Approximate solution in gasdynamics

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One-dimensional unsteady gasdynamics is considered. An approximation based mainly on the interaction of simple and entropy waves is adopted. A discussion supporting this approximation, based in part on shock expansion theory, is given. By the use of certain transformations the approximation leads to solution in terms of quadratures. Excellent agreement with exact numerical results is obtained over a wide range of cases.

I. INTRODUCTION

In this and the following paper[1] we consider the approximate and numerical integration of the gasdynamic equations. Specifically, we consider one-dimensional unsteady flows in which shock waves are present. The focus in this paper will be mainly analytical, while that of Ref. 1 will be mainly numerical. A goal of the present work is to render gasdynamic problems, by analytic means, into their most tractable form for subsequent numerical integration. We will rely mainly on the use of characteristics, Riemann invariants, and simple waves.[2-4]

The first step of our procedure is the determination of an accurate analytic approximation to the exact solution. A good approximation not only shortens the integration time, but without it one may be outside the "circle of convergence" of a perfectly valid numerical scehme. A number of analytical approximations are to be found in the literature. Notable are the simple wave method of Friedrichs[5] (see also Ref. 2), the method of nonlinearization of Whitham[6-8] and Landau,[9] and the method of perturbation of characteristics of Lin[10] (see also Fox[11]). All of these are, in effect, low-order approximations and lose accuracy with increasing Mach number. (For an exposition see Lighthill.[12]) Although less elegant in form, the analytical approximation presented here remains quite accurate even at very high Mach numbers.

For supersonic airfoil theory (and analogously for one-dimensional unsteady theory) the pressure on a body is accurately given by shock expansion theory.[13] This and simple wave theory enter in our calculations and we briefly review these.

For one-dimensional unsteady piston motions, shock expansion theory approximates the flow behind the shock by a simple wave. For a piston moving from left to right into a quiescent perfect gas, this amounts to taking $r^- = u - 2c/(\gamma - 1)$ and the entropy S to be constants. (For symbols and normalization see Sec. II.) Both constants are evaluated behind the shock at the initial instant. In reality [see Eq. (7)] r^- varies along the shock, and as it travels along backward characteristics, it is modified by entropy variations. Mahony[14] has shown that these two effects seek to annul each other so that r^- set to a constant at the piston is an accurate approximation. Since u is known at the piston, c from the approximation is also determined; and c together with S when substituted into the state equation

$$p = c^{2\gamma/(\gamma-1)} \exp(-S)$$

give the pressure. We note in passing that $2\gamma/(\gamma - 1) \approx 7$ for air, so that great accuracy in c is necessary in the computation of p.

Shock expansion theory becomes incorrect away from the piston. Both r^- and S which are fixed at their piston values should tend to their upstream values as the shock travels toward infinity. On the other hand, the simple wave theory of Fredrichs[5] is accurate toward infinity. In this theory, changes in entropy and r^-, which vary with third order in strength, are ignored. It therefore follows that Friedrichs' theory, while valid at infinity, is an approximation at the piston which becomes less accurate with increasing Mach number.

In Fig. 1, changes in r^- and S across a shock are shown. Changes in both S and r^- are smaller than the above-mentioned third-order estimate. In fact, if a small strength expansion is performed for an ideal gas of $\gamma = 1.4$, we find

$$(\Delta r^-/r_0^-) \approx 0.02(M^2 - 1)^3, \quad (\Delta S/R) \approx 0.16(M^2 - 1)^3 .$$

Thus, on a numerical basis $(\Delta r^-/r_0^-)$ is more than an order-of-magnitude smaller than expected.

Mahony[14] and Meyer[15] have considered extensions of shock expansion theory in which variations of S and r^- are included with the object of improving load calculations; but, in view of the extreme accuracy of shock expansion theory, both authors concluded that there was little reason to go beyond it.

