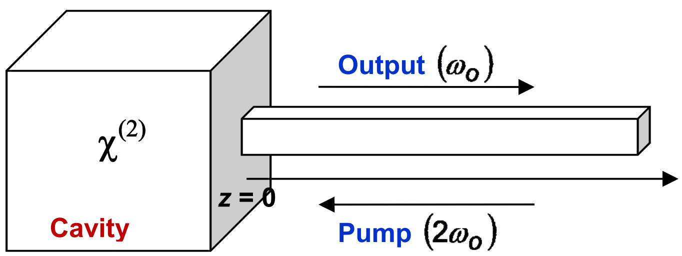
Chapter 14: Optical Parametric Oscillators

14.1 Introduction

In this Chapter we will discuss an optical parametric oscillator. A parametric oscillator is almost like a laser. The difference is that the optical gain inside the cavity comes not from a population inverted medium but from a nonlinear optical medium which possesses either the second order (χ^2) or the third order (χ^3) optical nonlinearity.

Consider an optical cavity that supports two optical modes, mode 1 and mode 2, with frequencies ω_o and $2\omega_o$, respectively, and contains a second order non-linear medium. Mode 1 can experience parametric gain from mode 2 due to stimulated down conversion, as discussed in Chapter 12. The cavity is pumped with coherent light at the frequency of mode 2. If the parametric gain for mode 1 becomes equals to the cavity loss then a large photon number population at the mode 1 frequency can build up inside the cavity in steady state. This is called parametric oscillation. In this Chapter, we will discuss a degenerate parametric oscillator.



14.2 Hamiltonian and Heisenberg Equations

The Hamiltonian, including the second order nonlinearity, is,

$$\hat{H} = \hbar\omega_o\, \hat{a}_1^{+} a_1 + 2\hbar\omega_o\, \hat{a}_2^{+}\hat{a}_2 - i\hbar\frac{\kappa}{2}\left[\hat{a}_2^{+}(\hat{a}_1)^2 - (\hat{a}_1^{+})^2\hat{a}_2\right]$$

The cavity lifetime for mode 1 photons is τ_{p1} and for mode 2 photons it is τ_{p2}. The finite cavity lifetimes are due to couplings of the cavity modes with the waveguide. The Heisenberg equations for mode 1 operators, including waveguide coupling, are,

$$\frac{d}{dt}\begin{bmatrix}\hat{a}_1(t)\\ \hat{a}_1^{+}(t)\end{bmatrix} = \begin{bmatrix} -i\omega_o - \dfrac{1}{2\tau_{p1}} & 0 \\ 0 & i\omega_o - \dfrac{1}{2\tau_{p1}}\end{bmatrix}\begin{bmatrix}\hat{a}_1(t)\\ \hat{a}_1^{+}(t)\end{bmatrix} + \kappa\begin{bmatrix}\hat{a}_1^{+}\hat{a}_2\\ \hat{a}_2^{+}\hat{a}_1\end{bmatrix} + \sqrt{\frac{1}{\tau_{p1}}}\begin{bmatrix}\hat{S}_{in}(t)e^{-i\omega_o t}\\ \hat{S}_{in}^{+}(t)e^{i\omega_o t}\end{bmatrix}$$

$$\left\langle \hat{S}_{in}^{+}(t)\hat{S}_{in}(t')\right\rangle = 0$$

$$\left\langle \hat{S}_{in}(t)\hat{S}_{in}^{+}(t')\right\rangle = \delta(t-t')$$

Here,

$$\hat{S}_{in}(t)e^{-i\omega_o t} = \sqrt{v_g}\,\hat{b}_L(z=0,t)e^{-i\omega_o t}$$
$$\hat{S}_{in}^{+}(t)e^{i\omega_o t} = \sqrt{v_g}\,\hat{b}_L^{+}(z=0,t)e^{i\omega_o t}$$

