exp[-(\xi+\epsilon)^2]}{\pi^{1/2}}\int\mathcal{G}\,d\xi \tag{41}$$

with boundary conditions

$$\lim_{y\to\infty}\mathcal{G}(y,\xi)=0\qquad \xi\geq 0, \tag{42}$$

$$\lim_{y\to\infty}[\mathcal{G}^{(0)}+\epsilon\mathcal{G}^{(1)}+\cdots+\epsilon^N\mathcal{G}^{(N)}]=N \text{ term inner limit of } G.$$

The zero-order solution is obviously

$$\mathcal{G}^{(0)}=0,\qquad \alpha_0=0. \tag{43}$$

Continuing to the next order we find

$$\xi\frac{\partial\mathcal{G}^{(1)}}{\partial y}+\mathcal{G}^{(1)}=\Omega_0\int\mathcal{G}^{(1)}\,d\xi,$$

$$\lim_{y\to 0}\mathcal{G}^{(1)}=0\qquad \xi\geq 0, \tag{44}$$

$$\lim_{y\to\infty}\mathcal{G}^{(1)}=\Omega_0(2y+\alpha_1-2\xi).$$

The boundary condition at $y\to\infty$ is easily interpreted in terms of velocity:

$$\lim_{y\to\infty}u^{(1)}(y)=\lim_{y\to\infty}\int\mathcal{G}^{(1)}\,d\xi=2y+\alpha_1,$$

which shows that Eq. (44) defines the Kramer's problem with a far field gradient of 2. The solution is well known and yields $\alpha_1$ as the macroscopic slip. Accurate numerical values for $\alpha_1$ have been obtained by Willis[8] and Albertoni.[6]

At the next order we find the inhomogeneous equation

$$\xi\frac{\partial\mathcal{G}^{(2)}}{\partial y}+\mathcal{G}^{(2)}=\Omega_0\int\mathcal{G}^{(2)}\,d\xi-2\xi\Omega_0\int\mathcal{G}^{(1)}\,d\xi,$$

$$\lim_{y\to 0}\mathcal{G}^{(2)}=0\qquad \xi\geq 0, \tag{45}$$

$$\lim_{y\to\infty}\mathcal{G}^{(2)}=\Omega_0(-2y^2-2y\alpha_1+\alpha_2).$$

We write

$$\int\mathcal{G}^{(1)}\,d\xi=u^{(1)}(y).$$

The problem becomes homogeneous on writing

$$\mathcal{G}^{(2)}=\mathcal{G}_H^{(2)}-2\Omega_0\int_0^y u^{(1)}(y')\,dy', \tag{46}$$

which gives, in place of (45),

$$\xi\frac{\partial\mathcal{G}_H^{(2)}}{\partial y}+\mathcal{G}_H^{(2)}=\Omega_0\int\mathcal{G}_H^{(2)}\,d\xi,$$

$$\lim_{y\to 0}\mathcal{G}_H^{(2)}=0\qquad \xi\geq 0, \tag{47}$$

$$\lim_{y\to\infty}\mathcal{G}_H^{(2)}=\Omega_0\left(-2y^2-2y\alpha_1+\alpha_2+\lim_{y\to\infty}2\int_0^y u^{(1)}(y')\,dy'\right).$$

Clearly,

$$u^{(1)}(y)=2y+\alpha_1+\psi(y),$$

where

$$\psi(y)\to 0\qquad \text{as } y\to\infty.$$

Substituting into (47) we find

$$\lim_{y\to\infty}\mathcal{G}_H^{(2)}=\Omega_0\left(\alpha_2+2\int_0^\infty\psi(y')\,dy'\right).$$

As at the zero order, the boundary condition implies that

$$\mathcal{G}_H^{(2)}\equiv 0,$$

which gives

$$\alpha_2=-2\int_0^\infty\psi(y')\,dy'. \tag{48}$$

This procedure may be continued to higher orders to obtain the constants $\alpha_i$. As far as we know, $\alpha_2$ has not been computed numerically. We can continue in this manner, and, in principle, determine all $\alpha_i$.

As we see from the above discussion, asymptotic boundary conditions are determined from a series of canonical kinetic theory boundary value problems. It is not necessary for us to continue further since all the $\alpha$'s have been determined from the exact solution in the last section. More generally, we can say the exact solution contains the solutions to the sequence of canonical problems.