II. FORMULATION

We consider one-dimensional, unsteady, gasdynamic motions in a semi-infinite domain in which the finite boundary, the piston, is in motion (see Fig. 2). The piston, with trajectory $P(t)$, moves abruptly into the gas at the initial instant at a speed $P'(0) = M_0$, which from the normalization to be given is a Mach number. A shock is generated and its time evolution is represented by $X_1(t)$. After the initial instant the piston loses speed, passing through zero to negative speeds. At some later instant, the piston speed is increased abruptly to its final zero value, producing the tail shock $X_2(t)$.

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A. Governing equations

The equations of gasdynamics, in characteristic form, are

$$\left(\frac{\partial}{\partial t}+(u\pm c)\frac{\partial}{\partial x}\right)\left(u\pm\frac{2c}{\gamma-1}\right)\mp\frac{c}{\gamma}\left(\frac{\partial}{\partial t}+(u\pm c)\frac{\partial}{\partial x}\right)S=0\,,$$

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial t}=0\,, \tag{1}$$

$$\frac{c^2}{\gamma}=\frac{p}{\rho}=\frac{1}{\rho}[c^{2\gamma/(\gamma-1)}\exp(-S)]\,.$$

The speed of sound c and the velocity u have been normalized by c_0, ρ by ρ_0, T by T_0, and S by R. The space dimension x has been normalized by a reference length L, and time t by L/c_0. For convenience, the equations have been specialized to an ideal gas, but a similar development can be applied to nonideal gases.

In addition to the set (1) we will need the following shock relations:

$$\frac{u_+-u_-}{c_-}=\frac{2(M_-^2-1)}{(\gamma+1)M_-}\,, \tag{2}$$

$$S_+-S_-=\ln\left[\left(1+\frac{2\gamma}{\gamma+1}(M_-^2-1)\right)^{1/\gamma-1}\left(\frac{(\gamma+1)M_-^2}{(\gamma-1)M_-^2+2}\right)^{-\gamma/\gamma-1}\right]\,, \tag{3}$$

$$\frac{c_+^2}{c_-^2}=1+\frac{2(\gamma-1)(\gamma M_-^2+1)(M_-^2-1)}{(\gamma+1)^2M_-^2}\,, \tag{4}$$

$$M_-=(U-u_-)/c_-\,. \tag{5}$$

Downstream quantities carry a positive subscript and upstream quantities a negative subscript. U denotes the normalized shock speed.

1. Boundary conditions

The only boundary condition in the problem is

$$u[P(t),t]=P'(t)\,. \tag{6}$$

[It is assumed that $P'(t)$ has been normalized by c_0, and $P(t)$ by the reference length L, so that Eq. (6) refers to a family of motions.]

2. Characteristic transformation

We introduce new coordinates (α,β) based on the characteristic curves of the entropy and the right-going Riemann invariant,

$$\alpha=\alpha(x,t)\,,\quad \beta=\beta(x,t)\,, \tag{7}$$

such that

$$\left(\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}\right)\beta=0\,,\quad\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\alpha=0\,. \tag{8}$$

Both β and α are undetermined up to an arbitrary function so two further conditions will be required to fix Eq. (7). If we introduce Eqs. (7) and (8) into (1), these become

$$S_\beta=0\,, \tag{9}$$

$$r_\alpha^+=(c/\gamma)S_\alpha\,, \tag{10}$$

$$\alpha_x r_\alpha^-+2\beta_x r_\beta^-=-(c/\gamma)\alpha_x S_\alpha\,, \tag{11}$$

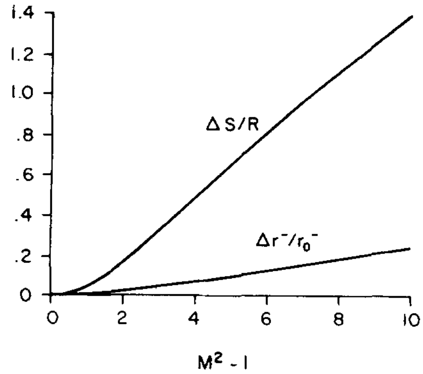

FIG. 1. Normalized change in entropy and cross Riemann invariant r^- across a normal shock versus the shock strength M^2-1. (M is the Mach number of the shock.)

