Supersonic flight in a stratified sheared atmosphere

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The problem of steady supersonic level flight in a stratified and sheared atmosphere is considered. The flow away from the body is shown to be governed by a Burgers-type equation. Under the inviscid limit exact analytical solutions are obtained showing the effects of shearing, the temperature, and the pressure variation. The solutions are in agreement with all known results under appropriate conditions.

I. INTRODUCTION

Among the unattractive features of supersonic flight is the ground level sonic boom generated by the shock wave which is carried by the aircraft. In contrast to the traditional problem of external aerodynamics, i.e., flow adjacent to the body, the sonic boom problem has forced investigators to examine the entire flow field—with special emphasis on the distant field.[1-13] (See Hayes[14,15] and Seebass[12] for comprehensive reviews.) These investigations are mainly based on geometrical acoustics, i.e., the formulation which results from linearized theory. Although the methods of geometrical acoustics are quite powerful,[16] linearized theory fails to give uniformly valid results at large distances from the aircraft. This was demonstrated by Whitman,[17,18] and Landau,[19] who also obtained the nonlinear corrections needed to extend the validity of linearized theory. As a result of this, there then developed a hybrid theory having geometrical acoustics as its base and containing the Whitham–Landau correction. This in essence is the theory used by Warren and Randall,[2] Randall,[5] Friedman *et al.*,[4] Hayes,[3] Hayes *et al.*,[9] George and Plotkin,[6] Seebass,[12] Kawamura and Makino,[13] and others.

A real atmosphere is stratified and this presents a number of difficulties not considered in the treatment of a homogeneous atmosphere by Whitham,[17,18] Landau,[19] and later the more comprehensive work by Rao.[20,21] (Turbulence will not be considered here; see Plotkin and George[22,23] and Crow.[24]) Thus, a real atmosphere contains variations in pressure, density, and temperature with altitude as well as shearing motions. Although these effects make analytical treatment difficult, various approximation and expansion procedures have led to fairly complete treatments by Randall,[5] George and Plotkin,[6] Seebass,[12] Kawamura and Makino,[13] and others. Shearing effects have, however, been neglected in these investigations.

Our investigation proceeds along entirely different lines. We employ the methods used in two earlier papers.[25,26] Starting from the Navier–Stokes equations, we derive asymptotic equations governing the flow away from a supersonic body. Both nonlinearity and dissipation enter and systematic higher orders can be obtained. With the exception of turbulence, all effects mentioned here are included in our analysis. In spite of all the complexities of the problem, our result has a remarkably simple form. We find that the far-field flow is governed by a modification of the Burgers' equation and if dissipation is neglected, the flow can be regarded as being governed by the standard equation

$$\frac{\partial G}{\partial \tau} + G\frac{\partial G}{\partial S} = 0,$$

where G is related to a typical flow variable and τ and S are respectively weighted radial and characteristic coordinates. The transformations (which include stratification and wind effects) leading from flow and configuration variables to G, τ, and S are relatively involved and are discussed in Sec. IV.

A number of comparisons with previous treatments are made. The far-field pressure jump due to a supersonic aircraft is computed and the result shown to agree, under appropriate limits, with the results of Randall,[5] George and Plotkin,[6] and Kawamura and Makino.[13] The most complete investigation of the sonic boom problem is that of Hayes *et al.*[9] who made extensive numerical calculations. When possible, comparison with their results is made. A number of other effects as well as further comparison with earlier investigations are mentioned in the text.

In view of the fact that a similar treatment of the homogeneous atmosphere exists in the literature, we have kept the analytical development to a minimum. Those interested in the details can refer to the two papers.[25,26] Also, we should mention that whereas our previous treatment considers the entire flow field, we have chosen to focus on the Mach wave and do not discuss the development of the wake in this paper.

II. FORMULATION OF PROBLEM

We consider steady level flight of a supersonic aircraft in a stratified atmosphere with winds. A basic small parameter in the flow is the thickness ratio of the body, denoted by ε. This arises naturally when distances are scaled with respect to the body length L. In this normalization, a second small parameter emerges, namely, the ratio of L to the atmospheric scale height. This we peg as ε^2 which proves to be convenient as well as realistic. Denoting the vertical height from ground by y, we write (dimensional quantities will be defined with tildas)

$$\tilde{\rho}^0 = \tilde{\rho}^0(\varepsilon^2 y),\ \tilde{p}^0 = \tilde{p}^0(\varepsilon^2 y),\ \tilde{T}^0 = \tilde{T}^0(\varepsilon^2 y),\ \tilde{\mathbf{U}}^0 = \tilde{\mathbf{U}}^0(\varepsilon^2 y). \tag{1}$$

These functions represent the unperturbed density, pressure, temperature, and velocity, respectively, relative to a coordinate system fixed in the aircraft. The vertical coordinate y, as well as the other independent variables are now regarded as normalized with respect to L. At the origin (i.e., at the body), we have $\tilde{\mathbf{U}}^0(0) = (\tilde{U}_0, 0, 0)$ with $\tilde{U}_0$ being a constant. The quantities defined in (1) are assumed to satisfy the Navier–Stokes equations with the gravity term $\tilde{\mathbf{g}} = (0, -\tilde{g}, 0)$.

We restrict our attention to axisymmetric bodies and, therefore, expect the flow field in a large neighborhood of the body to be axisymmetric to the lowest order. The θ dependent enters largely through the combination $\varepsilon^2 y = \varepsilon^2 r \cos\theta$, where $\theta = 0$ corresponds to the downward or the gravitational direction.

We will also assume

$$\frac{1}{r}\frac{\partial}{\partial\theta} = O(\varepsilon) \tag{2}$$

and later consider under what conditions this assumption holds. As we see momentarily, it follows from the fact that the circumferential velocity w is $O(\varepsilon^2)$. Before formulating the equations of motion, it will be useful to make several additional preparatory remarks.

In view of the slow variation introduced by stratification, a change of variable is advisable. We introduce $(r,\theta) \to (R, \Theta)$,

$$R = \int_0^r \phi(\varepsilon^2 s \cos\theta)\, ds, \quad \Theta = \theta, \tag{3}$$

where for the moment, ϕ is regarded as unknown. As (3) suggests, the slow variables

$$R_1 = \varepsilon^2 R, \qquad r_1 = \varepsilon^2 r,$$

will be of importance. One additional scale proves necessary

$$R_* = \nu_*(\varepsilon) R,$$

where ε^2 will be seen to be $o(\nu_*)$.

The method of multiple scales will eventually be used to eliminate secularities, i.e., cumulative effects, and to obtain a uniformly valid first-order description. Therefore from (3) and the above defined two scales, we write

$$\begin{aligned}\frac{\partial}{\partial r} &= \phi\frac{\partial}{\partial R} + \nu_*\phi\frac{\partial}{\partial R_*} + \varepsilon^2\phi\frac{\partial}{\partial R_1},\\ \frac{\partial}{\partial\theta} &= \frac{\partial R}{\partial\theta}\frac{\partial}{\partial R} + \frac{\partial}{\partial\Theta}.\end{aligned} \tag{4}$$

In the new coordinates the unperturbed flow is given by $\tilde{\rho}^0[r_1(R_1,\Theta)\cos\Theta]$, $\tilde{\mathbf{U}}^0[r_1(R_1,\Theta)\cos\Theta]$, $\tilde{T}^0[r_1(R_1,\Theta)\cos\Theta]$. For simplicity, the atmosphere is assumed to be governed by the perfect gas law. The relative wind is assumed to lie in the plane of the aircraft and parallel to the ground. Vertical wind components are assumed to vanish at the body. The relative wind parallel to the direction-of-flight of the aircraft will be denoted by $\tilde{U}$, and we also write

$$\tilde{\mathbf{U}}^0 = [\tilde{U}(r_1(R_1,\Theta)\cos\Theta), \varepsilon\tilde{v}^0(R_1,\Theta), \varepsilon\tilde{w}^0(R_1,\Theta)],$$

where v^0 and w^0 are the radial and θ components of the wind, respectively.

The perturbed flow field in normalized form is represented by $\mathbf{v} = (\rho, \mathbf{u}, T)$ with the normalization

$$\rho = \frac{\tilde{\rho}}{\tilde{\rho}^0},\ U = \frac{\tilde{U}}{\tilde{a}},\ \mathbf{u} = \frac{\tilde{\mathbf{u}}}{\tilde{a}},\ T = \frac{\tilde{T}}{\chi\tilde{T}^0},\ \varepsilon^2 g = \frac{\tilde{g}L}{\tilde{a}^2} \tag{5}$$

and

$$\tilde{a}^2(r_1\cos\theta) = \left.\frac{\partial\tilde{p}^0}{\partial\tilde{\rho}^0}\right|_T,\ \tilde{c}^2 = \left.\frac{\partial\tilde{p}^0}{\partial\tilde{\rho}^0}\right|_S,\ \gamma = \frac{\tilde{c}^2}{\tilde{a}^2},\ \chi^2 = \gamma - 1.$$

The insertion of ε^2 in the normalization of $\tilde{g}$ follows from straightforward estimates. It should be noted that all the perturbed quantities have nonconstant normalizations.

