Reply to "Observations regarding 'Coherence and chaos in a model of turbulent boundary layer' by X. Zhou and L. Sirovich [Phys. Fluids A 4, 2855 (1992)]"

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(Received 19 May 1993; accepted 17 November 1993)

In view of the "observations" of the Cornell group, reassessment of their and our models for wall-bounded turbulence has been made. Wide ranging evidence is presented for the existence and key role of propagating modes (streamwise dependent modes), absent in the original Cornell model but present in some of their later models. Evidence that the heteroclinic orbit (the *bursting mechanisms*) found in the original Cornell model is most likely an artifact of their Galerkin projection is presented. A thorough discussion detailing the physical and mathematical soundness, as well as the universality, of our models is presented.

I. INTRODUCTION

The paper by Zhou and Sirovich[1] (henceforth ZS), which is the basis of the "observations" by Berkooz *et al.*[2] (henceforth, Betal), is based on our view of how the turbulent boundary layer (TBL) should be modeled. This view differs substantially in regard to mathematical structure, physical content and background from that of Betal. We reply to Betal by considering each of these issues

II. BACKGROUND

In a paper that appeared in 1987,[3] it was shown that the use of the empirical eigenfunctions coupled with a Galerkin projection produced a remarkably faithful small dynamical representation of the Ginzburg–Landau partial differential equation. Later in the same year a thorough program for the similar treatment of the Navier–Stokes (NS) equations in various geometries, including channel flow, was presented.[4] Independently in the following year the Cornell group's model[5] appeared as did the Chambers *et al.* treatment of the Burgers equation.[6] A number of treatments using a similar formalism then followed.[7–10] ZS, which appeared last year attempted to remedy shortcomings of the Cornell model, as well as subsequent extensions,[11–13] and in particular to introduce more physics into the modeling.

III. ASPECTS OF THE CORNELL MODEL

The Cornell model[5] results from the projection of the NS equations onto a space of five complex (or ten real) modes in which streamwise dependence is eliminated and which is restricted to 40 wall units ($0<y^+\leqslant 40$). The latter restriction is a consequence of the limited data available from Herzog's experiments[14] for the determination of the empirical eigenfunctions. In subsequent models[11–13] streamwise variation was included.

In ignoring streamwise dependence, $\partial_x=0$, in the original model the Cornell group confer on their channel an infinite correlation in the streamwise direction. Since the resulting flow is two dimensional, it, therefore, also lacks the vortex stretching mechanism of true turbulence. These two related shortcomings and the mathematical ill poseness of their formulation cast some doubt on their treatment.

A. Moffatt's critique

The full impact of the unphysical properties implicit in the Cornell model was explicated by Moffatt[15] who demonstrated that $\partial_x=0$ forces the decoupling of cross-sectional velocities from the streamwise velocities in the exact NS equations. Moffatt then rigorously demonstrates that this leads to temporally decaying fluctuations and hence to the approach to Poiseuille flow.

The Cornell model does not inherit this property of decay (and the tendency to Poiseuille flow) from the NS equations. The source of this discrepancy is general and lies in the observation that a Galerkin procedure does not necessarily confer on its subspace of projection, properties of the full system. For example, dynamical systems generated by a Galerkin procedure do not in general respect properties such as conservation of energy, momentum, vorticity and so forth. In this vein as Moffatt forcefully points out the Cornell model[5] does not inherit the decay properties of the exact equations for the case of streamwise independence, $\partial_x=0$.

A rebuttal of Moffatt's criticism appeared in 1991.[16] The authors imply that the vector nature of the empirical eigenfunctions is the cause of the problem. However, the empirical eigenfunctions can be obtained by a unitary transformation of an orthonormal complete basis and this certainly cannot alter a basic property of the NS equations and certainly not the rigorous result of Moffatt. Their arguments appear unconvincing to us, and the authors themselves are forced to the conclusion that in their model, energy is removed from the mean flow, in contrast to the exact result of Moffatt.