where

$$r^\pm=u\pm 2c/(\gamma-1)\,. \tag{12}$$

x and t are now dependent variables which from (7) and (8) satisfy

$$\frac{\partial x}{\partial\alpha}=(u+c)\frac{\partial t}{\partial\alpha}\,,\quad\frac{\partial x}{\partial\beta}=u\frac{\partial t}{\partial\beta}\,. \tag{13}$$

To eliminate α_x and β_x from (12) introduce

$$(\alpha_x,\beta_x)=(-t_\beta,t_\alpha)/(t_\alpha x_\beta-x_\alpha t_\beta)\,,$$

so that

$$(2t_\alpha r_\beta^- - t_\beta r_\alpha^-)=(ct_\beta/\gamma)S_\alpha\,. \tag{14}$$

Also, if (10) is substituted into (14), then

$$t_\alpha u_\beta - t_\beta u_\alpha=[2/(\gamma-1)]t_\alpha c_\beta\,, \tag{15}$$

or

$$t_\alpha r_\beta^-=t_\beta u_\alpha\,. \tag{16}$$

Also, if the entropy from (1) is substituted into (10), then

$$u_\alpha+cp_\alpha/\gamma p=0\,. \tag{17}$$

Next, if we eliminate x from the set (13) we have

$$ct_{\alpha\beta}+u_\beta t_\alpha-u_\alpha t_\beta+c_\beta t_\alpha=0\,,$$

which from (15) can be put into the form

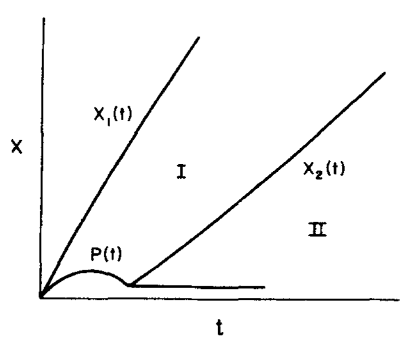

FIG. 2. Typical flow in the physical plane. $X_1(t)$ and $X_2(t)$ represent front and tail shocks, respectively, and $P(t)$ is the piston position.

$$t_{\alpha\beta}+\frac{\gamma+1}{\gamma-1}(\ln c)_\beta t_\alpha=0. \tag{18}$$

Equation (18) may be easily integrated to give

$$t(\alpha,\beta)=t(0,\beta)+\int_0^{\infty}\frac{A(\alpha')\,d\alpha'}{[c(\alpha',\beta)]^{(\gamma+1)/(\gamma-1)}}, \tag{19}$$

where the functions $t(0,\beta)$ and $A(\alpha)$ are to be determined.

The set of equations (9), (10), (15), and (19) have been derived without approximation. These form the basis of the approximate method of this paper and the exact numerical method of the subsequent paper.[1]

III. APPROXIMATE SOLUTION

Figure 1 suggests that over a considerable range of strengths, r^- may be replaced by a constant. If r^- is replaced by its value upstream, this gives inaccuracies at the piston, and to replace it by its value at the starting shock gives inaccuracies at infinity. In either case, one replaces Eq. (14) by $r^-=$const and then solves the remaining equations.

According to Mahony,[14] r^- is approximately a constant on the piston or for that matter any particular path; therefore in our notation

$$r^-=r^-(\alpha). \tag{20}$$

To obtain this approximation we can replace (16) by the condition

$$r_\beta^-\approx 0. \tag{21}$$

If (20) itself is substituted into (17), it suggests that u and, hence from (17), p are constant on β characteristics. Meyer[15] adopts a procedure based on this approach, but this leads to other difficulties.[16]

If (20) or its equivalent

$$u=r^-(\alpha)+2c/(\gamma-1) \tag{22}$$

is substituted into (10), we obtain after a simple integration,

$$\begin{aligned} c(\alpha,\beta)&=\left[F(\beta)-\int_0^\alpha r_\alpha^-(\alpha')\times\exp\left(-\frac{\gamma-1}{4\gamma}S(\alpha')\right)d\alpha'\right]\exp\left(\frac{\gamma-1}{4\gamma}S(\alpha)\right),\\ c(\alpha,\beta)&=\mathscr{F}(\alpha,\beta)\exp\left(\frac{\gamma-1}{4\gamma}S(\alpha)\right), \end{aligned} \tag{23}$$

where the integration "constant" $F(\beta)$ is to be determined. Substitution of (23) in (22) then furnished $u(\alpha,\beta)$.