The equations governing the perturbed flow field can be written in the following symbolic form:

$$\left(\mathbf{A}\frac{\partial}{\partial x} + \phi\mathbf{B}\frac{\partial}{\partial R} + \frac{1}{r}\mathbf{C}\right)\mathbf{v} = \mathbf{X}(\mathbf{v}) + \mathbf{Y}(\mathbf{v}) + \mathbf{N} + \mathbf{D}(w) \tag{6}$$

$$+ \mathbf{W}_R(\mathbf{v}) + \mathbf{W}_\theta(\mathbf{v}) + \mathbf{G}(\mathbf{v}) + \mathbf{Z}(\mathbf{v}) - \nu_*\phi\mathbf{B}\frac{\partial}{\partial R_*}\mathbf{v} - \varepsilon^2\phi\mathbf{B}\frac{\partial}{\partial R_1}\mathbf{v},$$

where $\mathbf{v} = (\rho, u, v, T)$, u is the x component velocity perturbation, v is the radial component, w is the θ component,

$$\mathbf{A} = \begin{bmatrix}\gamma^{1/2}M & 1 & 0 & 0\\ 1 & \gamma^{1/2}M & 0 & \chi\\ 0 & 0 & \gamma^{1/2}M & 0\\ 0 & \chi & 0 & \gamma^{1/2}M\end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix}0&0&1&0\\0&0&0&0\\1&0&0&\chi\\0&0&\chi&0\end{bmatrix},\quad \mathbf{C} = \begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&\chi&0\end{bmatrix} \tag{7}$$

and the Mach number $M = \gamma^{-1/2}U = M(r_1\cos\theta)$. The terms on the right-hand side of (6) are given in Appendix B. Briefly, we mention that $\mathbf{X}$ represents all quadratic nonlinearities; $\mathbf{Y}$ all linear dissipative terms; $\mathbf{N}$ represents contributions from derivatives with respect to the nonuniform normalization; $\mathbf{D}$ contains θ derivatives; $\mathbf{W}_\theta$, $\mathbf{W}_R$ refers to wind terms; $\mathbf{G}$ is the gravity term; $\mathbf{Z}$ represents higher order nonlinearities; and the remaining two terms represent derivatives with respect to the already introduced multiscales. In addition to the above equations, we also have the θ-momentum equation which, to the lowest order, is

$$\gamma^{1/2}M\frac{\partial}{\partial x}w + \frac{1}{r}\frac{\partial}{\partial\theta}\rho + \frac{\chi}{r}\frac{\partial}{\partial\theta}T = 0. \tag{8}$$

In order to solve (6) and (8), we write

$$\begin{aligned}\mathbf{v} &= \varepsilon\mathbf{v}_1 + \varepsilon_2(\varepsilon)\mathbf{v}_2 + \varepsilon_3(\varepsilon)\mathbf{v}_3 + \cdots,\\ w &= \varepsilon^2 w_1 + \cdots,\end{aligned} \tag{9}$$

where the $\varepsilon_i(\varepsilon)$ denote an asymptotic sequence of gauge functions. The expansion of the equations in the perturbation series, although not difficult, is tedious and we leave the details to Appendix B. Further notational comments are also to be found in that appendix.

III. A UNIFORM FIRST-ORDER SOLUTION

The lowest order equations which emerge from substituting (9) into (6) are

$$\left(\mathbf{A}\frac{\partial}{\partial x} + \phi\mathbf{B}\frac{\partial}{\partial R} + \frac{1}{r}\mathbf{C}\right)\mathbf{v}_1 = 0. \tag{10}$$

It should be recalled that (3) defines $r = r(R, \Theta)$. We introduce a linearly independent set of eigenvectors $\{\mathbf{l}_i\}$ and their biorthogonal set $\{\boldsymbol{\alpha}_i\}$ corresponding to eigenvalues $\lambda_{1,2,3,4} = 0, 0, \pm(M^2-1)^{-1/2}$ and such that[25,26]

$$\lambda_i \boldsymbol{\alpha}_i = \lambda_i \mathbf{A}\mathbf{l}_i = \mathbf{B}\mathbf{l}_i, \qquad i = 1, 2, 3, 4.$$

The eigenvalues and eigenvectors corresponding to λ_1 and λ_2 give rise to wakes carrying entropy and vorticity. This has been considered in earlier papers,[25,26] and we do not pursue these effects here. Also, the mode corresponding to λ_4 only contributes an "inward traveling wave" and this may be neglected in the lowest order. We therefore focus on the lowest order "Mach cone" flow field and tailor our analysis to this end.

The only vectors which figure in our analysis are

$$\mathbf{l}_3 = [1, -\gamma^{1/2}/M, \gamma^{1/2}(M^2-1)^{1/2}/M, \chi],$$
$$\boldsymbol{\alpha}_3 = [\gamma^{1/2}(M^2-1)/M, 0, \gamma(M^2-1)^{1/2}, \chi\gamma^{1/2}(M^2-1)/M].$$

Multiplying (10) by $\mathbf{l}_3$, we obtain

$$\frac{\partial}{\partial x}\boldsymbol{\alpha}_3\cdot\mathbf{v}_1 + \lambda_3\phi\frac{\partial}{\partial R}\boldsymbol{\alpha}_3\cdot\mathbf{v}_1 + \frac{1}{r}\mathbf{l}_3\cdot\mathbf{C}\mathbf{v}_1 = 0.$$

Since $\lambda_3 = (M^2-1)^{-1/2}$, we choose

$$\phi = [(M^2-1)/(M_0^2-1)]^{1/2},$$

where the zero subscript signifies evaluation at the body. On considering the solution of (10), for R large, we obtain[26]

$$\mathbf{v}_1 = r^{-1/2}\mathbf{l}_3 f(\sigma) + \frac{1}{8}r^{-3/2}(M^2-1)^{-1/2}(\mathbf{l}_3 - 2\mathbf{l}_4)f_{-1}(\sigma) + o(r^{-3/2}), \tag{11}$$

where σ is the linear "unstratified" characteristic given by

$$\sigma = x - (M_0^2-1)^{1/2}R, \tag{12}$$

f is to be determined, and $f_{-1}(\sigma) = \int^{\sigma} f(s, R_*, R_1)\,ds$.

Returning to Eq. (8) governing the circumferential velocity and inserting the solution (11), we obtain

$$\varepsilon^2\gamma^{1/2}M\frac{\partial}{\partial x}w_1 + \frac{\varepsilon}{r}\frac{\partial}{\partial\theta}(r^{-1/2}f) + \frac{\varepsilon\chi}{r}\frac{\partial}{\partial\theta}(\chi r^{-1/2}f) = 0,$$

which verifies the earlier estimate (9). The solution to w is then

$$w \sim \varepsilon^2 w_1 = \varepsilon\left(\frac{1}{r}\frac{\partial R}{\partial\theta}\right)\frac{\gamma^{1/2}}{Mr^{1/2}}(M_0^2-1)^{1/2}f. \tag{13}$$

In view of the appearance of fractional powers of r in the denominator in the expansion of $\mathbf{v}_1$, the limit $\varepsilon\to 0$ although in itself clear cut fails to take into consideration all vanishingly small effects due to $R\to\infty$. To take account of this in what follows we will often perform a limit in which $\nu(\varepsilon)R$, $[\nu = o(1)]$, is held fixed while $\varepsilon\to 0$. It, therefore, becomes necessary to avoid segregating terms according to the power of ε in their coefficients. As we shall see, several different apparent orders contribute at the same order when considering, e.g., the limit R_1 fixed, $\varepsilon\to 0$.

The next order equation obtained by substituting (9) into (6) is

$$\varepsilon_2\left(\mathbf{A}\frac{\partial}{\partial x} + \phi\mathbf{B}\frac{\partial}{\partial R} + \frac{1}{r}\mathbf{C}\right)\mathbf{v}_2 = -\varepsilon\phi\left(\varepsilon v^0\frac{\partial}{\partial R}\mathbf{v}_1 + \nu_*\frac{\partial}{\partial R_*}\mathbf{B}\mathbf{v}_1\right), \tag{14}$$

where $\mathbf{B}\mathbf{v}_1 = (v_1, 0, \rho_1, +\chi T_1, \chi v_1)$ and recall that v^0 is the radial component of wind. To solve (14), we first multiply from the left by $\mathbf{l}_3$. This results in a directional derivative along the Mach cone and on integration the right-hand side leads to secularity as $R\to\infty$. This would, in fact, occur if we had not the added latitude of the R_* dependence. This is now used to force the right-hand side to vanish. Specifically, we require

$$\lim_{\varepsilon\to 0}\ \mathbf{l}_3\cdot\left\{\varepsilon v^0\frac{\partial}{\partial R}\mathbf{v}_1 + \nu_*\frac{\partial}{\partial R_*}\mathbf{B}\mathbf{v}_1\right\} = 0, \tag{15}$$

where the limit is taken with σ and R_* held fixed.

In order to perform this limit we first note that the wind is zero at the body. Hence, since we are provisionally assuming $\varepsilon^2 = o(\nu_*)$, we have

$$v^0 = O(\varepsilon^2 R) = O\left(\frac{\varepsilon^2}{\nu_*}R_*\right). \tag{16}$$

Similarly, since from (3) $\nu_* r = O(R_*)$, we have

$$\mathbf{v}_1 \sim r^{-1/2}\mathbf{l}_3 f = O(\nu_*^{1/2}).$$

Therefore, we have

$$\varepsilon v^0\frac{\partial}{\partial R}\mathbf{v}_1 = O(\varepsilon^3\nu_*^{-1/2}) \text{ and } \nu_*\mathbf{B}\frac{\partial}{\partial R_*}\mathbf{v}_1 = O(\nu_*^{3/2}).$$

Equating these two yields $\nu_* = \varepsilon^{3/2}$. Finally, since the left-hand side of (14) is $O(\varepsilon_2\nu_*^{1/2})$, we have $\varepsilon_2 = \varepsilon^{5/2}$.

