Wave propagation on the von Karman trail

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The presence of wave propagation on vortex trails has been pointed out by Tritton [J. Fluid Mech. **6**, 547 (1959)] who measured their speeds in the wake of a cylinder at moderate Reynolds numbers. It is shown here that the von Karman model of the vortex trail leads to such disturbance waves and, moreover, that they can be of growing amplitude. The theoretical values of the wave speeds are found to lie within the experimental error bounds.

Tritton[1] has reported the appearance of irregularities in the vortex trail behind a circular cylinder at moderate Reynolds number. This was confirmed in later experiments[2,3] and more recently Sreenivasan[4] has shown the irregularities to contain a rich substructure. From simultaneous recordings taken at two downstream locations, Tritton in particular demonstrated the presence of a signal traveling along the vortex trail in the upstream direction with a speed

$$U_p \approx 0.3\ U_0 \tag{1}$$

relative to the vortex trail. In this discussion U_0 represents the upstream velocity, taken to be positive. It should be noted that the street moves upstream relative to the fluid and Tritton[1] gives

$$U_s \approx 0.8\ U_0 \tag{2}$$

as a nominal value for the street velocity. Tritton reports considerable variation in both (1) and (2).

The purpose of this Letter is to demonstrate that the phenomenon of wave propagation can be accounted for within the framework of the von Karman model of the vortex trail.[5–8] Moreover, as will be seen, the values of propagation speed as found from the theory show a reasonable agreement with those of Tritton given above.

We recall that the equilibrium von Karman vortex trail is a double row of vortices of opposite circulation located at

$$Z_m^{\pm} = (m \mp \tfrac{1}{4})l + U_s t \pm ih/2, \tag{3}$$

where

$$U_s = (\kappa/2l)\tanh \pi h/l \tag{4}$$

is the street velocity and κ the circulation. If we denote perturbations from equilibrium by

$$z_m^{\pm} = x_m^{\pm} + iy_m^{\pm}, \tag{5}$$

then these are governed by

$$\frac{d}{dt}\bar{z}_m^{\pm} = \mp \frac{i\kappa}{2\pi}\sum_k{}' \frac{z_m^{\pm} - z_k^{\pm}}{(m-k)^2 l^2} \pm \frac{i\kappa}{2\pi}\sum_k \frac{z_m^{\pm} - z_k^{\mp}}{[(m-k\mp\frac{1}{2})l \pm ih]^2}, \tag{6}$$

where the overbar denotes complex conjugation. To solve (6), it is convenient to introduce the generating functions

$$(X^+,Y^+) = \sum_m (x_m^+, y_m^+)\exp(im\phi),$$
$$(X^-,Y^-) = \sum_m (x_m^-, y_m^-)\exp\left[i\left(m+\frac{1}{2}\right)\phi\right] \tag{7}$$

and the transformation

$$\alpha = \frac{X^+ + X^-}{2},\quad \beta = \frac{Y^+ - Y^-}{2},$$
$$\mu = \frac{X^+ - X^-}{2},\quad \nu = \frac{Y^+ + Y^-}{2}. \tag{8}$$

It then follows that

$$\frac{d}{d\tau}\begin{bmatrix}\alpha & \bar{\nu}\\ \beta & \bar{\mu}\end{bmatrix} = \begin{bmatrix} ib & a-c \\ a+c & ib\end{bmatrix}\begin{bmatrix}\alpha & \bar{\nu}\\ \beta & \bar{\mu}\end{bmatrix}, \tag{9}$$

where time has been normalized by

$$\tau = \kappa t/2\pi l^2, \tag{10}$$

and a, b, c are functions of ϕ and can be found in Lamb.[9]

In this notation one may show that the solution to (9) is given by

$$\begin{bmatrix}\alpha & \bar{\nu}\\ \beta & \bar{\mu}\end{bmatrix} = \exp(ib\tau)\begin{bmatrix}\cos K\tau & (a-c)\sin K\tau \\ [(a+c)/K]\sin K\tau & \cos K\tau\end{bmatrix}\begin{bmatrix}\alpha_0 & \bar{\nu}_0\\ \beta_0 & \bar{\mu}_0\end{bmatrix}, \tag{11}$$

where zero subscripts indicate initial data and $K = \sqrt{c^2 - a^2}$. Thus the solution can be expressed in terms of the four frequencies

$$\pm\omega^{\pm}(\phi) = \pm(b \pm K). \tag{12}$$

Plots of $\omega^{\pm}$ are given in Ref. 6 and we remark that group speeds are given by the slopes of these curves. To complete the solution we need only invert transforms (7), e.g.,

$$x_m^+(\tau) = \frac{1}{2\pi}\int_0^{2\pi} X^+(\tau,\phi)\exp(-im\phi)\,d\phi. \tag{13}$$

For periodic initial data, the generating functions are proportional to delta functions. On the other hand, for signals of finite extent, the generating functions are smooth in the wavenumber ϕ.