One of the two conditions on the transformation (7) is fixed by

$$t(0,\beta)=\beta, \tag{24}$$

which from the boundary condition (6) yields

$$u(0,\beta)=P'(\beta). \tag{25}$$

(The piston position, which is an isentropic curve, is taken as $\alpha=0$.) If we evaluate (23) at $\alpha=0$ and substitute (22) and (25), we obtain

$$F(\beta)=\frac{\gamma-1}{2}[P'(\beta)-r^-(0)]\exp\left(-\frac{\gamma-1}{4\gamma}S_0\right), \tag{26}$$

where $S_0=S(0)$. Next, $t(\alpha,\beta)$ is determined from (19), in which $A(\alpha)$ remains to be determined.

If we denote a shock trajectory in the $\alpha-\beta$ plane by $\beta=\beta(\alpha)$, then its relationship to the shock velocity U in the physical plane is given by

$$U=\frac{dx}{dt}=\frac{x_\alpha+\beta' x_\beta}{t_\alpha+\beta' t_\beta}=\frac{(u+c)t_\alpha+\beta' u t_\beta}{t_\alpha+\beta' t_\beta}, \tag{27}$$

where (13) has been substituted and evaluation is at the shock.

At this point we are at liberty to specify the shock trajectory. This, in fact, is the choice in the following numerical investigation; and is one of the strengths of that method. In the present instance, more convenient approaches are possible. In the following we forego a general approach and treat the two regions of Fig. 2 separately.

IV. LEADING SHOCK–REGION I

We consider $X_1(t)$ and region I of Fig. 2. The shock transition is characterized by

$$(u_-,c_-,S_-)=(0,1,S_0),\quad (u_+,c_+,S_+)=(u,c,S)$$

(for simplicity, we avoid subscripts). The relative Mach number, Eq. (5), is

$$U=M_-=M(\alpha),$$

the Mach number of the shock. Initially, the gas velocity is the piston Mach number $P'(0)=M_0$ and from Eq. (2) the shock Mach number is

$$M(\alpha=0)=M^0=\frac{\gamma+1}{4}M_0+\left[\left(\frac{\gamma+1}{4}M_0\right)^2+1\right]^{1/2}. \tag{28}$$

Ultimately, M must approach unity.

A convenient second condition to fix the transformation (8) is obtained by specification of $M=M(\alpha)$ on the shock. We take

$$M(\alpha)=M^0-\alpha(M^0-1), \tag{29}$$

so that for $\alpha=0$ we obtain Eq. (28) and for $\alpha=1$ we obtain $M(1)=1$, which corresponds to $t\uparrow\infty$ in the physical plane. If we write

$$\mathscr{U}(\alpha)=u[\alpha,\beta(\alpha)],$$

then from (2)

$$\mathscr{U}(\alpha)=[2B(\alpha)/(\gamma+1)M(\alpha)];\quad B(\alpha)=M^2(\alpha)-1. \tag{30}$$

Similarly, (3) determines the entropy $S(\alpha)$

$$S(\alpha)=\ln\left[\left(1+\frac{2\gamma}{\gamma+1}B(\alpha)\right)^{1/\gamma-1}\left(\frac{(\gamma+1)M^2(\alpha)}{(\gamma-1)M^2(\alpha)+2}\right)^{-\gamma/\gamma-1}\right], \tag{31}$$

which, as mentioned earlier, is indeed a known function. Finally, from (4)

$$\mathscr{C}(\alpha)=c[\alpha,\beta(\alpha)]=\left(1+\frac{2(\gamma-1)(\gamma M^2(\alpha)+1)B(\alpha)}{(\gamma+1)^2M^2(\alpha)}\right)^{1/2}. \tag{32}$$

Equations (30 and (32) determine $r^-(\alpha)$,

$$r^-(\alpha) = \mathscr{U}(\alpha) - 2\mathscr{C}(\alpha)/(\gamma - 1). \tag{33}$$