Substituting the leading term of $\mathbf{v}_1$ from (11) into (15), we get

$$\frac{\partial}{\partial R_*}f - \frac{M}{\gamma^{1/2}}\left(\frac{M_0^2-1}{M^2-1}\right)^{1/2}v^0\frac{\partial}{\partial\sigma}f = 0. \tag{17}$$

The solution to (17) is

$$f = f(s, R_1), \tag{18}$$

where

$$s = \sigma + \int_0^{R_*}\frac{M}{\gamma^{1/2}}\left(\frac{M_0^2-1}{M^2-1}\right)^{1/2}v^0\,dR_*. \tag{19}$$

It should be noted that the derivation shows that v^0 in

(19) can be replaced by its expansion. This, however, requires that we carry

$$\left[v^0\frac{\partial}{\partial R}\mathbf{v}_1-\lim_{\substack{\varepsilon\to 0\\ R_*\,\text{fixed}}}\left(v^0\frac{\partial}{\partial R}\mathbf{v}_1\right)\right]$$

at the next order. On the other hand, keeping the integrand intact in (19) avoids this step.

Going to the next order in (6), we have

$$\varepsilon_3\left(\mathbf{A}\frac{\partial}{\partial x}+\phi\mathbf{B}\frac{\partial}{\partial R}+\frac{1}{r}\mathbf{C}\right)\mathbf{v}_3=\varepsilon^2\mathbf{X}(\mathbf{v}_1)+\varepsilon\mathbf{Y}(v_1)+\varepsilon^3\mathbf{N}$$

$$+\varepsilon^2\mathbf{D}(w_1)+\varepsilon^2\mathbf{W}_\Theta(\mathbf{v}_1)+\varepsilon^3\mathbf{G}(\mathbf{v}_1)-\varepsilon^3\phi\mathbf{B}\frac{\partial}{\partial R_1}\mathbf{v}_1$$

$$+\varepsilon\left(\mathbf{A}\frac{\partial}{\partial x}+\phi\mathbf{B}\frac{\partial}{\partial R}+\frac{1}{r}\mathbf{C}\right)(r^{-1/2}\mathbf{l}_3f-\mathbf{v}_1),\tag{20}$$

where the last term is due to the second term in the expansion of $\mathbf{v}_1$ given by (11). The other terms have been discussed in connection with (6) and their explicit forms are given in Appendix B. [The earlier remark, on retaining terms of different virtual orders should be recalled in viewing (20).]

As before, (20) leads to secularity for $R\to\infty$. The second slow variable R_1 is now employed to eliminate this. Specifically, it is required that the right-hand side of (20) be orthogonal to $\mathbf{l}_3$ under the limit (R_1, s) fixed and $\varepsilon\to 0$. In this limit

$$\mathbf{v}_1\sim r^{-1/2}\mathbf{l}_3f(s,R_1)=O(\varepsilon),$$

since $\varepsilon^2 r=O(R_1)$; see (3). From this we see that $\varepsilon^2\mathbf{X}$, $\varepsilon^3\mathbf{N}$, $\varepsilon^2\mathbf{D}$, $\varepsilon^2\mathbf{W}_\Theta$, $\varepsilon^3\mathbf{G}$ are all $O(\varepsilon^4)$. The solution $\mathbf{v}_3$ on the left-hand side of (20) has a complementary part which is similar to $\mathbf{v}_1$ and is $O(\varepsilon)$ as $\varepsilon\to 0$. Therefore, the left-hand side of (20) is $O(\varepsilon\varepsilon_3)$. Setting the two sides of (20) to the same order gives $\varepsilon_3=\varepsilon^3$. Next, we note that although $\varepsilon(r^{-1/2}\mathbf{l}_3\mathbf{f}-\mathbf{v}_1)=O(\varepsilon r^{-3/2})=O(\varepsilon^4)$, we have

$$\varepsilon\mathbf{l}_3\cdot\left(\mathbf{A}\frac{\partial}{\partial x}+\phi\mathbf{B}\frac{\partial}{\partial R}+\frac{1}{r}\mathbf{C}\right)(r^{-1/2}\mathbf{l}_3f-\mathbf{v}_1)=O(\varepsilon^6).$$

Hence, the asymptotic orthogonality condition is

$$\lim_{\substack{\varepsilon\to 0\\ s,R_1\,\text{fixed}}}\mathbf{l}_3\cdot\left(\varepsilon^2\mathbf{X}+\varepsilon\mathbf{Y}+\varepsilon^3\mathbf{N}+\varepsilon^2\mathbf{D}+\varepsilon^2\mathbf{W}_\Theta+\varepsilon^3\mathbf{G}\right.$$

$$\left.-\varepsilon^3\phi\mathbf{B}\frac{\partial}{\partial R_1}\mathbf{v}_1\right)=0,\tag{21}$$

where s and R are held fixed. Substituting $\mathbf{v}_1$ as given by (11) and (18) into (21), we find that

$$\phi\frac{\partial f}{\partial R_1}-\left[\frac{1}{2\varepsilon}\left(\frac{1}{r}\frac{\partial R}{\partial\theta}\right)^2\frac{(M_0^2-1)^{1/2}}{(M^2-1)^{1/2}}+\frac{Mw^0}{\gamma^{1/2}\phi}\left(\frac{1}{\varepsilon r}\frac{\partial R}{\partial\theta}\right)\right.$$

$$\left.+\frac{(\gamma+1)M^2}{2(M^2-1)^{1/2}r_1^{1/2}}f\right]\frac{\partial f}{\partial s}=\frac{[\gamma(\zeta+\eta)+\chi^2\xi]}{2\varepsilon^2\gamma^{3/2}(M^2-1)^{1/2}}\frac{\partial^2 f}{\partial s^2}$$

$$+\frac{1}{2}\left(\frac{1}{\gamma}g\cos\theta-\frac{(\gamma+1)}{\gamma}\phi\frac{\partial\ln\tilde{p}^0}{\partial R_1}\right.$$

$$+\left(\frac{M^2-2}{M^2-1}\right)\phi\frac{\partial\ln M}{\partial R_1}-\frac{1}{\varepsilon r}\left(\frac{R}{\phi r}-1\right)\sec^2\theta$$

$$-\frac{1}{\varepsilon}\left\{\left[\left(\frac{R}{\phi r}-1\right)+\frac{M^2}{M^2-1}\right]\right.$$

$$\left.\left.\times\phi\frac{\partial\ln M}{\partial R_1}+\frac{1}{\varepsilon r}\left(\frac{R}{\phi r}-1\right)\right\}\tan^2\theta\right)f.\tag{22}$$

A number of transformations may be made which simplify the form of (22). To begin with, we see that it is no longer necessary to retain the formal small parameter, ε. To eliminate it we simply set $\varepsilon=1$. Also, it is no longer necessary to employ variable $R=R(r)$, and we transform back to r by means of (3). Carrying out these steps we obtain

$$\frac{\partial f}{\partial r}-\frac{k}{r^{1/2}}f\frac{\partial f}{\partial S}=\frac{1}{\mathfrak{R}}\frac{\partial^2 f}{\partial S^2}+b(r,\theta)f,\tag{23}$$

where

$$k=\frac{(\gamma+1)M^2}{2(M^2-1)^{1/2}},$$

$$\mathfrak{R}=\frac{2\gamma^{3/2}(M^2-1)^{1/2}}{[\gamma(\zeta+\eta)+\chi^2\xi]M^3},$$

$$b(r,\theta)=-\frac{1}{2}\frac{\partial\ln\tilde{p}^0}{\partial r}+\frac{1}{2}\left(\frac{M^2-2}{M^2-1}\right)\frac{\partial\ln M}{\partial r}$$

$$-\frac{1}{2\phi r}\left(\frac{R}{r}-\phi\right)\sec^2\theta-\frac{1}{2}\tan^2\theta\left\{\left[\frac{1}{\phi}\left(\frac{R}{r}-\phi\right)\right.\right.$$

$$\left.\left.+\frac{M^2}{M^2-1}\right]\frac{\partial\ln M}{\partial r}+\frac{1}{\phi r}\left(\frac{R}{r}-\phi\right)\right\},$$

$$S=x-\int_0^r\left\{(M^2-1)^{1/2}-\left[\frac{1}{2\phi}\left(\frac{1}{r}\frac{\partial R}{\partial\theta}\right)^2+\frac{Mv^0}{\gamma^{1/2}}\right.\right.$$

$$\left.\left.+\frac{Mw^0}{\gamma^{1/2}\phi}\left(\frac{1}{r}\frac{\partial R}{\partial\theta}\right)\right]\right\}dr,\tag{24}$$

where $\mathfrak{R}$ is the Reynolds number based on body length. In writing the form for $b(r,\theta)$, we have used the approximate atmospheric equation

$$\partial\tilde{p}^0/\partial\tilde{r}=\tilde{\rho}^0\tilde{g}\cos\theta.$$

We now pause to consider the assumption made in (2) on the smallness of $r^{-1}(\partial/\partial\theta)$. Note first that θ appears in the variation of equilibrium quantities, e.g., $\tilde{\rho}^0=\tilde{\rho}^0(\varepsilon^2 r\cos\theta)$, and also in the argument of R. In the former case, $r^{-1}(\partial/\partial\theta)$ is $O(\varepsilon^2)$ and (2) is certainly conservative. To consider the latter case note that

$$\frac{1}{r}\frac{\partial R}{\partial\theta}=\left(\frac{R}{r}-\phi\right)\tan\theta$$

$$=\left[\frac{1}{r}\int_0^r\left(\frac{M^2-1}{M_0^2-1}\right)^{1/2}dr-\left(\frac{M^2-1}{M_0^2-1}\right)^{1/2}\right]\tan\theta.\tag{25}$$