The mode 1 photons coming out of the cavity are described by the equations,

$$\hat{S}_{out}(t)e^{-i\omega_o t} = \sqrt{v_g}\,\hat{b}_R(z=0,t)e^{-i\omega_o t} = \sqrt{\frac{1}{\tau_{p1}}}\,\hat{a}_1(t) - \sqrt{v_g}\,\hat{b}_L(z=0,t)e^{-i\omega_o t} = \sqrt{\frac{1}{\tau_{p1}}}\,\hat{a}_1(t) - \hat{S}_{in}(t)e^{-i\omega_o t}$$
$$\hat{S}_{out}^{+}(t)e^{i\omega_o t} = \sqrt{v_g}\,\hat{b}_R^{+}(z=0,t)e^{i\omega_o t} = \sqrt{\frac{1}{\tau_{p1}}}\,\hat{a}_1^{+}(t) - \sqrt{v_g}\,\hat{b}_L^{+}(z=0,t)e^{i\omega_o t} = \sqrt{\frac{1}{\tau_{p1}}}\,\hat{a}_1^{+}(t) - \hat{S}_{in}^{+}(t)e^{i\omega_o t}$$

The equations for mode 2 operators are,

$$\frac{d}{dt}\begin{bmatrix}\hat{a}_2(t)\\ \hat{a}_2^{+}(t)\end{bmatrix} = \begin{bmatrix}-i2\omega_o - \dfrac{1}{2\tau_{p2}} & 0\\ 0 & i2\omega_o - \dfrac{1}{2\tau_{p2}}\end{bmatrix}\begin{bmatrix}\hat{a}_2(t)\\ \hat{a}_2^{+}(t)\end{bmatrix} - \frac{\kappa}{2}\begin{bmatrix}\hat{a}_1^{2}(t)\\ (\hat{a}_1^{+}(t))^2\end{bmatrix} + \sqrt{\frac{1}{\tau_{p2}}}\begin{bmatrix}\hat{F}_{in}(t)e^{-i2\omega_o t}\\ \hat{F}_{in}^{+}(t)e^{i2\omega_o t}\end{bmatrix}$$

Here,

$$\hat{F}_{in}(t)e^{-i2\omega_o t} = \sqrt{v_g}\,\hat{d}_L(z=0,t)e^{-i2\omega_o t}$$
$$\hat{F}_{in}^{+}(t)e^{i2\omega_o t} = \sqrt{v_g}\,\hat{d}_L^{+}(z=0,t)e^{i2\omega_o t}$$

Since the input radiation at the frequency $2\omega_o$ is continuous wave coherent state, we have,

$$\left\langle \hat{F}_{in}(t)e^{-i2\omega_o t}\right\rangle = \sqrt{r_p}\,e^{i\theta_p}\,e^{-i2\omega_o t} \qquad \left\langle \hat{F}_{in}^{+}(t)e^{i2\omega_o t}\right\rangle = \sqrt{r_p}\,e^{-i\theta_p}\,e^{i2\omega_o t}$$

where r_p (units: 1/sec) is the flux of pump photons at frequency $2\omega_o$ coming into the cavity from the waveguide, and θ_p is the phase of the pump. Note the following averages,

$$\left\langle \hat{F}_{in}(t)\hat{F}_{in}(t')\right\rangle = r_p\,e^{2i\theta_p} \qquad \left\langle \hat{F}_{in}^{+}(t)\hat{F}_{in}^{+}(t')\right\rangle = r_p\,e^{-2i\theta_p}$$
$$\left\langle \hat{F}_{in}^{+}(t)\,\hat{F}_{in}(t')\right\rangle = r_p \qquad \left\langle \hat{F}_{in}(t)\hat{F}_{in}^{+}(t')\right\rangle = r_p + \delta(t-t')$$

14.3 Semi-Classical Solution for Steady State Operation

We first find steady state solutions by replacing the operator equations of the previous Section with their average values. We assume,

$$\langle \hat{a}_1(t)\rangle = \beta_1(t)e^{-i\omega_o t}$$
$$\langle \hat{a}_2(t)\rangle = \beta_2(t)e^{-i2\omega_o t}$$