At this point the flow field is fully determined in the α-β plane, i.e., with the substitution of (33) and (31) in (23) we have $c(\alpha, \beta)$ and from Eq. (22) we have $u(\alpha, \beta)$. The shock trajectory $\beta(\alpha)$ is determined implicitly by evaluating (23) at the shock,

$$F[\beta(\alpha)] = \mathscr{C}(\alpha) \exp\left(-\frac{\gamma-1}{4\gamma} S(\alpha)\right) + \int_0^\alpha r_\alpha^-(\alpha') \exp\left(-\frac{\gamma-1}{4\gamma} S(\alpha')\right) d\alpha'. \tag{34}$$

Finally, it remains for us to determine $A(\alpha)$ which occurs in Eq. (19). This is determined from Eq. (27), which, after some rearrangement, may be written as

$$t_\alpha + b(\alpha) t_\beta = 0, \tag{35}$$

where

$$b(\alpha) = \beta'(\alpha) \frac{\mathscr{U}(\alpha) - M(\alpha)}{\mathscr{U}(\alpha) - M(\alpha) + \mathscr{C}(\alpha)} \tag{36}$$

is known. If we substitute (23) into (19) and the result into (35), we obtain

$$0 = 1 + \frac{A(\alpha)\exp\left(-\frac{\gamma+1}{4\gamma} S(\alpha)\right)}{b(\alpha)\mathscr{F}[\alpha, \beta(\alpha)]} - \frac{\gamma+1}{\gamma-1} F_\beta[\beta(\alpha)] \times \int_0^\alpha \frac{A(\alpha')\exp\left(-\frac{\gamma+1}{4\gamma} S(\alpha')\right) d\alpha'}{\{\mathscr{F}[\alpha', \beta(\alpha)]\}^{2\gamma/\gamma-1}}. \tag{37}$$

Although this integral equation is somewhat unwieldy, it is trivially solved by numerical means.

The remaining portion of the transformation to the physical plane is $x(\alpha, \beta)$, which from Eq. (15) is

$$x(\alpha, \beta) = P(\beta) + \int_0^\alpha [u(\alpha', \beta) + c(\alpha', \beta)] \times \frac{A(\alpha')}{[c(\alpha', \beta)]^{(\gamma+1)/(\gamma-1)}} d\alpha'. \tag{38}$$

Thus, with the exception of the trivial integral equation (37), the entire solution has been reduced to quadratures.

V. TRAILING SHOCK—REGION II

The trailing shock is generated when the piston is brought to rest. In this case, $P'(\beta) = 0$, so that $F(\beta)$ is a constant and from that $c = c(\alpha)$ and $u = u(\alpha)$ as well as $S = S(\alpha)$, i.e., all quantities are constant on particle paths. In view of the fact that $u = 0$ at $\alpha = 0$ and $\alpha = 1$, it is tempting to take $u \approx 0$ and in fact, this is easily seen to follow from Eq. (15). On taking $u \approx 0$, Eq. (10) yields

$$\frac{\partial c}{\partial c} = \frac{\gamma - 1}{2\gamma} c \frac{\partial S}{\partial \alpha}, \tag{39}$$

which in turn gives

$$c_2 = c_2(\alpha) = \exp[(\gamma - 1) S_2(\alpha)/2\gamma], \tag{40}$$

since $c_2 = 1$ when $S_2 = 0$. Note that we have introduced the

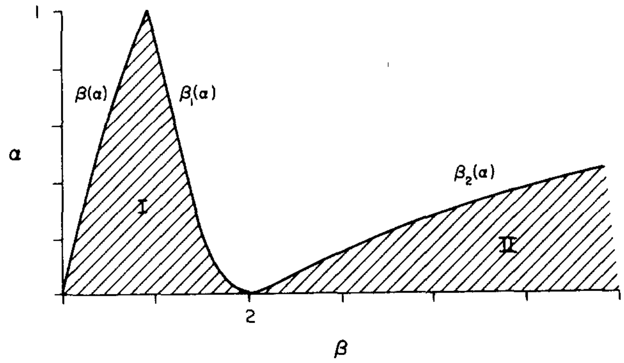

FIG. 3. Typical flow in the (α, β) plane. Regions I and II of Fig. 2 map into disjoint regions. $\beta(\alpha)$ is the map of $X_1(t)$ while $X_2(t)$ maps into $\beta_1(\alpha)$ and $\beta_2(\alpha)$. (This figure is based on the $M_0 = 1$ case discussed in Sec. VI.)