Therefore, the smallness of (25) depends on the difference of the average of $[(M^2-1)/(M_0^2-1)]^{1/2}$ and its value at the end point. Since $M=M(\varepsilon^2 r\cos\theta)$, the smallness of (25) follows immediately if $r\ll 1/\varepsilon^2$. On the other hand, when $r=O(1/\varepsilon^2)$, the smallness of (25) and hence w, (13), comes into question. In fact, for an arbitrary function $M(\varepsilon^2 r \cos\theta)$ we can expect $r^{-1}(\partial R/\partial\theta) = O(1)$ if $\varepsilon^2 r = O(1)$. In order to pursue this further, we must make use of the special properties of $\phi = [(M^2 - 1)/(M_0^2 - 1)]^{1/2}$ as is found in a standard atmosphere.[27] In Fig. 1, we sketch the variation of ϕ for the atmosphere vs altitude for several values of Mach numbers. From these sketches we see that the variation is slow and there is no systematic growth. This suggests that a quantity such as in (25) remains small. Denoting $r^{-1}\int^r \phi\, dr$ by $\langle\phi\rangle$, in Fig. 2 we sketch $\langle\phi\rangle - \phi$ for several Mach numbers. As can be seen, for a specific Mach number, the maximum is achieved at the ground and the greatest difference occurs at the lowest Mach number. This difference decreases systematically with Mach number, e.g., the difference at $M_0 = 1.2$ is 0.423, at $M_0 = 1.5$ is 0.200, at $M_0 = 5$ is 0.110. From these, we conclude that although this difference is small, it is certainly not $O(\varepsilon)$. Hence, as altitude increases $r^{-1}(\partial R/\partial\theta)$ grows but still remains relatively small. We make immediate use of this in connection with the expression for $b(r,\theta)$ given in (24). In that expression, the coefficient of $(\tan^2\theta)(\partial \ln M/\partial R)$

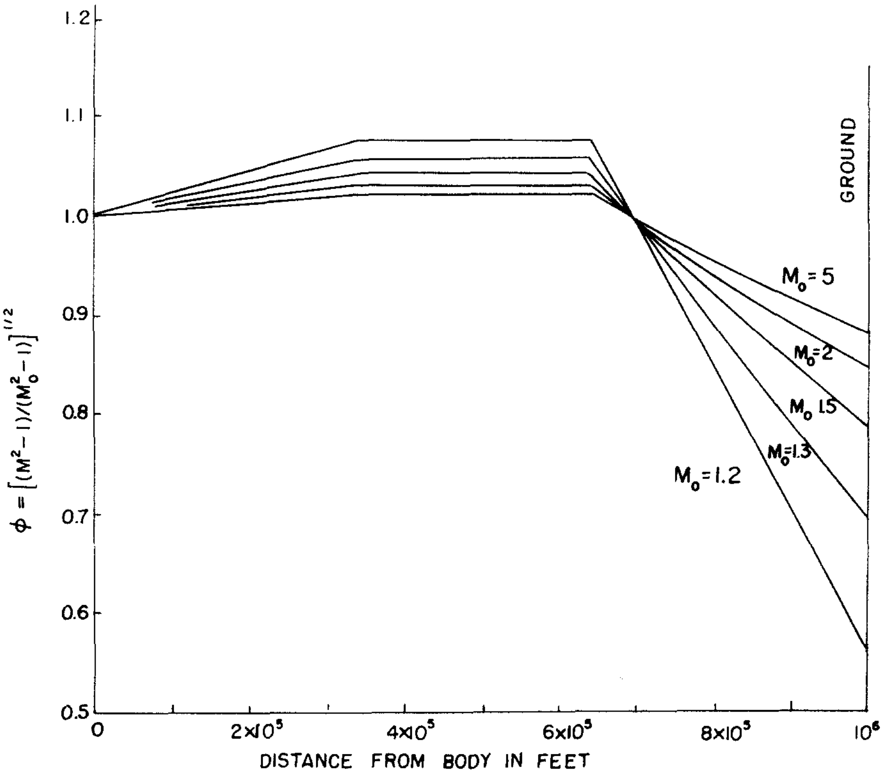

FIG. 1. Variation of ϕ in the real atmosphere for an aircraft flying at an altitude of 100 000 ft.

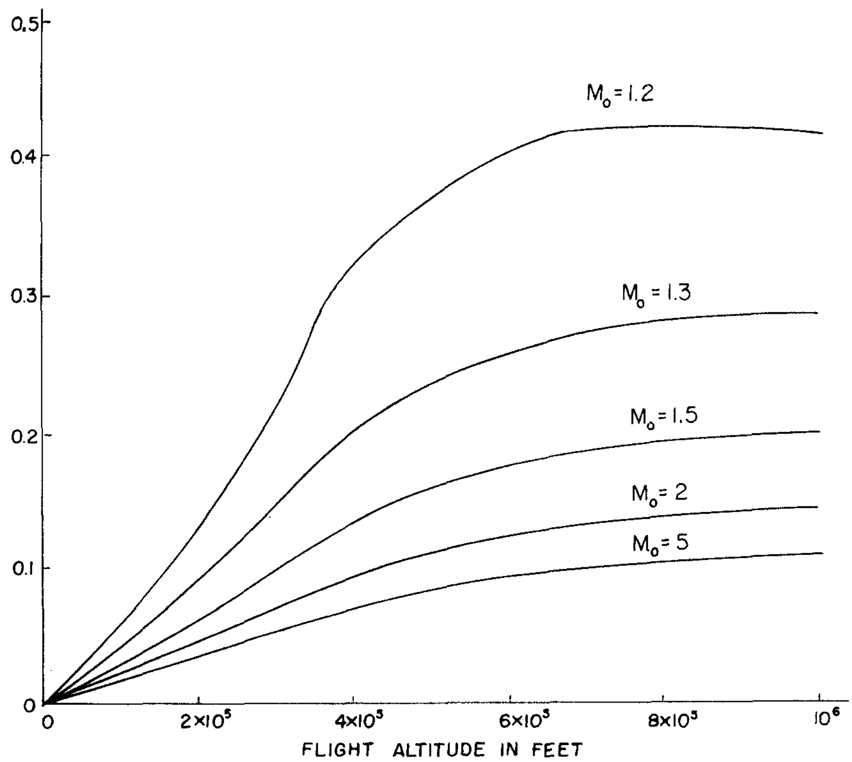

FIG. 2. Plot of $(\langle\phi\rangle - \phi)$ evaluated on the ground (where the maximums occur) vs flight altitude.

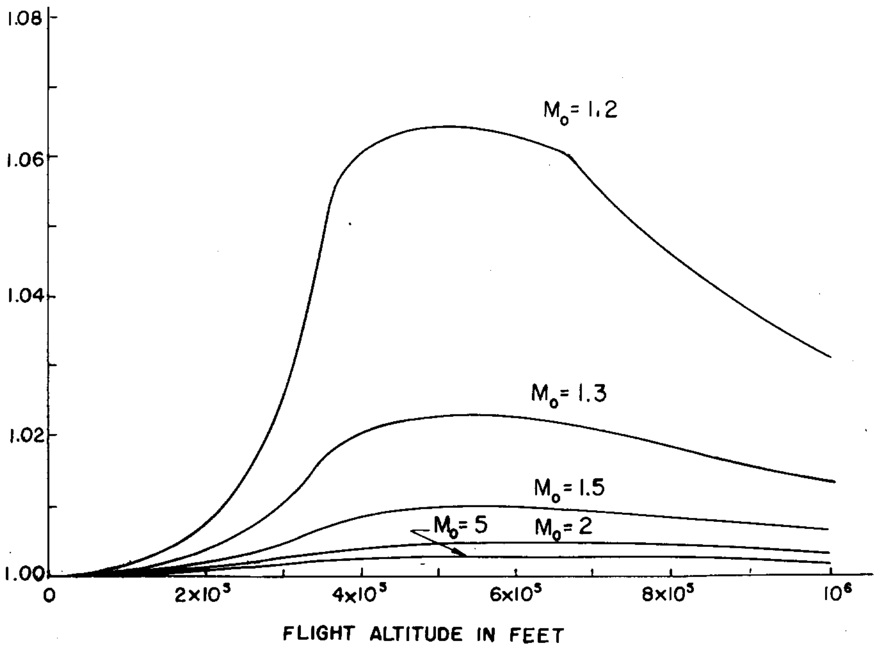

FIG. 3. Plot of $\langle\phi\rangle\cdot\langle 1/\phi\rangle$ evaluated on the ground (where maximum departure from unity occurs) vs flight altitude.

contains two terms, and from our previous discussion, we see that the first of these is negligible compared with the second.