Taking the average of the operator equations derived in the previous Section we obtain,

$$\frac{d\beta_1(t)}{dt} = -\frac{1}{2\tau_{p1}}\beta_1 + \kappa\beta_1^{*}\beta_2$$
$$\frac{d\beta_2(t)}{dt} = -\frac{1}{2\tau_{p2}}\beta_2 - \frac{\kappa}{2}\beta_1^{2} + \sqrt{\frac{1}{\tau_{p2}}}\sqrt{r_p}\,e^{i\theta_p}$$

In steady state, we must have,

$$\frac{d\beta_1(t)}{dt} = \frac{d\beta_2(t)}{dt} = 0$$

Solution for Mode 1: After setting the right hand side of the second equation above to zero, one can obtain $\beta_2(t)$ in terms of $\beta_1(t)$, and then one can substitute this value in the first equation to get,

$$\frac{d\beta_1(t)}{dt} = -\frac{1}{2\tau_{p1}}\beta_1 - \tau_{p2}\,\kappa^2|\beta_1|^2\,\beta_1 + 2\kappa\sqrt{\tau_{p2}\,r_p}\,e^{i\theta_p}\;\beta_1^*$$

Now, just as we did in the case of the laser, if we write the complex number $\beta_1(t)$ as $x_1(t) + i\,x_2(t)$ and then define a vector,

$$\vec{r}(t) = x_1(t)\hat{x} + x_2(t)\hat{y}$$
$$= |\vec{r}(t)|\,[\cos(\phi)\hat{x} + r\sin(\phi)\hat{y}]$$

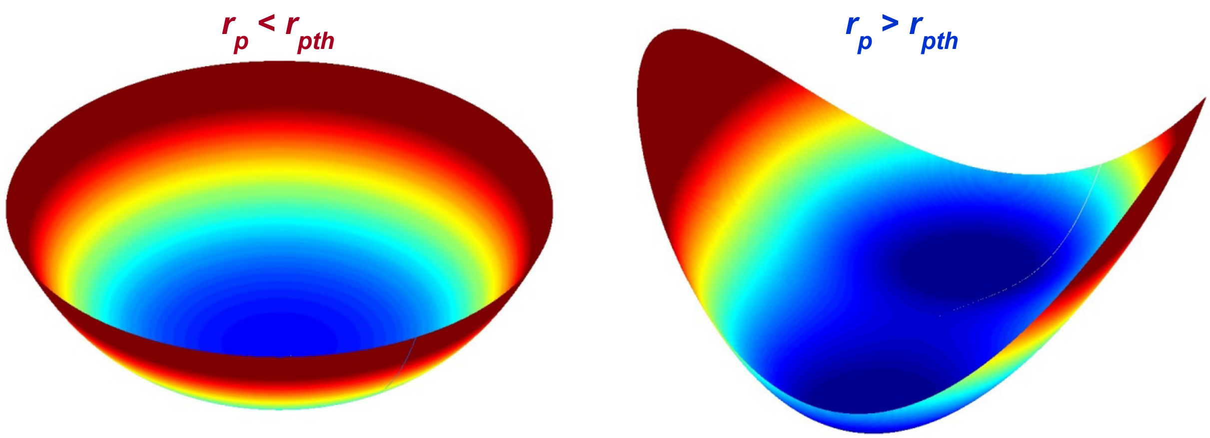
then the above equation for $\beta_1(t)$ can be written as,

$$\frac{d\,\vec{r}(t)}{dt} = -\nabla V(\vec{r}(t))$$

Here,

$$V(\vec{r}) = V(x_1, x_2) = \frac{(x_1^2 + x_2^2)}{4\tau_{p1}} + \tau_{p2}\kappa^2\,\frac{(x_1^2 + x_2^2)^2}{4} - \kappa\sqrt{\tau_{p2}\,r_p}\left[\frac{(x_1^2 - x_2^2)}{2}\cos\theta_p + x_1 x_2\sin\theta_p\right]$$

$$= \frac{r^2}{4\tau_{p1}} + \tau_{p2}\kappa^2\,\frac{r^4}{4} - \kappa\sqrt{\tau_{p2}\,r_p}\,\frac{r^2}{2}\cos(2\phi - \theta_p)$$