In a similar vein consider the mean flow $U(y)$ as given by

$$U(y)=\frac{1}{\nu}\int_0^y \langle uv\rangle dy+\frac{u_*^2}{\nu}\left(\frac{y-y^2}{H}\right), \tag{1}$$

where $\langle uv\rangle$ denotes integration over horizontal planes. In

1070-6631/94/6(4)/1579/4/$6.00

both the Cornell model and in ZS it is found that $U(y)$ actually shows sharp temporal variations. Only in the limit of an infinite number of modes, i.e., for the full NS equations does the ergodic assumption implicit in writing (1) become valid. [Even in a numerical integration where $O(10^5)$ modes enter one still finds a slow time variation.[17]] We mention in passing that for the earlier model of channel flow suggested in Ref. 4, the mean velocity is forced to be time independent through temporal averaging.

IV. HETEROCLINIC CYCLES: REALITY AND RELEVANCY

The obverse of the above observations is also of importance, namely, the Galerkin projection of the NS equations onto a subspace may introduce features and properties unique to the subspace and not possessed by the NS equations. Of importance in this regard is a stable heteroclinic cycle found to exist in a four-dimensional invariant subspace of the Cornell model. They also conjecture that this structure persists in going to the full ten,[11–13] dimensional space of their model and also in their subsequent extensions. Perhaps the best evidence for the role of this structure in the ten-dimensional space is shown in Fig. 7 of ZS.[1] In fact, the structure is shown there in physical space, as is its structural change with Re. However, it is also demonstrated in ZS[1] (see Fig. 10 and discussion on p. 2869) that adding streamwise varying modes, thus creating a more realistic model, eventually removes all vestiges of the heteroclinic cycle. From the point of view of the NS equations the heteroclinic orbit appears to be an artifact of the formulation, i.e., the chosen Galerkin subspace. This is further supported by the calculation discussed next.

A. Simulation

To investigate the possible role of the heteroclinic cycle we performed a large-scale simulation of channel flow at a Reynolds number of 125 based on half-channel width. (See Ref. 17 for computational details.) As initial conditions we removed all streamwise dependence from a fully turbulent simulation. The flow was then followed in time in a fully three-dimensional framework, i.e., streamwise variation was permitted to develop. Figure 1 shows a select set of modes over the course of 6000 viscous times. Only a gentle exchange of energy occurs. No bursting associated with a heteroclinic orbit is in evidence. We regard this as convincing evidence that the heteroclinic orbit is an artifact of the particular Galerkin projection. Only after numerical noise (single precision was used) stimulated propagating modes did transition to turbulence occur.

1. Numerical trouble

The case of the subspace of two complex ordinary differential equations was considered by Armbruster, Guckenheimer, and Holmes[18] who proved the existence of a stable heteroclinic cycle. This case was further considered by Krupa and Melbourne[19] who have also presented a rigorous proof using other methods.

As observed in ZS convergence to equilibrium can be lost due to numerical roundoff. This has also been observed