In the same vein, we mention another estimate based on the standard atmosphere,

$$\left(\frac{1}{r}\int_0^r \phi\, dr\right)\left(\frac{1}{r}\int_0^r \frac{1}{\phi}\, dr\right)^{-1} = \langle\phi\rangle\langle 1/\phi\rangle \approx 1. \quad (26)$$

This is also suggested by Fig. 1. Figure 3 contains a sketch indicating this property. We see from the sketch that even for small Mach numbers, the estimate is extremely good and falls within $O(\varepsilon)$. This approximation will be used in the next section and results in a certain amount of analytical simplification.

IV. INVISCID THEORY

For the remainder of this paper we focus on the inviscid limit. A discussion of the effect of viscosity on solutions to equations of the form (23) has been discussed in an earlier paper.[26] Setting $\mathfrak{R} \to \infty$ in (23) we obtain

$$\frac{\partial f}{\partial r} - \frac{k}{r^{1/2}} f \frac{\partial f}{\partial S} = b(r, \theta) f. \quad (27)$$

The solution to (27), subject to the data $f(r \to 0) = f_0(x)$, is

$$f = \mu(r, \theta) f_0 \left[S + \frac{f}{\mu(r, \theta)} \int_0^r \frac{k(r)\mu(r, \theta)}{r^{1/2}} dr\right], \quad (28)$$

where

$$\mu = \exp\left[\int_0^r b(r, \theta)\, dr\right]. \quad (29)$$

On setting

$$G = -f/\mu,$$

$$\tau = \int_0^r (k\mu/r^{1/2})\, dr,$$

we have

$$\frac{\partial G}{\partial \tau} + G\frac{\partial G}{\partial S} = 0, \quad (30)$$

which is sometimes called the Hopf equation.

We note at this point that our analysis so far applies only to an axisymmetric body. For a nonaxisymmetric body, one can consider each meridian plane θ = const separately. The flow far from the body near the Mach cone is essentially axisymmetric and the θ dependence of the flow is introduced by the F function.[18] Thus, to obtain results corresponding to a nonaxisymmetric body, one inserts an F function parameterized in θ according to the supersonic area rule.[28] We do not carry this modification in the subsequent analysis.

It can be shown[25,26] that the well-known Whitham theory[17,18] deals with the solution of this equation. At this point, in order to solve for the flow past a body, one constructs the Whitham F function and solves (27) subject to the F function as data. These solutions generally give rise to shock waves. In this section we outline the solution to the boundary value problem and in the next section, the shock results are developed.

To solve the boundary value problem, we note that linearized theory in a uniform atmosphere yields the solution

$$v_1 \sim \frac{\gamma^{1/2} M_0 (M_0^2 - 1)^{1/4}}{(2r)^{1/2}} F[x - (M_0^2 - 1)^{1/2} r] \quad (31)$$

for r large. In (31), v_1 refers to the radial velocity perturbations,

$$F(\sigma) = \int_0^\sigma \frac{[r_b(t) r_b'(t)]'}{(\sigma - t)^{1/2}} dt$$

is the Whitham F function, r_b is the normalized body radius of the slender body, and the prime denotes differentiation. On the other hand, from (11) and (28), we have

$$v_1 \sim \frac{\gamma^{1/2}(M^2-1)^{1/2}}{M r^{1/2}} \mu f_0 \left(S + \frac{f}{\mu} \int_0^r \frac{k\mu}{r^{1/2}} dr \right). \qquad (32)$$

Comparing (32) with (31) in the appropriate limit (say limit εr fixed and $\varepsilon \to 0$), we obtain

$$v_1 \sim \frac{\gamma^{1/2} M_0^2 (M^2-1)^{1/2} \mu}{(2r)^{1/2} M (M_0^2 - 1)^{1/4}} \times F\left[S + \frac{M r^{1/2} v_1}{\gamma^{1/2}(M^2-1)^{1/2}\mu} \int_0^r \frac{k\mu}{r^{1/2}} dr \right], \qquad (33)$$

where

$$\mu = \left(\frac{\tilde{p}_0}{\tilde{p}^0} \right)^{1/2} \left(\frac{r}{\hat{r}} \right)^{(\frac{1}{2} + \tan^2\theta)} \left(\frac{M}{M_0} \right) \left(\frac{M_0^2 - 1}{M^2 - 1} \right)^{\frac{1}{4}\sec^2\theta} \qquad (34)$$

and

$$\left(\frac{\hat{r}}{r} \right) = \exp\left[\int_0^r \frac{1}{\phi r} \left(\frac{1}{r} \int_0^r \phi \, dr - \phi \right) dr \right] \approx \frac{1}{r} \int_0^r \frac{1}{\phi} dr. \qquad (35)$$

Note that the second part of (35) comes from the approximation (26). In the case $M = M_0$, $(\hat{r}/r)$ is simply 1.

Shock theory

The description of shock waves generated by the above theory follows from the governing equation (27)—or rather from its integral form which we can write as

$$\frac{\partial}{\partial r} \int f \, dS = \int k \frac{\partial}{\partial S} \left(\frac{f^2}{2r^{1/2}} \right) dS + \int b f \, dS.$$

Assuming a shock with a trajectory $\mathbb{S}(r)$, it follows that

$$\frac{d\mathbb{S}}{dr} = -\frac{k}{2r^{1/2}} (f^+ + f^-),$$

where the $\pm$ superscripts denote conditions in front and behind the shock. Considering the bow shock, $f^+ = 0$ and on substituting (33) we have

$$\frac{d\mathbb{S}}{dr} = -\frac{M_0^2 k\mu}{2^{3/2} r^{1/2} (M_0^2 - 1)^{1/4}} \times F\left(\mathbb{S} + \frac{M r^{1/2} v_1}{\gamma^{1/2}(M^2-1)^{1/2}\mu} \int_0^r \frac{k\mu}{r^{1/2}} dr \right).$$

Next, following Whitham,[17] we obtain the asymptotic form for the pressure jump across the shock. We define

$$\tau = \mathbb{S} + \frac{M r^{1/2} v_1}{\gamma^{1/2}(M^2-1)^{1/2}\mu} \int_0^r \frac{k\mu}{r^{1/2}} dr.$$

Then,

$$\frac{d\tau}{dr} = \frac{M_0^2 F'}{2^{1/2}(M_0^2-1)^{1/4}} \frac{d\tau}{dr} \int_0^r \frac{k\mu}{r^{1/2}} dr + \frac{M_0^2 k \mu F}{2^{3/2}(M_0^2-1)^{1/4} r^{1/2}},$$

which upon multiplying by F and integrating gives

$$\int_0^\tau F(\tau)\, d\tau = \frac{M_0^2 F^2(\tau)}{2^{3/2}(M_0^2-1)^{1/4}} \int_0^r \frac{k\mu}{r^{1/2}} dr.$$

Now since $\int_0^r k\mu r^{-1/2} dr \to \infty$ as $r \to \infty$, we must have $\tau \to \tau_0$ as $r \to \infty$, where τ_0 is the first zero of F. Therefore,

$$F(\tau) \sim \left[\frac{2^{3/2}(M_0^2-1)^{1/4}}{M_0^2 \int_0^r (k\mu/r^{1/2})\, dr} \int_0^{\tau_0} F(\tau)\, d\tau \right]^{1/2} \qquad (36)$$

and substituting into (11),

$$v_1 \sim \frac{\gamma^{1/2}}{M} (M^2-1)^{1/2} \frac{f}{r^{1/2}} \sim \frac{2^{1/4} \gamma^{1/2} M_0 (M^2-1)^{1/2} \mu}{r^{1/2} M (M_0^2-1)^{1/8}} \left(\frac{\int_0^{\tau_0} F(\tau)\, d\tau}{\int_0^r k\mu r^{-1/2} dr} \right)^{1/2} \qquad (37)$$

and

$$\frac{\Delta\tilde{p}}{\tilde{p}^0} \sim \frac{\gamma f}{r^{1/2}} \sim \frac{2^{1/4}\gamma M_0 \mu}{r^{1/2}(M_0^2-1)^{1/8}} \left(\frac{\int_0^{\tau_0} F(\tau)\, d\tau}{\int_0^r k\mu r^{-1/2} dr} \right)^{1/2}. \qquad (38)$$

In order to avoid complicated expressions, we henceforth restrict our attention to the case of small θ. (We hasten to mention that the general case offers no real difficulty.)