Since in steady state,

$$\frac{d\beta_1(t)}{dt} = 0 \quad \Rightarrow \quad \frac{d\,\vec{r}(t)}{dt} = 0$$

the steady state value of $\beta_1(t)$ corresponds to that point in the complex x_1, x_2 plane at which the potential $V(\vec{r})$ has the minimum value. The shape of the potential differs depending on the strength of the pump r_p. When,

$$r_p < \frac{1}{16\tau_{p1}^2\tau_{p2}\kappa^2}$$

the potential has a single minimum at $\vec{r} = 0$. When,

$$r_p > \frac{1}{16\tau_{p1}^2\tau_{p2}\kappa^2}$$

the potential has two separate minima, both occur at the same non-zero value of $|\vec{r}|$ but differ in the value of the angle ϕ,

$$|\vec{r}|^2 = \frac{2}{\kappa\sqrt{\tau_{p2}}}\left(\sqrt{r_p} - \sqrt{\frac{1}{16\tau_{p1}^2\tau_{p2}\kappa^2}}\right) \qquad \phi = \frac{\theta_p}{2}$$

$$|\vec{r}|^2 = \frac{2}{\kappa\sqrt{\tau_{p2}}}\left(\sqrt{r_p} - \sqrt{\frac{1}{16\tau_{p1}^2\tau_{p2}\kappa^2}}\right) \qquad \phi = \frac{\theta_p}{2} + \pi$$

The pumping rate $\left(16\tau_{p1}^2\tau_{p2}\kappa^2\right)^{-1}$ is called the threshold pumping rate r_{pth},

$$r_{pth} = \frac{1}{16\tau_{p1}^2\tau_{p2}\kappa^2}$$

When $r_p > r_{pth}$, and the potential has two minima, the steady state value of $\beta_1(t)$ can correspond to any one of these two minima. We will assume that in steady state the minimum where $\phi = \theta_p/2$ is chosen. If the value of β_1 is steady state is $\beta_{1s} = |\beta_{1s}|e^{i\phi_{1s}}$ then we can write,

$$\beta_{1s} = \begin{cases} 0 & \text{for } r_p \le r_{pth} \\ \sqrt{\frac{2}{\kappa\sqrt{\tau_{p2}}}\left(\sqrt{r_p} - \sqrt{r_{pth}}\right)}\, e^{i\frac{\theta_p}{2}} & \text{for } r_p > r_{pth} \end{cases} \qquad \left\{ \phi_{1s} = \frac{\theta_p}{2} \right.$$

The average field strength $|\beta_{1s}|$ is zero below threshold, but increases with the pumping rate above threshold.

Solution for Mode 2: In steady state, let,

$$\beta_2(t) = \beta_{2s} = |\beta_{2s}|e^{i\phi_{2s}}$$

Then in steady state the equation,

$$\frac{d\beta_2(t)}{dt} = -\frac{1}{2\tau_{p2}}\beta_2 - \frac{\kappa}{2}\beta_1^2 + \sqrt{\frac{1}{\tau_{p2}}}\sqrt{r_p}\,e^{i\theta_p}$$

gives,

$$\beta_{2s} = -\tau_{p2}\kappa\beta_{1s}^2 + 2\sqrt{\tau_{p2}\, r_p}\; e^{i\theta_p}$$

which can be written as,

$$\beta_{2s} = \begin{cases} 2\sqrt{\tau_{p2}\, r_p}\; e^{i\theta_p} & \text{for } r_p \le r_{pth} \qquad \{\phi_{2s} = \theta_p \\ \dfrac{e^{i\theta_p}}{2\tau_{p1}\kappa} & \text{for } r_p > r_{pth} \qquad \{\phi_{2s} = \theta_p \end{cases}$$