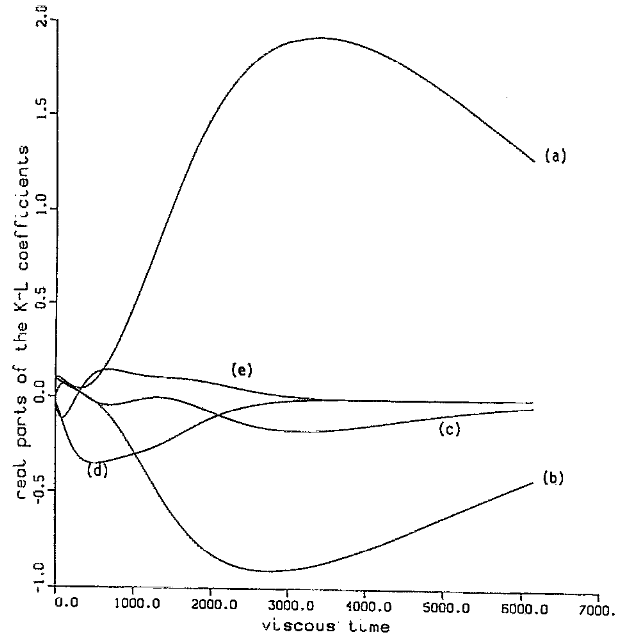

FIG. 1. A full simulation with turbulent initial conditions without streamwise dependence projected on Kahunen–Loeve eigenfunctions. Streamwise wave number $k_x=0$ and vertical quantum number $q=1$. (a) $k_z=1$; (b) $k_z=2$; (c) $k_z=3$; (d) $k_z=4$; and (e) $k_z=5$. (k_z is the spanwise wave number.)

by Silber,[20] as well as by Betal.[2] About this point there is no dispute. Krupa and Melbourne[21] have also generalized their proof to systems of more than two complex equations but this is not directly applicable to the case of the five complex equations; i.e., there is no rigorous proof that the larger system possesses a stable heteroclinic cycle. While this point needs further study, to quote from ZS: "If it were always noise that triggered a spike when a solution is near to an equilibrium point, it would push the solution randomly to either one of the two opposite directions of the unstable manifold, and only irregular spike solutions on the time traces would be seen. Obviously, this is not the case." Thus it seems likely that the case of five-complex ordinary differential equations is not the same as two complex ordinary differential equations as the Cornell group suggested.

It is our view that more evidence is necessary in order to substantiate the role of heteroclinic cycles in turbulent bursting. Our study of the system of five complex roll modes was undertaken solely for comparison with the Cornell model. The entire issue of numerical effects becomes moot with the addition of streamwise dependent modes, since the latter dominate the results of integration of the resulting system.

Recently we have presented an entirely different model based on three complex dynamical equations,[22] which appears to bear no relation to the Cornell model[5] and its extensions.[11–13] This model follows the Ruelle–Takens[23] route to chaos and the heteroclinic cycle is absent. This model appears to capture the essential dynamics of wall-bounded turbulence.[24]

V. A MATHEMATICALLY ILL-POSED FORMULATION

A flaw of the Cornell model pointed out in ZS and acknowledged in Betal is that their derivation starts with a mathematically ill-posed formulation. In brief, since the domain of the experimental data terminates at $y^+=40$, the Cornell group argue that the pressure must also be supplied at $y^+=40$. Thus they seek to solve the NS equations with one boundary condition at $y^+=40$, instead of three conditions at $y^+=40$, as is mandated from general mathematical arguments. It was demonstrated in ZS that their formulation is mathematically ill-posed. This we deem to be a noncontroversial point. The Cornell group's assertion that the resulting ODE problem is perfectly well posed only shows how the Galerkin procedure can be misused

As a result of this incorrect information an improper inhomogeneous pressure term is carried along. The statement that this term is necessary to drive the system is discussed and rejected in ZS.

VI. UNIVERSALITY

Betal suggest that their restriction to $0<y^+<40$ confers a universality on their results not present in ZS, since we use a formulation based on a channel flow. It is our contention that the reverse is true, viz., that our approach has a generality that is absent in their models. We now amplify on this. Based on the belief that only a neighborhood of the wall is of importance one can postulate a weight function $\omega(y)$ and on that basis construct a suitable orthonormal set. Our choice of

$$\omega(y)=\begin{cases}1, & 0<y^+<40,\\ 0, & y^+>40,\end{cases}$$

was made solely to facilitate comparison with the Cornell group's work. Our eigenfunctions would be identical in the selected wall layer, to those of the Cornell group's if they had employed the data of Kim, Moin, and Moser[24] used by us. The resulting system $\{\phi^n\}$, in our notation, is then defined in the full domain. (Because of limited data, the Cornell eigenfunctions are only known for $y^+<40$ and this lies at the heart of some of their problems.) To give perspective to this transformation, in ZS we have explained the quite simple and straightforward manner in which the weighted empirical eigenfunctions are related to their unweighted counterparts by a nonsingular linear transformation.