Wind effects

Before discussing the pressure jump, we comment on the effect of winds and temperature variation. We recall that the wind effect was resolved into a component parallel and normal to the direction of flight. The parallel wind component enters into the normalization (5) and hence appears directly in the Mach number. The same can be said for temperature variation which is also incorporated in the Mach number and also explicitly in the pressure ratio in (38). In addition, both these effects as well as cross winds appear in the definition of $\hat{r}$ and in the argument of F in (33). Dealing, for example, with the shock wave, we see that the location of the shock is affected by all the wind effects as well as the temperature. In addition, the strength of the shock wave is directly and in a pronounced way affected by the longitudinal wind component and the temperature variation since they appear explicitly in the Mach number. For small winds, we can ignore the wind effect in all except the Mach number terms appearing as the coefficient in (38). Making this approximation we can compare over-pressure in a standard atmosphere with that in a sheared atmosphere. In fact from (38), with $\theta = 0$, we have

$$\frac{\Delta\tilde{p}_w}{\Delta\tilde{p}} \approx \left(\frac{M_w}{M} \right) \left(\frac{M^2-1}{M_w^2-1} \right)^{1/4},$$

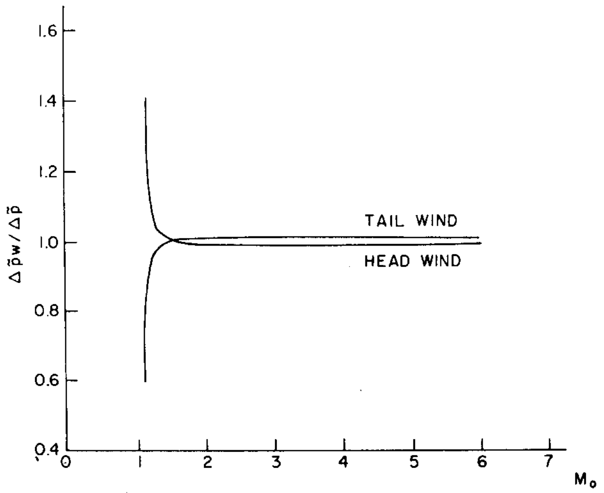

FIG. 4. Variation of pressure correction factor with Mach numbers for an aircraft flying at 30 000 ft. with head and tail winds.

where $M_w(r)$ is the Mach number with wind $[M_w(0) = M_0]$. Using the mean zonal wind given in Hayes *et al.*,[9] this pressure ratio is shown in Fig. 4. In both cases the aircraft is flying at 30,000 ft. in a standard atmosphere. Figure 4 shows good agreement with Fig. 18 given in Hayes *et al.*[9] As is clear from the figure, a tail wind is beneficial while a head wind is detrimental to the sonic boom effect. We next consider our results, (37) and (38), for special atmospheres.

Uniform atmosphere

In this case $v^0 = w^0 = 0$, $\tilde{T}^0(r) = \tilde{T}_0$, $M = M_0$, $\tilde{p}^0(r) = \tilde{p}_0$. It then follows from (33), (36), and (38), that

$$v_1 \sim \frac{\gamma^{1/2} M_0 (M_0^2 - 1)^{1/4}}{(2r)^{1/2}} F\left[x - (M_0^2 - 1)^{1/2} r + \frac{(\gamma + 1) M_0^3 r}{\gamma^{1/2}(M_0^2 - 1)} v_1\right] \qquad (39)$$

and

$$\frac{\Delta \tilde{P}}{\tilde{p}^0} \sim \frac{2^{1/4}\gamma (M_0^2 - 1)^{1/8}}{(\gamma + 1)^{1/2} r^{3/4}} \left[\int_0^{\tau_0} F(\tau)\, d\tau\right]^{1/2}. \qquad (40)$$

In this last expression we have used $\tilde{P}$ to denote pressure in the uniform atmosphere calculation since it is customary to normalize other pressure jump calculations by (40). Combining (38) and (40) gives

$$\frac{\Delta \tilde{p}}{\Delta \tilde{P}} \sim \left(\frac{\tilde{p}_0}{\tilde{p}^0(r)}\right)^{1/2} \left(\frac{M}{M_0}\right) \left(\frac{M_0^2 - 1}{M^2 - 1}\right)^{1/4} \times \left(\frac{2r^{1/2}(r/\hat{r})}{\int_0^r \frac{1}{\hat{r}^{1/2}} \left(\frac{\tilde{p}_0}{\tilde{p}^0(r)}\right)^{1/2} \left(\frac{M}{M_0}\right)^3 \left(\frac{M_0^2 - 1}{M^2 - 1}\right)^{3/4} dr}\right)^{1/2} \qquad (41)$$

which is also obtained by Kawamura and Makimo.[13]

Isothermal atmosphere

In this case we take $v^0 = w^0 = 0$, $\tilde{T}^0(r) = \tilde{T}_0$, $M = M_0$, and we have

$$v_1 \sim \frac{\gamma^{1/2} M_0 (M_0^2 - 1)^{1/4}}{(2r)^{1/2}} \left(\frac{\tilde{\rho}_0}{\tilde{\rho}^0(r)}\right)^{1/2} F\left[x - (M_0^2 - 1)^{1/2} r + \frac{(\gamma + 1) M_0^3 r^{1/2} v_1}{2\gamma^{1/2}(M_0^2 - 1)} \left(\frac{\tilde{\rho}^0(r)}{\tilde{\rho}_0}\right)^{1/2} \int_0^r \frac{1}{r^{1/2}} \left(\frac{\tilde{\rho}_0}{\tilde{\rho}^0(r)}\right)^{1/2} dr\right] \qquad (42)$$

and

$$\frac{\Delta \tilde{p}_i}{\tilde{p}^0} \sim \frac{2^{3/4}\gamma (M_0^2 - 1)^{1/8}}{[(\gamma + 1) r]^{1/2}} \left(\frac{\tilde{\rho}_0}{\tilde{\rho}^0(r)}\right)^{1/2} \times \left(\frac{\int_0^{\tau_0} F(\tau)\, d\tau}{\int_0^r \frac{1}{r^{1/2}} \left(\frac{\tilde{\rho}_0}{\tilde{\rho}^0(r)}\right)^{1/2} dr}\right)^{1/2}, \qquad (43)$$

where $\tilde{\Delta} p_i$ is the pressure jump of the front shock in an isothermal atmosphere. And hence, combining (40) and (43), we have

$$\frac{\Delta \tilde{p}_i}{\Delta \tilde{P}} \sim \left(\frac{\tilde{\rho}_0}{\tilde{\rho}^0(r)}\right)^{1/2} \left(\frac{2r^{1/2}}{\int_0^r \frac{1}{r^{1/2}} \left(\frac{\tilde{\rho}_0}{\tilde{\rho}^0(r)}\right)^{1/2} dr}\right)^{1/2}. \qquad (44)$$

Note that (43) gives, for r large,

$$\Delta \tilde{p}_i = O(\tilde{p}^0 / r^{1/2})$$

a result quoted by George and Seebass.[7]

Polytropic atmosphere

In this case we take $v^0 = w^0 = 0$, $U = \text{const}$, and $\tilde{p}^0 \propto (\tilde{\rho}^0)^\alpha$. Then

$$v_1 \sim \frac{\gamma^{1/2} M_0 (M^2 - 1)^{1/4}}{2\hat{r}^{1/2}} \left(\frac{M}{M_0}\right)^{1+m} \times F\left(S + \frac{(\gamma + 1)\hat{r}^{1/2} v_1}{2\gamma^{1/2}(M^2 - 1)^{1/4} M^{1+m}} \times \int_0^r \frac{M^{4+m}}{r^{1/2}(M^2 - 1)^{3/4}} dr\right) \qquad (45)$$

and

$$\frac{\Delta \tilde{p}_p}{\tilde{p}^0} \sim \frac{2^{3/4}\gamma (M_0^2 - 1)^{1/8}}{[(\gamma + 1)\hat{r}]^{1/2}} \left(\frac{M}{M_0}\right)^{2+m} \left(\frac{M_0^2 - 1}{M^2 - 1}\right)^{1/4} \times \left(\frac{\int_0^{\tau_0} F(\tau)\, d\tau}{\int_0^r \frac{1}{r^{1/2}} \left(\frac{M}{M_0}\right)^{4+m} \left(\frac{M_0^2 - 1}{M^2 - 1}\right)^{3/4} dr}\right)^{1/2}, \qquad (46)$$

where the subscript p denotes polytropic atmosphere and $m = 1/(\alpha - 1)$. Hence from (40), we have

$$\frac{\Delta \tilde{p}_p}{\Delta \tilde{P}} \sim \left(\frac{M}{M_0}\right)^{2+m} \left(\frac{M_0^2 - 1}{M^2 - 1}\right)^{1/4} \times \left(\frac{2r^{1/2}(r/\hat{r})}{\int_0^r \frac{1}{\hat{r}^{1/2}} \left(\frac{M}{M_0}\right)^{4+m} \left(\frac{M_0^2 - 1}{M^2 - 1}\right)^{3/4} dr}\right)^{1/2}. \qquad (47)$$

This agrees with the result obtained by Kawamura and Makino.[13]

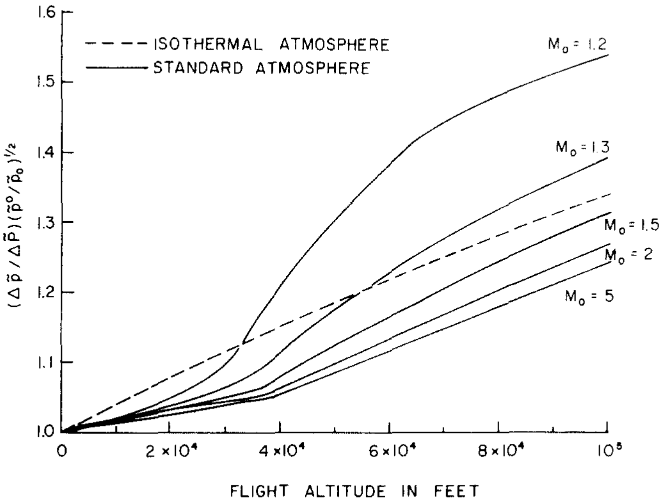

FIG. 5. Pressure correction factors for standard and isothermal atmospheres.

Real atmosphere

In this case no simplification can be expected and the full expression for the pressure jump, (41), must be employed.

Using data taken from the standard atmosphere tables,[27] in Fig. 5 we have plotted the pressure correction factor $(\Delta\tilde{p}/\Delta\tilde{P})(\tilde{p}^0/\tilde{p}_0)^{1/2}$ for several Mach numbers. For comparison, we have also sketched (44). These sketches are in good agreement with those obtained by George and Plokin,[6] Kane and Palmer,[10] Randall,[5] and Kawamura and Makino[13] under similar conditions. The reader is referred to these papers for further discussion.