$|\beta_{2s}|$ increases with the pumping rate below threshold, but is fixed at a value independent of the pumping rate above threshold. In steady state,

14.4 Physical Interpretation of the Steady State Solution

Many features of parametric oscillation are similar to those of a laser. This is best illustrated by looking at the equation for the average photon number in mode 1 given by $|\beta_1(t)|^2$,

$$\left\langle \hat{a}_1^{+}(t)\hat{a}_1(t)\right\rangle \approx |\beta_1(t)|^2$$

We suppose,

$$\beta_1(t) = |\beta_1(t)|\, e^{i\phi_1(t)}$$

$$\beta_2(t) = |\beta_2(t)|\, e^{i\phi_2(t)}$$

From the equation,

$$\frac{d\beta_1(t)}{dt} = -\frac{1}{2\tau_{p1}}\beta_1 + \kappa\beta_1^{*}\beta_2$$

We can derive an equation for the photon number in mode 1,

$$\frac{d|\beta_1(t)|^2}{dt} = \left\{-\frac{1}{\tau_{p1}} + 2\kappa|\beta_2(t)|\cos[\phi_2(t) - 2\phi_1(t)]\right\}|\beta_1(t)|^2$$

The first term in the brackets on the right hand side represents cavity loss and the second term describes the parametric gain due to stimulated down conversion. Note that the gain depends on the number of pump photons in the cavity. The gain is maximum when $\phi_2(t) = 2\phi_1(t)$ and therefore the phase of the mode 1 field acquires this optimum value in steady state.

Below threshold, when $|\beta_{2s}| = 2\sqrt{\tau_{p2}\, r_p}$, the gain is less than the loss, and therefore $|\beta_{1s}|^2 = 0$. At threshold, the gain equals the loss,

$$2\kappa|\beta_{2s}| = \frac{1}{\tau_{p1}}$$

The above relation can be used to find the threshold pumping rate,

$$2\kappa|\beta_{2s}| = \frac{1}{\tau_{p1}}$$

$$\Rightarrow 4\kappa\sqrt{\tau_{p2}\, r_{pth}} = \frac{1}{\tau_{p1}}$$

$$\Rightarrow r_{pth} = \frac{1}{16\tau_{p1}^2\tau_{p2}\kappa^2}$$

Above threshold, just like in a laser, the gain cannot exceed the loss. Since the gain is determined by the value of $|\beta_{2s}|$ or, equivalently, the number of photons in mode 2, the number of photons in mode 2 remain fixed at their threshold value even when the pumping rate r_p is increased above the threshold pumping rate r_{pth}. Therefore, above threshold, $|\beta_{2s}| = 2\sqrt{\tau_{p2}\, r_{pth}}$.

14.5 Quantum Fluctuations and Noise

To study the quantum properties of the light coming out of an optical parametric oscillator, we expand the field operators as follows,

$$\hat{a}_1(t) = \left[|\beta_{1s}| + \Delta\hat{b}_1(t) \right] e^{-i\omega_o t + i\frac{\theta_p}{2}}$$
$$\hat{a}_2(t) = \left[|\beta_{2s}| + \Delta\hat{b}_2(t) \right] e^{-i2\omega_o t + i\theta_p}$$

where,

$$\left[\Delta\hat{b}_1(t), \Delta\hat{b}_1^{+}(t)\right] = \left[\Delta\hat{b}_2(t), \Delta\hat{b}_2^{+}(t)\right] = 1$$

14.5.1 Operation below Threshold and Squeezing

Field inside the Cavity: We first consider the case below threshold. We have,

$$r_p < r_{pth} \;\Rightarrow\; |\beta_{1s}| = 0 \qquad |\beta_{2s}| = 2\sqrt{\tau_{p2} r_p}$$