It follows from the functional forms of a, b, and c that $K = \sqrt{c^2 - a^2} = O(\phi)$ and $a - c = O(1)$ as $\phi \to 0$. From this and the form of (11), it is seen that the most significant term of the solution (13) has the form

$$I = \int_0^{\pi} I_0(\phi)\,\frac{a-c}{K}\exp(ib\tau - im\phi)\sin(K\tau)\,d\phi, \tag{14}$$

where $I_0(\phi)$ is representative of the initial data. [Symmetry conditions allow us to take the interval $(0, \pi)$ instead of $(0, 2\pi)$.] One can argue that $I_0(\phi = 0) \neq 0$ for the irregular-

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ities reported by Tritton. Therefore, the largest contribution to (14) can be expected to come from the neighborhood of the origin. Thus in order to capture the main contribution it is sufficient to consider the integral

$$J = 2i \int_0^{\pi} \frac{\sin K\tau}{\phi} \exp(ib\tau - im\phi) d\phi$$
$$= \int_0^{\pi} \frac{[\exp(i\omega^+\tau) - \exp(i\omega^-\tau)]}{\phi} \exp(-im\phi) d\phi. \qquad (15)$$

To evaluate (15) we consider the limit $\tau \to \infty$, in order to use Kelvin's stationary phase formula.[10] If the stationary point ϕ_0 is not the origin, the standard procedure considers each integral in the second form of (15) and yields a wave traveling with the group speed $(d\omega^{\pm}/d\phi)(\phi_0)$ and decaying as $O(\tau^{-1/2})$ for each. However, if the stationary point coincides with a singularity, a larger signal can be expected. With this in mind, we consider (15) for the case when the group speed derives from a stationary point at the origin. In this case each of the exponential integrals of (15) is divergent and the standard method does not apply. To treat this case we consider

$$J_\epsilon = \int_0^{\pi} d\phi \frac{\exp(i\omega^+\tau - im\phi)}{\phi^{1-\epsilon}} - \int_0^{\pi} d\phi \frac{\exp(i\omega^-\tau - im\phi)}{\phi^{1-\epsilon}}, \qquad (16)$$

under the limit $\epsilon \to 0$. By expanding $\omega^{\pm}$ in powers of ϕ we can write each of the above integrals in the following way:

$$J_\epsilon^{\pm} = \int_0 \frac{d\phi \exp(-im\phi)}{\phi^{1-\epsilon}}$$
$$\times \exp[i(b_1\phi + b_2\phi^2)\tau \pm i(K_1\phi + K_2\phi^2)\tau], \qquad (17)$$

where the coefficients are

$$b_1 = \frac{\pi}{\sqrt{2}} + \frac{h\pi^2}{2l}, \quad b_2 = \pi \frac{h}{l},$$
$$K_1 = \left[\pi^2\left(\frac{\pi^2 h^2}{4l^2} + \frac{\pi h}{l\sqrt{2}} - \frac{1}{2}\right)\right]^{1/2}, \qquad (18)$$
$$K_2 = -\left(\frac{\pi^2 h^2/2l^2 + \pi h/l\sqrt{2} - \frac{1}{2}}{\pi^2 h^2/4l^2 + \pi h/l\sqrt{2} - \frac{1}{2}}\right)\frac{K_1}{\pi}.$$

The stationary point at the origin corresponds to two values of the group velocity namely,

$$b_1 \pm K_1 = m/\tau. \qquad (19)$$

Since the analysis for both cases are similar, we will only discuss one of them, viz., $b_1 + K_1 = m/\tau$. In this case application of a standard procedure[10] yields

$$J_\epsilon^+ \approx \int_0 \frac{d\phi}{\phi^{1-\epsilon}} \exp[i\tau(b_2 + K_2)\phi^2]$$
$$\approx \frac{\exp(i\pi\epsilon/4)}{\epsilon} [\tau(b_2 + K_2)]^{-\epsilon/2}, \qquad (20a)$$

$$J_\epsilon^- \approx \int_0 \frac{d\phi}{\phi^{1-\epsilon}} \exp(-2i\tau K_1\phi) \approx \frac{\exp(i\pi\epsilon/2)}{\epsilon}(2K_1\tau)^{-\epsilon}. \qquad (20b)$$

Under the limit being considered we obtain

$$J = \lim_{\epsilon \to 0}(J_\epsilon^+ - J_\epsilon^-)$$
$$\approx \lim_{\epsilon \to 0} \frac{1}{\epsilon}\{\exp[-(\epsilon/2)(\ln\tau + \ln C_1)]$$
$$-\exp[-\epsilon(\ln\tau + \ln C_2)]\}, \qquad (21)$$

where the constants C_1, C_2 are derived from the coefficients in (20a) and (20b). Expansion of the exponentials yields

$$I = O(\ln\tau). \qquad (22)$$

The theoretical wave speeds therefore correspond to the slopes of the dispersion curve at $\phi = 0$ and are given by

$$\frac{d\omega^{\pm}(0)}{d\phi} = b'(0) \pm K'(0) = 5.3,\ 1.9. \qquad (23)$$

The units are vortex spaces l, per unit time, τ as given by (10). Therefore, in dimensional units, the speeds are

$$U_p = \frac{5.3\kappa}{2\pi l},\quad \frac{1.9\kappa}{2\pi l}. \qquad (24)$$

To evaluate κ/l we appeal to the relation that gives the street velocity relative to the cylinder

$$U_s = U_0 - \kappa/(2^{3/2}l) \approx 0.8U_0, \qquad (25)$$

where we have substituted Tritton's value (2). The two wave speeds therefore become

$$U_p = 0.47U_0,\ 0.17\ U_0, \qquad (26)$$

which is to be compared with Tritton's value (1). In view of the loose connection between theory and experiment, as well as the wide error bounds of the experiment, this comparison must be regarded as encouraging. Further, the result that there is a growing front may very well be significant in the downstream development of the wake.

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