APPENDIX A

For the sake of completeness, we outline the two-dimensional calculations. In this case, only one "slow" variable is needed in the vertical direction, namely $y_1 = \epsilon y$. Following the transformation (3), we define

$$Y = \int_0^y \phi(\epsilon y)\,dy, \qquad Y_1 = \epsilon Y, \tag{A1}$$

where y is positive downward from the aircraft. Our unperturbed flow is then $[\tilde{p}^0(Y_1), \tilde{U}(Y_1), \epsilon\tilde{v}^0(Y_1), \tilde{T}^0(Y_1)]$, where $\tilde{v}^0$ is the vertical wind. Normalizing variables according to (5) and introducing an expansion similar to (9), the Navier–Stokes equations give

$$\epsilon\left(\mathbf{A}\frac{\partial}{\partial x} + \phi\mathbf{B}\frac{\partial}{\partial Y}\right)\mathbf{v}_1 = 0, \tag{A2}$$

$$\epsilon_2\left(\mathbf{A}\frac{\partial}{\partial x} + \phi\mathbf{B}\frac{\partial}{\partial Y}\right)\mathbf{v}_2 = \epsilon^2\left[\mathbf{X}(\mathbf{v}_1) + \mathbf{Y}(\mathbf{v}_1) + \mathbf{N} + \mathbf{W}(\mathbf{v}_1) + \mathbf{G}(\mathbf{v}_1) - \phi\mathbf{B}\frac{\partial}{\partial Y_1}\mathbf{v}_1\right], \tag{A3}$$

where $\mathbf{v}_i = (\rho_i, u_i, v_i, T_i)$ is the ith order perturbation; $\mathbf{X}$ is the quadratic nonlinear term; $\mathbf{Y}$ is the linear dissipative term; $\mathbf{N}$ is due to the nonconstant normalization; $\mathbf{W}$ is wind; $\mathbf{G}$ is gravity; and the last term represents the derivative with respect to the "slow" scale.

The solution to (A2) in the lower half-plane is

$$\mathbf{v}_1 = \mathbf{l}_3 f(x - (M_0^2 - 1)^{1/2} Y, Y_1) \tag{A4}$$

and

$$\phi[(M^2 - 1)/(M_0^2 - 1)]^{1/2}$$

where we have ignored the upstream wave $\mathbf{l}_4\hat{f}(x + (M_0^2 - 1)^{1/2} Y)$. Multiplying (A3) from the left by $\mathbf{l}_3$ and integrating we find that the right-hand side gives rise to secularity in $\mathbf{v}_2$ unless

$$\mathbf{l}_3 \cdot \left(\mathbf{X} + \mathbf{Y} + \mathbf{N} + \mathbf{W} + \mathbf{G} - \phi\mathbf{B}\frac{\partial}{\partial Y_1}\mathbf{v}_1\right) = 0. \tag{A5}$$

On substituting (A4) into (A5) and simplifying we finally arrive at the equation [corresponding to Eq. (23) for the three-dimensional case]

$$\frac{\partial f}{\partial y} - kf\frac{\partial f}{\partial s} = \frac{1}{\Re}\frac{\partial^2 f}{\partial s^2} + bf, \tag{A6}$$

where

$$\begin{aligned} k &= \frac{(\gamma+1)M^2}{2(M^2-1)^{1/2}}, \\ \frac{1}{\Re} &= \frac{[\gamma(\zeta+\eta) + \chi^2\xi]M^3}{2\gamma^{3/2}(M^2-1)^{1/2}}, \\ b &= \frac{1}{2}\left[-\frac{\partial \ln \tilde{p}^0}{\partial y} + \left(\frac{M^2-2}{M^2-1}\right)\frac{\partial \ln M}{\partial y}\right], \\ s &= x - \int_0^y \left[(M^2-1)^{1/2} - \frac{Mv^0}{\gamma^{1/2}}\right] dy. \end{aligned} \tag{A7}$$

For a flow past a thin body given by

$$y = \begin{cases} F(x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

The boundary condition is

$$v_1(x,y)|_{y=0} = \gamma^{1/2} M_0 F'. \tag{A8}$$

Hence, (A4) and (A8) give

$$v_1 \sim \frac{\gamma^{1/2}M_0^2}{M}\left(\frac{M^2-1}{M_0^2-1}\right)^{1/2} \times \mu F'\left(s + \frac{Mv_1}{\gamma^{1/2}(M^2-1)^{1/2}\mu}\int_0^y k\mu\,dy\right), \tag{A9}$$

$$\mu = \exp\left(\int_0^y b\,dy\right) = \left(\frac{\tilde{p}_0}{\tilde{p}^0}\right)^{1/2}\left(\frac{M}{M_0}\right)\left(\frac{M_0^2-1}{M^2-1}\right)^{1/4}.$$

Performing a shock analysis similar to the three-dimensional case, we find that the pressure jump across the front shock is (for y large)

$$\frac{\Delta\tilde{p}}{\tilde{p}^0} \sim \frac{2^{1/2}\gamma M_0\mu}{(M_0^2-1)^{1/4}}\left(\frac{F_0}{\int_0^y k\mu\,dy}\right)^{1/2}, \tag{A10}$$

where x_0 is the first zero of $F'(x)$, and $F(x_0) = F_0$. For a uniform atmosphere, (A10) reduces to

$$\frac{\Delta\tilde{P}}{\tilde{p}^0} \sim \frac{2\gamma}{(\gamma+1)^{1/2}}\left(\frac{F_0}{y}\right)^{1/2}. \qquad \text{(A11)}$$

Hence,

$$\frac{\Delta\tilde{p}}{\Delta\tilde{P}} \sim \left(\frac{\tilde{p}_0}{\tilde{p}^0(y)}\right)^{1/2}\left(\frac{M}{M_0}\right)\left(\frac{M_0^2-1}{M^2-1}\right)^{1/4}$$

$$\times\left(\frac{y}{\int_0^y \left(\frac{\tilde{p}_0}{\tilde{p}^0}\right)^{1/2}\left(\frac{M}{M_0}\right)^3\left(\frac{M_0^2-1}{M^2-1}\right)^{3/4} dy}\right)^{1/2}. \qquad \text{(A12)}$$

For an isothermal atmosphere, (A12) gives

$$\frac{\Delta\tilde{p}_i}{\Delta\tilde{P}} \sim \left(\frac{\tilde{\rho}_0}{\tilde{\rho}^0}\right)^{1/2}\left(\frac{y}{\int_0^y \left(\frac{\tilde{\rho}_0}{\tilde{\rho}^0}\right)^{1/2} dy}\right)^{1/2}$$

$$= \left(\frac{gy}{2e^{gy}(1-e^{-gy/2})}\right)^{1/2} \qquad \text{(A13)}$$

and for polytropic atmosphere, $(\tilde{\rho}^0/\tilde{\rho}_0) = (\tilde{p}^0/\tilde{p}_0)^\alpha$,

$$\frac{\Delta\tilde{p}_p}{\Delta\tilde{P}} \sim \left(\frac{M}{M_0}\right)^{2+m}\left(\frac{M_0^2-1}{M^2-1}\right)^{1/4}$$

$$\times\left(\frac{y}{\int_0^y \left(\frac{M}{M_0}\right)^{m+4}\left(\frac{M_0^2-1}{M^2-1}\right)^{3/4} dy}\right)^{1/2} \qquad \text{(A14)}$$

where $m = 1/(\alpha - 1)$.

APPENDIX B

Referring to Eq. (6), the terms on the right-hand side are

$$\mathbf{X} = -\begin{bmatrix} \frac{\partial}{\partial x}\rho u + \frac{\rho v}{r} + \phi\frac{\partial}{\partial R}\rho v \\ \gamma^{1/2}M\rho\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \chi\frac{\partial}{\partial x}\rho T + \phi v\frac{\partial u}{\partial R} \\ \gamma^{1/2}M\rho\frac{\partial v}{\partial x} + u\frac{\partial v}{\partial x} + \phi\frac{\partial}{\partial R}\left(\frac{v^2}{2} + \chi\rho T\right) \\ \gamma^{1/2}M\rho\frac{\partial T}{\partial x} + u\frac{\partial T}{\partial x} + \chi(\rho+\chi T)\frac{\partial u}{\partial x} \\ \qquad + \phi\left[v\frac{\partial T}{\partial R} + \chi(\rho+\chi T)\left(\frac{\partial v}{\partial R} + \frac{v}{\phi r}\right)\right] \end{bmatrix},$$

$$\mathbf{Y} = \Big[0, \zeta D^2 u + \eta\left(\frac{\partial^2 u}{\partial x^2} + \phi\frac{\partial^2 v}{\partial x\partial R} + \frac{1}{r}\frac{\partial v}{\partial x}\right),$$

$$\zeta\left(D^2 v - \frac{v}{r}\right) + \eta\left(\phi\frac{\partial^2 u}{\partial x\partial R} + \frac{\phi}{r}\frac{\partial v}{\partial R} + \phi^2\frac{\partial^2 v}{\partial R^2} - \frac{v}{r^2}\right),$$

$$\xi D^2 T\Big],$$

where

$$D^2 = \frac{\partial^2}{\partial x^2} + \frac{\phi}{r}\frac{\partial}{\partial R} + \phi^2\frac{\partial^2}{\partial R^2},$$