The linearized equation for the operator $\Delta\hat{b}_1(t)$ becomes,

$$\frac{d\Delta\hat{b}_1(t)}{dt} = -\frac{1}{2\tau_{p1}}\Delta\hat{b}_1(t) + \kappa|\beta_{2s}|\Delta\hat{b}_1^{+}(t) + \sqrt{\frac{1}{\tau_{p1}}}\,\hat{S}_{in}(t)\, e^{-i\frac{\theta_p}{2}}$$

Let,

$$\Delta\hat{b}_1(t) = \Delta\hat{x}_{\theta_p/2}(t) + i\,\Delta\hat{x}_{\theta_p/2+\pi/2}(t)$$

then,

$$\frac{d\Delta\hat{x}_{\theta_p/2}(t)}{dt} = -\left[\frac{1}{2\tau_{p1}} - \kappa|\beta_{2s}|\right]\Delta\hat{x}_{\theta_p/2}(t) + \sqrt{\frac{1}{\tau_{p1}}}\left[\frac{\hat{S}_{in}(t)e^{-i\frac{\theta_p}{2}} + \hat{S}_{in}^{+}(t)e^{+i\frac{\theta_p}{2}}}{2}\right]$$

$$\frac{d\Delta\hat{x}_{\theta_p/2+\pi/2}(t)}{dt} = -\left[\frac{1}{2\tau_{p1}} + \kappa|\beta_{2s}|\right]\Delta\hat{x}_{\theta_p/2+\pi/2}(t) + \sqrt{\frac{1}{\tau_{p1}}}\left[\frac{\hat{S}_{in}(t)e^{-i\frac{\theta_p}{2}} - \hat{S}_{in}^{+}(t)e^{+i\frac{\theta_p}{2}}}{2i}\right]$$

The above equations can be solved by direct integration to find the following steady state mean square quadrature fluctuations,

$$\left\langle \Delta\hat{x}^2_{\theta_p/2}(t) \right\rangle = \frac{1}{4}\frac{1}{\left(1 - \sqrt{\frac{r_p}{r_{pth}}}\right)}$$

$$\left\langle \Delta\hat{x}^2_{\theta_p/2+\pi/2}(t) \right\rangle = \frac{1}{4}\frac{1}{\left(1 + \sqrt{\frac{r_p}{r_{pth}}}\right)}$$

Much below threshold (i.e. for $r_p << r_{pth}$), the fluctuations in both quadratures equal $1/4$, as expected. As the pumping rate is increased and the threshold is approached, the fluctuations in one quadrature increase and the fluctuations in the other quadrature decrease. Since,

$$\left\langle \Delta\hat{x}^2_{\theta_p/2}(t) \right\rangle \left\langle \Delta\hat{x}^2_{\theta_p/2+\pi/2}(t) \right\rangle = \frac{1}{16}\frac{1}{\left(1 - \frac{r_p}{r_{pth}}\right)} >> \frac{1}{16}$$

the quantum state of mode 1 inside the cavity is not a squeezed state or a two-photon coherent state.

Field outside the Cavity: The field outside the cavity is not the same as the field inside the cavity. Recall that,

$$\hat{S}_{out}(t)e^{-i\omega_o t} = \sqrt{v_g}\,\hat{b}_R(z=0,t)e^{-i\omega_o t} = \sqrt{\frac{1}{\tau_{p1}}}\,\hat{a}_1(t) - \hat{S}_{in}(t)e^{-i\omega_o t}$$

$$= \sqrt{\frac{1}{\tau_{p1}}}\Delta\hat{b}_1(t)\; e^{-i\omega_o t + i\frac{\theta_p}{2}} - \hat{S}_{in}(t)e^{-i\omega_o t}$$

$$\hat{S}^{+}_{out}(t)e^{i\omega_o t} = \sqrt{v_g}\,\hat{b}^{+}_R(z=0,t)e^{i\omega_o t} = \sqrt{\frac{1}{\tau_{p1}}}\,\hat{a}^{+}_1(t) - \hat{S}^{+}_{in}(t)e^{i\omega_o t}$$

$$= \sqrt{\frac{1}{\tau_{p1}}}\Delta\hat{b}^{+}_1(t)\; e^{i\omega_o t - i\frac{\theta_p}{2}} - \hat{S}^{+}_{in}(t)e^{i\omega_o t}$$