The TBL contains two important fiducial locations. One is the locus of maximal turbulence production, $y^+\approx 14$. The second is the locus of maximal Reynolds stress. This last location is at $y^+\sim\sqrt{2.5\ \mathrm{Re}}$ (Ref. 21), a fact that is also well documented experimentally.[25,26] As Jordinson[27] has shown for transitional flow, peak Reynolds stress occurs in the critical layer. There is ample evidence from our own work as well as others[25,26] that the location of maximal stress plays an analogous central role for the TBL. As Sreenivasan[25] has pointed out $y^+<40$ is roughly 10^{-3} of the total TBL thickness at $\mathrm{Re}\approx 10^6$. One may reasonably feel incredulous that in this case the innermost 1/10% can drive the entire TBL.

The available evidence is that the important turning points of the eigenfunctions lie in the neighborhood of $d/dy^+\langle uv\rangle=0$ $(d^2U/dy^{+2}\approx 0)$. The restriction to $y^+<40$ or any fixed domain excludes this for $\mathrm{Re}_\tau\gg 1$. The Cornell model loses universality precisely because $y^+<40$ does not include this turning point as $R_\tau\uparrow\infty$. On the other hand, the log layer, which contains this locus, is always present in channel flow.

VII. PROPAGATING AND NONPROPAGATING MODES

In ZS the essential role played by propagating modes in wall-bounded turbulence is central to the creation of models of wall turbulence. Since doubt has been cast on their importance, we briefly summarize the evidence for their existence and importance. In particular it should be noted that (a) the evidence for these modes and their role come from full-scale simulations, not model calculations; (b) the propagating modes have a group speed equal to the convection speed at the location of maximal Reynolds stress. (This is contrary to the statement by Betal that they move "at local convection speeds.") In addition, it has been proven that "spanwise propagating modes' do not exist.[28,29] (c) In the ZS framework the propagating modes are plane waves, this is a consequence of the channel geometry. Propagation occurs in general as a consequence of translation invariance in the streamwise direction, and appear when this occurs. (d) The physical time scale of bursting comes from the propagating modes and not the roll modes. This has been verified in full-scale simulations and comparison made with experiment.[30] (e) References 28 and 29 should be consulted for the arguments that show propagating modes trigger bursts. (d) In the recent model extensions by Sanghi and Aubry[13] propagation was found. Support for the presence of propagating structures in the wall region is wide ranging. Abundant experimental evidence is available[31–38] and a recent numerical investigation by Kim and Hussein[39] also reveals the presence of propagating structures.

Recently, Schmid and Henningson[40] made a careful study of the transition to turbulence. They demonstrate that the key element in the transition is the interaction of roll and propagating modes. Unless the roll modes give up energy to the propagating modes, transition does not take place.

VIII. CONCLUSION

An ever increasing body of evidence indicates that propagating modes are essential for the description of near-wall turbulence. In neglecting such modes the original Cornell model fails to provide a satisfactory framework for the description of such turbulence; however, subsequent models fix this defect. Thus we feel that the heteroclinic orbit in their models, which is at the heart of their claim for producing bursting, is probably an artifact of their formulation. It is also our belief that the Cornell restriction to $0<y^+<40$ and the lack of focus on the role of propagating modes (except in Ref. 13) make their model less realistic than the ZS models.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the use of the Pittsburgh Supercomputing Center. We express grateful appreciation to Bruce Knight and Chris Jones for valuable discussion and suggestions. We also thank Robert Handler for helping with the simulation. This work was supported by the National Science Foundation, NSF IRI-911-645.