$$\mathbf{N} = -\varepsilon^2\phi\Big[v\frac{\partial}{\partial R_1}\ln\tilde{\rho}^0\tilde{a}, \gamma^{1/2}Mv\frac{\partial}{\partial R}\ln\tilde{a}M,$$

$$(\rho+\chi T)\frac{\partial}{\partial R}\ln\tilde{\rho}^0\tilde{T}^0, \quad \frac{v}{\chi}\frac{\partial}{\partial R_1}\ln\tilde{T}^0\tilde{a}^{\gamma-1}\Big],$$

$$\mathbf{D} = -(1,0,0,\chi)\frac{1}{r}\frac{\partial w}{\partial\theta},$$

$$\mathbf{W} = \mathbf{W}_r + \mathbf{W}_\theta = -\varepsilon\phi v^0\frac{\partial}{\partial R}\mathbf{v} - \varepsilon\frac{w^0}{r}\frac{\partial}{\partial\theta}\mathbf{v},$$

$$\mathbf{G} = \varepsilon^2(0,0,g\rho\cos\theta,0),$$

$$\zeta = \tilde{\mu}/\tilde{\rho}^0\tilde{a}L, \ \eta = (\tilde{\beta}+\tilde{\mu}/3)/\tilde{\rho}^0\tilde{a}L, \ \xi = \tilde{\kappa}/c_v\tilde{\rho}^0\tilde{a}L.$$

In the last set; $\tilde{\mu}$ is the viscosity, $\tilde{\beta}$ is the bulk viscosity, and $\tilde{\kappa}$ is the heat conduction coefficient.

Substituting expansions (9) into (6), we obtain the set of perturbed equations

$$\varepsilon\left\{\gamma^{1/2}M\frac{\partial\rho_1}{\partial x} + \frac{\partial u_1}{\partial x} + \phi\frac{\partial v_1}{\partial R} + \frac{v_1}{r}\right\}$$

$$+ \varepsilon_2\left\{\gamma^{1/2}M\frac{\partial\rho_2}{\partial x} + \frac{\partial u_2}{\partial x} + \phi\frac{\partial v_2}{\partial R} + \frac{v_2}{r}\right\}$$

$$+\varepsilon_3\left(\gamma^{1/2}M\frac{\partial\rho_3}{\partial x} + \frac{\partial u_3}{\partial x} + \phi\frac{\partial v_3}{\partial R} + \frac{v_3}{r}\right) + \cdots$$

$$+\varepsilon^2\left(\frac{\partial}{\partial x}\rho_1 u_1 + \frac{\rho_1 v_1}{r} + \phi\frac{\partial}{\partial R}\rho_1 v_1\right) + \varepsilon^3\left(v_1\phi\frac{\partial\ln\tilde{\rho}^0\tilde{a}}{\partial R_1}\right)$$

$$+\varepsilon^2\left(\frac{1}{r}\frac{\partial w_1}{\partial\theta}\right) + \varepsilon^2\left\{v^0\phi\frac{\partial\rho_1}{\partial R} + w^0\frac{1}{r}\frac{\partial\rho_1}{\partial\theta}\right\} + \varepsilon v_*\left(\phi\frac{\partial v_1}{\partial R_*}\right)$$

$$+\varepsilon^3\left(\phi\frac{\partial v_1}{\partial R_1}\right) + \cdots = 0, \qquad \text{(B1)}$$

$$\varepsilon\left(\gamma^{1/2}M\frac{\partial u_1}{\partial x} + \frac{\partial\rho_1}{\partial x} + \chi\frac{\partial T_1}{\partial x}\right)$$

$$+ \varepsilon_2\left(\gamma^{1/2}M\frac{\partial u_2}{\partial x} + \frac{\partial\rho_2}{\partial x} + \chi\frac{\partial T_2}{\partial x}\right)$$

$$+\varepsilon_3\left(\gamma^{1/2}M\frac{\partial u_3}{\partial x} + \frac{\partial\rho_3}{\partial x} + \chi\frac{\partial T_3}{\partial x}\right) + \varepsilon^2\Big(\gamma^{1/2}M\rho_1\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_1}{\partial x}$$

$$+\chi\frac{\partial}{\partial x}\rho_1 T_1 + \phi v_1\frac{\partial u_1}{\partial R}\Big) - \varepsilon\Big[\zeta D^2 u_1 + \eta\Big(\frac{\partial^2 u_1}{\partial x^2} + \phi\frac{\partial^2 v_1}{\partial x\partial R}$$

$$+\frac{1}{r}\frac{\partial v_1}{\partial x}\Big)\Big] + \varepsilon^3\left(\gamma^{1/2}v_1\phi\frac{\partial M}{\partial R_1} + \gamma^{1/2}Mv_1\phi\frac{\partial\ln\tilde{a}}{\partial R_1}\right)$$

$$+\varepsilon^2\left(\phi v^0\frac{\partial u_1}{\partial R} + w^0\frac{1}{r}\frac{\partial}{\partial\theta}u_1\right) + \cdots = 0, \qquad \text{(B2)}$$

$$\varepsilon\left[\gamma^{1/2}M\frac{\partial v_1}{\partial x} + \phi\left(\frac{\partial\rho_1}{\partial R} + \chi\frac{\partial T_1}{\partial R}\right)\right] + \varepsilon_2\Big[\gamma^{1/2}M\frac{\partial v_2}{\partial x} + \phi\Big(\frac{\partial\rho_2}{\partial R}$$

$$+\chi\frac{\partial T_2}{\partial R}\Big)\Big] + \varepsilon_3\left[\gamma^{1/2}M\frac{\partial v_3}{\partial x} + \phi\left(\frac{\partial\rho_3}{\partial R} + \chi\frac{\partial T_3}{\partial R}\right)\right] + \cdots$$

$$+\varepsilon^2\left[\gamma^{1/2}M\rho_1\frac{\partial v_1}{\partial x} + u_1\frac{\partial v_1}{\partial x} + \phi\left(v_1\frac{\partial v_1}{\partial R} + \chi\frac{\partial}{\partial R}\rho_1 T_1\right)\right]$$

$$-\varepsilon\left[\zeta\left(D^2 v_1 - \frac{v_1}{r}\right) + \eta\left(\phi\frac{\partial^2 u_1}{\partial x\partial R} + \frac{\phi}{r}\frac{\partial v_1}{\partial R} + \phi^2\frac{\partial^2 v_1}{\partial R^2} - \frac{v_1}{R^2}\right)\right]$$

$$+\varepsilon^3\left\{\phi\left(\rho_1+\chi T_1\right)\frac{\partial\ln\tilde{\rho}^0\tilde{T}^0}{\partial R_1}\right\}$$

$$+\varepsilon^2\left(\phi v^0\frac{\partial v_1}{\partial R}+w^0\frac{1}{r}\frac{\partial}{\partial\theta}v_1\right)$$

$$-\varepsilon^3\left(g\rho_1\cos\theta\right)+\varepsilon\nu_*\left[\phi\left(\frac{\partial\rho_1}{\partial R_*}+\chi\frac{\partial T_1}{\partial R_*}\right)\right]+\varepsilon^3\left[\phi\left(\frac{\partial\rho_1}{\partial R_1}+\chi\frac{\partial T_1}{\partial R_1}\right)\right]+\cdots=0, \tag{B3}$$

$$\varepsilon\left(\gamma^{1/2}M\frac{\partial T_1}{\partial x}+\chi\frac{\partial u_1}{\partial x}+\phi\chi\frac{\partial v_1}{\partial R}+\chi\frac{v_1}{r}\right)+\varepsilon_2\left(\gamma^{1/2}M\frac{\partial T_2}{\partial x}+\chi\frac{\partial u_2}{\partial x}+\phi\chi\frac{\partial v_2}{\partial R}+\chi\frac{v_2}{r}\right)+\varepsilon_3\left(\gamma^{1/2}M\frac{\partial T_3}{\partial x}+\chi\frac{\partial u_3}{\partial x}+\phi\chi\frac{\partial v_3}{\partial R}+\chi\frac{v_3}{r}\right)$$

$$+\varepsilon^2\left\{\gamma^{1/2}M\rho_1\frac{\partial T_1}{\partial x}+\chi(\rho_1+\chi T_1)\frac{\partial u_1}{\partial x}+u_1\frac{\partial T_1}{\partial x}+\phi\left[v_1\frac{\partial T_1}{\partial R}+\chi(\rho_1+\chi T_1)\left(\frac{\partial v_1}{\partial R}+\frac{v_1}{\phi r}\right)\right]\right\}-\varepsilon(\xi D^2T_1)$$

$$+\varepsilon^3\left(\frac{v_1}{\chi}\frac{\partial}{\partial R_1}\ln\tilde{T}^0\tilde{a}^{\gamma-1}\right)+\varepsilon^2\left(\chi\frac{1}{r}\frac{\partial}{\partial\theta}w_1\right)$$

$$+\varepsilon^2\left(v^0\phi\frac{\partial T_1}{\partial R}+w^0\frac{1}{r}\frac{\partial}{\partial\theta}T_1\right)+\varepsilon\nu_*\left(\phi\chi\frac{\partial v_1}{\partial R_*}\right)$$

$$+\varepsilon^3\left(\chi\phi\frac{\partial v_1}{\partial R_1}\right)+\cdots=0. \tag{B4}$$

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