## VI. ASYMPTOTIC BOUNDARY CONDITIONS FOR THE CHAPMAN–ENSKOG–HILBERT PROCEDURE

Aside from its own intrinsic interest an exact solution in kinetic theory has a much wider significance; for from it, one can extract information which may be used in the detailed solution of a potentially large class of problems. To see this let us consider near continuum flow past an arbitrary body. One then proceeds by means of the Chapman–Enskog–Hilbert procedure to solve the outer flow problem. The boundary data are determined by a sequence of inner, kinetic theory, problems—just as was indicated in the previous section. These inner problems are determined by matching them to the outer solutions. This involves only the derivatives of the outer solution at the boundaries. In this sense all boundary value problems are the same and the resolution of one such problem eliminates the necessity of repeating the steps determining the boundary conditions in other problems. We formally demonstrate this in the present section, and employ this to obtain general slip boundary conditions from the suction flow problem.

Consider flow past a body, and let $\epsilon$, small, represent the ratio of the relevant molecular scale to the relevant macroscopic scale. (Of course, there may be several of the latter, and we are therefore assuming that we have fixed on a region where only one is relevant.) For convenience, we consider the linear, two-dimensional case. Writing the Boltzmann equation in the outer variables yields

$$\left(\frac{\partial}{\partial t}+\xi_1\frac{\partial}{\partial x}+\xi_2\frac{\partial}{\partial y}\right)f=\epsilon^{-1}L(f) \tag{49}$$

and assuming

$$f_c=\sum(\epsilon^i/i!)f_c^{(i)},$$

the formal Chapman–Enskog solution results. This we write as

$$f_c=f_c(\mathbf{v},\nabla\mathbf{v},\nabla\nabla\mathbf{v},\cdots;\xi,\epsilon), \tag{50}$$

where $\mathbf{v}=(\rho,\mathbf{u},t)$ represents the hydrodynamical variables, since as is well known only hydrodynamical derivatives occur. Note that Eq. (50) states that $f_c$ depends on $\mathbf{x}$ and $t$ only implicitly through the hydrodynamical variables. The solution to (50) is completed by solving the conservation equations for the hydrodynamical variables. These are

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0,$$

$$\rho\frac{\partial u}{\partial t}+\rho(\mathbf{u}\cdot\nabla)u_i=-\frac{\partial}{\partial x_j}\int f_c(\mathbf{v},\nabla\mathbf{v},\cdots;\xi,\epsilon)\times(\xi_i-u_i)(\xi_j-u_j)\,d\boldsymbol{\xi}, \tag{51}$$

$$\tfrac{3}{2}\rho R\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)T=-\frac{\partial}{\partial x_j}\frac{1}{2}\int|\boldsymbol{\xi}-\mathbf{u}|^2(\xi_i-u_i)f_c\,d\boldsymbol{\xi}-\frac{\partial u_i}{\partial x_j}\int(\xi_i-u_i)(\xi_j-u_j)f_c\,d\boldsymbol{\xi}.$$

The boundary conditions take the form

$$u(0)=u_0=\epsilon c_1+\epsilon^2c_2+\cdots,$$
$$T(0)=T_0=\epsilon d_1+\epsilon^2d_2+\cdots, \tag{52}$$

where for convenience we take the origin as the boundary point in question. The formal solution therefore leads to the following representation,

$$f_c=f_c(c_1,\cdots;d_1,\cdots;\mathbf{x},t,\xi,\epsilon). \tag{53}$$

Next, let us consider both representations of $f_c$, (51) and (53) at the boundary point $\mathbf{x}=0$. The comparison suggests that the infinite sequence of constants $c_i$, $d_i$, $i=1,\infty$ can be expressed in terms of the derivatives of $\mathbf{v}$ at the wall. Writing $\mathbf{v}=(\rho,u,v,T)$ this leads to the following representation for $u(0)$:

$$u(0)=\sum\alpha_{ij}\epsilon^{i+j}\frac{\partial^{i+j}}{\partial x^i\partial y^j}u(0)+\sum\beta_{ij}\epsilon^{i+j}\frac{\partial^{i+j}v(0)}{\partial x^i\partial y^j}+\sum\gamma_{ij}\epsilon^{i+j}\frac{\partial^{i+j}}{\partial x^i\partial y^j}T(0)+\sum\delta_{ij}\epsilon^{i+j}\frac{\partial^{i+j}}{\partial x^i\partial y^j}p(0), \tag{54}$$

with a similar expression for the temperature boundary condition. (We recall that $x$, $y$ are outer or hydrodynamical variables.) The underlying assumption is that $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ do not depend on the choice of normalization or on the geometry. In general, they will depend on the assumed molecular structure of the surface; for example, whether the reflection is specular or diffuse. It is hypothesized that it is possible to determine the coefficients $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ by solving elementary or canonical problems, and examining the obtained solutions in the hydrodynamical limit. In other words, once having a solution to any problem in kinetic theory, we can extract from it certain of the coefficients in (34). Explicitly, let $f(\mathbf{x},\xi,t;\epsilon)$ represent the exact solution to (49) for the problem under consideration. From this we may determine the terms in the expansion of the Chapman–Enskog solution under the following limiting procedure. Holding $x$ fixed:

$$\lim_{\epsilon\to0}\frac{\partial^i}{\partial\epsilon^i}f(x,\xi,t)=f_c^{(i)}(x,\xi,t).$$