subscript 2 to indicate quantities evaluated in region II. It should be further noted that Eq. (40) is the condition that the pressure be constant, as it should be when $u = 0$. The transformation equations (15) become

$$x_\alpha = c t_\alpha, \quad x_\beta = 0.$$

Hence, t satisfies the wave equation

$$t_{\alpha\beta} = 0,$$

which with the boundary condition (24) yields

$$t_2 = \beta + \mathscr{A}(\alpha) \tag{41}$$

with $\mathscr{A}(\alpha)$ to be determined.

To complete the solution we first recognize that the mapping of regions I and II to the (α, β) plane does not leave these regions contiguous. The trailing shock maps into two different trajectories in the (α, β) plane, one each for the upstream and downstream sides of the shock. The situation is sketched in Fig. 3, where we have designated the upstream and downstream shocks by $\beta_1(\alpha)$ and $\beta_2(\alpha)$.

If we attach subscripts 1 to quantities calculated in region I (see Sec. IV), then Eq. (2) and the condition that $u_2 = 0$ yield

$$\frac{u_1}{c_1} = -\frac{2}{\gamma+1}\left(\frac{M_1^2 - 1}{M_1}\right), \tag{42}$$

where from (5)

$$M_1 = \frac{U_1 - u_1}{c_1} = \frac{(dx/dt - u_1)}{c_1}, \tag{43}$$

and $U_1 = dx/dt$ refers to the trailing shock trajectory in the physical plane. We observe that from (27) this is given by

$$\frac{dx}{dt} = \frac{(u_1 + c_1) t_{1\alpha} + \beta_1' u_1 t_{1\beta}}{t_{1\alpha} + \beta_1' t_{1\beta}} \Bigg|_{[\alpha, \beta_1(\alpha)]} \tag{44}$$

in region I. If, in this expression, the subscript 1 is replaced by 2, we obtain the representation in terms of the solution in region II.

If we solve Eq. (42) for M_1, then

$$M_1 = M_1(\alpha) = -\frac{\gamma+1}{4} \frac{u_1}{c_1} + \left[\left(\frac{\gamma+1}{4} \frac{u_1}{c_1}\right)^2 + 1\right]^{1/2}, \tag{45}$$

where it is understood that u_1 and c_1 are evaluated at

$\beta_1(\alpha)$. If (45) and (43) are substituted in (44), we find

$$\beta_1'(\alpha) = -\frac{t_{1\alpha}}{t_{1\beta}}\left(\frac{\frac{\gamma+1}{4}u_1 - \left[\left(\frac{\gamma+1}{4}u_1\right)^2 + c_1^2\right]^{1/2} + c_1}{\frac{\gamma+1}{4}u_1 - \left[\left(\frac{\gamma+1}{4}u_1\right)^2 + c_1^2\right]^{1/2}}\right), \quad (46)$$

where $t_{1\alpha}$, $t_{2\beta}$, u_1, and c_1 are all evaluated at $\beta = \beta_1(\alpha)$. This is a first-order ordinary differential equation in β_1 and is easily integrated by numerical means. With the solution (46) for $\beta_1(\alpha)$, the shock trajectory in the physical plane is also obtained by substitution of $\beta = \beta_1(\alpha)$ into (19) and (38).