[1]X. Zhou and L. Sirovich, "Coherence and chaos in a model of turbulent boundary layer," Phys. Fluids A **4**, 2855 (1992).
[2]G. Berkooz, P. Holmes, J. L. Lumley, N. Aubry, and E. Stone, Phys. Fluids **6**, 1574 (1994).
[3]L. Sirovich and J. D. Rodriguez, "Coherent structures and chaos: A model problem," Phys. Lett. A **120**, 211 (1987).
[4]L. Sirovich, "Turbulence and the dynamics of coherent structures; part I: Coherent structures; part II: Symmetries and transformations; part III: Dynamics and scaling," Q. Appl. Math. **XLV**, 583 (1987).
[5]N. Aubry, P. Holmes, J. L. Lumley, and E. Stone, "The dynamics of coherent structures in the wall region of turbulent boundary layer," J. Fluid Mech. **192**, 115 (1988).
[6]D. H. Chambers, R. J. Adrian, P. Moin, D. S. Stewart, and H. J. Sung, "Karhunen–Loève expansion of Burgers' model of turbulence," Phys. Fluids **31**, 2573 (1988).
[7]A. E. Deane and C. Mavriplis, "Low-dimensional description of the dynamics in separated flow past thick airfoils," AIAA 22nd Fluid Dynamics, Plasma Dynamics and Lasers Conference, June 1991 (AIAA, Washington, DC, 1992).
[8]A. E. Deane, I. G. Kevrekidis, G. E. Karniadakis, and S. A. Orszag, "Low-dimensional models for complex geometry flows. Application to channels and circular cylinders," Phys. Fluids A **3**, 2337 (1991).
[9]C. C. Chen and H. C. Chang, "Accelerated distubances damping of an unknown distributed system by nonlinear dominant-mode feedback," AIChE J. **38**, 1461 (1992).
[10]D. Rempfer, *Low-Dimensional Models of a Flat Plate Boundary Layer in Near-Wall Turbulent Flows*, edited by R. M. C. So, C. G. Speziale, and B. E. Launder (Elsevier, New York, 1993).
[11]N. Aubry and S. Sanghi, "Bifurcations and bursting of streaks in the turbulent wall layer," in *Turbulence and Coherent Structures*, edited by O. Metais and M. Lesieur (Kluwer, Dordrecht, 1991), pp. 227–251.
[12]N. Aubry and S. Sanghi, "Streamwise and spanwise dynamics in the turbulent wall layer," in *Chaotic Dynamics in Fluid Mechanics*, edited by K. N. and U. Ghia (American Society of Mechanical Engineering, New York, 1989), pp. 110–118.
[13]S. Sanghi and N. Aubry, "Mode interaction model for near wall turbulence," J. Fluid Mech. **247**, 455 (1993).
[14]S. Herzog, "The large scale structure in the near-wall region of turbulent pipe flow," Ph.D. thesis, Cornell University, 1986.
[15]H. K. Moffatt, "Fixed points of turbulent dynamical systems and suppression of nonlinearity," in *Whither Turbulence? Turbulence at the Crossroads*, Lecture Notes in Applied Physics, edited by J. L. Lumley (Springer-Verlag, Berlin, 1990), Vol. 357, pp. 250–257.
[16]G. Berkooz, P. Holmes, and J. L. Lumley, "Intermittent dynamics in simple models of the turbulent wall layer," J. Fluid Mech. **230**, 75 (1991).
[17]R. A. Handler, E. W. Hendricks, and R. I. Leighton, "Low Reynolds number calculation of turbulent channel flow: A general discussion," NRL Memorandum Report No. 6410, 1989.
[18]D. Armbruster, J. Guckenheimer, and P. Holmes "Heteroclinic cycles and modulated traveling waves in systems with $O(2)$ symmetry," Physica D **29**, 257 (1988).
[19]M. Krupa and I. Melbourne, "Asymptotic stability of heteroclinic cycles in systems with symmetry," Research Report, Department of Mathematics, University of Houston, 18 December 1991.