This leads to the following representation for $f_c$:

$$f_c(x,\xi,t;\epsilon)=\sum\frac{\epsilon^i}{i!}\frac{\partial^if}{\partial\epsilon^i}\bigg|_{\epsilon=0}.$$

In particular, we can compute $u(0)$ from this expression, as well as $\nabla\mathbf{v}(0)$, $\nabla\nabla\mathbf{v}(0)$, $\cdots$. Hence, we can determine the coefficients in (53).

Any problem can serve to define a canonical problem of the above type, and we carry out this program for the suction problem. The hydrodynamical variable in the normal direction is $z$. Thus, (54) becomes

$$u(0)=\sum\alpha_{0j}\epsilon^j\frac{d^ju}{dz^j}. \tag{55}$$

The hydrodynamical solution given in (32) is expanded in $\epsilon$ as

$$u(z)=u^{(0)}(z)+\epsilon u^{(1)}(z)+\cdots$$

and on substituting into (55) and equating coefficients of $\epsilon$, we find

$$u^{(0)}(0)=0,$$
$$u^{(1)}(0)=\alpha_{01}\frac{du^{(0)}(0)}{dz}, \qquad (56)$$
$$\vdots$$

which explicitly determines the $\alpha_{0j}$. We now rewrite (55) with the normalization removed to give

$$u(0)=c_0 l\frac{du(0)}{d\bar{y}}-(\tfrac{3}{2}-c_1+\tfrac{1}{2}c_0^2)l^2\frac{d^2u(0)}{d\bar{y}^2}$$
$$+(\tfrac{1}{6}c_0^3-c_0c_1+c_0+c_2-\tfrac{1}{8}\beta_0)l^3\frac{d^3u}{d\bar{y}^3}(0)+\cdots, \qquad (57)$$

$$\beta_0=-\frac{2}{\pi^{1/2}}$$
$$\times\int_0^\infty s\exp(s^2)\bigg/\left[\pi s^2+\left(\exp(s^2)-2s\int_0^s\exp(y^2)\,dy\right)^2\right]ds,$$

where $\bar{y}$ is the dimensional space variable, $l$ is the mean free path $[=\nu^{-1}(2RT)^{1/2}]$, and $c_N$ were defined in (33).

## VII. SLIP FLOW DUE TO A PRESSURE FIELD

As an application of the previous section we obtain the velocity slip at a wall due to a pressure field along the wall. This as we shall see can be unraveled from an earlier kinetic theory calculation.

Consider a planar flow field. In this case the slip velocity (54) is given by

$$u(0)=\alpha_{01}l\frac{du(0)}{d\bar{y}}+\alpha_{02}l^2\frac{d^2u(0)}{d\bar{y}^2}+\cdots$$
$$+\delta_{10}l\frac{dp(0)}{d\bar{x}}+\cdots+\gamma_{01}\frac{dT(0)}{d\bar{y}}+\cdots. \qquad (58)$$

In this section we compute the pressure gradient slip coefficient $\delta_{10}$.

Cercignani[5] considers Poiseuille flow and shows that

$$u(0)=c_0 l\frac{du(0)}{d\bar{y}}-(\tfrac{1}{4}+\tfrac{1}{2}c_0^2)l^2\frac{d^2u}{d\bar{y}^2}. \qquad (59)$$

This form, however, is really misleading. For in the Poiseuille flow problem, the hydrodynamical flow is governed to lowest order by the Stokes equation. In particular, this implies that at the wall

$$\frac{dp(0)}{dx}=l\frac{d^2u(0)}{dy^2}. \qquad (60)$$

In the previous section we demonstrated that

$$\alpha_{01}=c_0, \qquad \alpha_{02}=-\tfrac{3}{2}+c_1-\tfrac{1}{2}c_0^2. \qquad (61)$$

Therefore, comparison of (58) and (59) gives the pressure gradient slip coefficient $\delta_{10}$ as

$$\delta_{10}=\tfrac{5}{4}-c_1.$$

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