The solution in region II now follows. Substitution of $\beta_1(\alpha)$ in (45) yields $M_1(\alpha)$, and substitution of this into (3) gives

$$S_2(\alpha) = S_1(\alpha) + \frac{1}{\gamma-1}\ln\frac{\left(1 + \frac{2\gamma}{\gamma+1}[M_1^2(\alpha) - 1]\right)}{\left(\frac{(\gamma+1)M_1^2(\alpha)}{(\gamma-1)M_1(\alpha)+2}\right)^{\gamma}}. \quad (47)$$

Finally, the substitution of (47) into (40) gives $c_2(\alpha)$. This completes the solution since it is unnecessary, in this instance, to determine the transformation, and, in particular, $\mathscr{A}(\alpha)$ in (41). To see this, we recall that S and c which have been computed in region II are constant on particle paths $x = \text{const}$.

VI. EXPLICIT EXAMPLES—COMPARISON WITH EXACT RESULTS

We illustrate these methods for the case of a parabolic piston motion

$$P(t) = M_0 t(1 - t/2), \quad (48)$$

where, we recall, M_0 is the Mach number of the piston at $t = 0$. If (48) is substituted into (26), then

$$F(\beta) = \frac{\gamma-1}{2}[M_0(1-\beta) - r^-(0)]\exp\left(-\frac{\gamma-1}{4\gamma}S_0\right). \quad (49)$$

Also from (34) the shock trajectory in the (α, β) plane is explicitly given by

$$\beta = 1 - \frac{1}{M_0}\left\{r^-(0) + \frac{2}{\gamma-1}\left[\mathscr{C}(\alpha)\exp\left(-\frac{\gamma-1}{4\gamma}S(\alpha)\right) + \int^{\alpha} r^-(\alpha')\exp\left(-\frac{\gamma-1}{4\gamma}S(\alpha')\right)d\alpha'\right]\exp\left(\frac{\gamma-1}{4\gamma}S_0\right)\right\}, \quad (50)$$

where the velocity and entropy on the shock, $\mathscr{C}(\alpha)$ and $S(\alpha)$, are given by (30) and (32). The additional simplifications afforded by (48) are not significant and we do not deem it necessary to repeat the remaining expressions found in Secs. IV and V with (49) and (50) inserted.

The details of the flow field are given parametrically in α and β. For each value of (α, β), we obtain a point in the physical plane from substitution in (19) and (38). The values of u and c at such points are given by substitution in (49) and then substitution into (22) and (23), while the entropy is obtained from (31).

The leading shock is, of course, Eq. (50) and the trailing shock is obtained by integrating Eq. (46). Figure 3, in fact, contains the precise graphs of these two shocks when $M_0 = 1$. It also contains $\beta_2(\alpha)$ which may also be calculated.

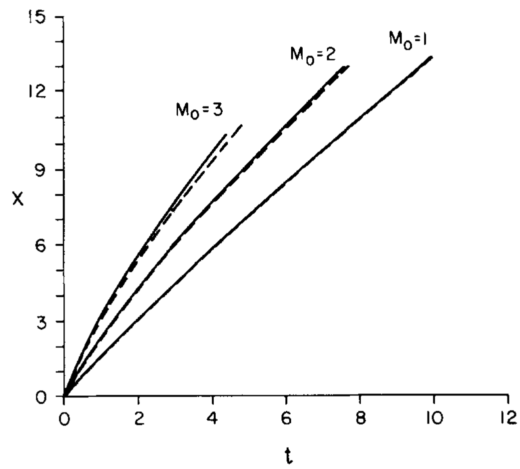

FIG. 4. Leading shocks in the physical plane for the cases $M_0 = 1, 2, 3$ as indicated. Solid lines correspond to results from the approximate theory. Dashed line corresponds to results from exact calculation.

Leading shock trajectories in the physical plane are plotted in Fig. 4 for the three cases indicated. Comparison with exact results (dashed curves) found in Ref. 1 is also given. As can be seen, the agreement is good. The initial shock strengths $[(M^0)^2 - 1]$ are, respectively, 2.12, 6.63, and 13.89, or in terms of initial pressure ratio 2.47, 7.73, and 16.21, so that we are dealing with relatively strong shock waves.

Graphical comparison between approximate and exact flow quantities is not possible since they are virtually indistinguishable. In the worst case, at the piston, the exact and approximate results differ by less than 1%. The full flow field is presented in the following paper[1] and further comparison is given there.

ACKNOWLEDGMENTS

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