[20]M. Silber, H. Riecke, and L. Kramer, "Symmetry-break Höpf bifurcation in anisotropic systems," preprint (1992).
[21]M. Krupa and I. Melbourne, "Asymptotic stability of heteroclinic cycles in systems with symmetry," Research Report, Department of Mathematics, University of Houston, 23 December 1992.
[22]L. Sirovich and X. Zhou, "A dynamical model of wall-bounded turbulence," Phys. Rev. Lett. **72**, 34 (1994).
[23]D. Ruelle and F. Takens, "On the nature of turbulence," Commun. Math. Phys. **20**, 167 (1991).
[24]J. Kim, P. Moin, and R. Moser, "Turbulence Statistics in fully developed channel flow at low Reynolds number," J. Fluid Mech. **177**, 135 (1987).
[25]K. Sreenivasan, "The turbulent boundary layer," in *Frontiers in Experimental Fluid Mechanics*, edited by M. Gad-el-Hak (Springer-Verlag, Berlin, 1989).
[26]K. Sreenivasan, "A unified view of the origin and morphology of the turbulent boundary layer structure," in *Turbulence Management and Relaminzrization*, IUTAM Symposium, 1987, edited by H. W. Liepmann and R. Narasimha (Springer-Verlag, Berlin, 1988).
[27]R. Jordinson, "The flat plate boundary layer Pt. 1. Numerical integration of the Orr–Sommerfeld equations," J. Fluid Mech. **43**, 801 (1970).
[28]L. Sirovich, K. S. Ball, and R. A. Handler, "Propagating structures in wall-bounded turbulent flows," Theor. Comput. Fluid Dyn. **2**, 307 (1991).
[29]L. Sirovich, K. S. Ball, and L. R. Keefe, " Plane waves and structure in turbulent channel flow," Phys. Fluids A **2**, 2217 (1990).
[30]K. Ball, D. G. Bogard, and C. Gan (private communication, 1991).
[31]G. Comte-Bellot and S. Corrsin, " Simple Eulerian correlation of full- and narrow-band velocity signals in grid-generated, 'isotropic' turbulence," J. Fluid Mech. **48**, 273 (1971).
[32]P. Bradshaw, " 'Inactive' motion and pressure fluctuation in turbulent boundary layer," J. Fluid Mech. **27**, 209 (1967).
[33]W. W. Willmarth and C. E. Wooldridge, "Measurements of the fluctuating pressure at the wall beneath a thick turbulent boundary layer," J. Fluid Mech. **14**, 187 (1962).
[34]A. Dinkelacker, M. Hessel, G. E. A. Meier, and G. Schewe, "Investigation of pressure fluctuations beneath a turbulent boundary layer by means of an optical method," Phys. Fluids **20**, S216 (1977).
[35]G. Schewe, "On the structure and resolution of wall-pressure fluctuations associated with turbulent boundary layer flow," J. Fluid Mech. **134**, 311 (1983).
[36]J. A. B. Wills, "On convection velocities in turbulent shear flows," J. Fluid Mech. **20**, 417 (1964).
[37]W. R. B. Morrison, K. J. Bullock, and R. E. Kronauer, "Experimental evidence of waves in the sublayer," J. Fluid Mech. **47**, 639 (1971).
[38]V. W. Goldschmidt, H. F. Young, and E. S. Ott, "Turbulent convective velocities broadband and wavenumber dependent in a plane jet," J. Fluid Mech. **105**, 327 (1981).
[39]J. Kim and F. Hussain, "Propagation velocity of perturbations in turbulent channel flow," Phys. Fluid A **5**, 695 (1993).
[40]P. Schmid and D. Henningson, "Nonlinear energy density growth during oblique transion in plane Poiseuille flow," Technical report TRITA-MEK 1993:5, Royal Institute of Technology, Stockholm, 1993.