Notice that the field outside the cavity also consists of the reflected vacuum fluctuations (last terms on the left hand sides in the above equations). The quadratures of the outside propagating field are defined as follows,

$$\hat{x}^{out}_{\theta_p/2}(z,t) = \frac{\hat{b}_R(z,t)\, e^{-i\frac{\theta_p}{2}} + \hat{b}^{+}_R(z,t)\, e^{i\frac{\theta_p}{2}}}{2}$$

$$\hat{x}^{out}_{\theta_p/2+\pi/2}(z,t) = \frac{\hat{b}_R(z,t)\, e^{-i\frac{\theta_p}{2}} - \hat{b}^{+}_R(z,t)\, e^{i\frac{\theta_p}{2}}}{2i}$$

To find the field outside the cavity, it is best to work in the frequency domain. We the equations for the field quadratures inside the cavity in the frequency domain and use the relations,

$$\hat{S}_{in}(t) = \sqrt{v_g}\,\hat{b}_L(z=0,t)$$

$$\hat{S}^{+}_{in}(t) = \sqrt{v_g}\,\hat{b}^{+}_L(z=0,t)$$

to obtain,

$$\Delta\hat{x}_{\theta_p/2}(\omega) = \frac{\sqrt{\frac{1}{\tau_{p1}}}}{-i\omega + \frac{1}{2\tau_{p1}}\left(1-\sqrt{\frac{r_p}{r_{pth}}}\right)}\sqrt{v_g}\left[\frac{\hat{b}_L(z=0,\omega)e^{-i\frac{\theta_p}{2}} + \hat{b}^{+}_L(z=0,\omega)e^{i\frac{\theta_p}{2}}}{2}\right]$$

$$\Delta\hat{x}_{\theta_p/2+\pi/2}(\omega) = \frac{\sqrt{\frac{1}{\tau_{p1}}}}{-i\omega + \frac{1}{2\tau_{p1}}\left(1+\sqrt{\frac{r_p}{r_{pth}}}\right)}\sqrt{v_g}\left[\frac{\hat{b}_L(z=0,\omega)e^{-i\frac{\theta_p}{2}} - \hat{b}^{+}_L(z=0,\omega)e^{i\frac{\theta_p}{2}}}{2i}\right]$$

The quadratures of the field outside the cavity are therefore,

$$\Delta\hat{x}^{out}_{\theta_p/2}(z=0,\omega) = \frac{+i\omega + \frac{1}{2\tau_{p1}}\left(1+\sqrt{\frac{r_p}{r_{pth}}}\right)}{-i\omega + \frac{1}{2\tau_{p1}}\left(1-\sqrt{\frac{r_p}{r_{pth}}}\right)}\left[\frac{\hat{b}_L(z=0,\omega)e^{-i\frac{\theta_p}{2}} + \hat{b}^+_L(z=0,\omega)e^{i\frac{\theta_p}{2}}}{2}\right]$$

$$\Delta\hat{x}^{out}_{\theta_p/2+\pi/2}(z=0,\omega) = \frac{+i\omega + \frac{1}{2\tau_{p1}}\left(1-\sqrt{\frac{r_p}{r_{pth}}}\right)}{-i\omega + \frac{1}{2\tau_{p1}}\left(1+\sqrt{\frac{r_p}{r_{pth}}}\right)}\left[\frac{\hat{b}_L(z=0,\omega)e^{-i\frac{\theta_p}{2}} - \hat{b}^+_L(z=0,\omega)e^{i\frac{\theta_p}{2}}}{2i}\right]$$

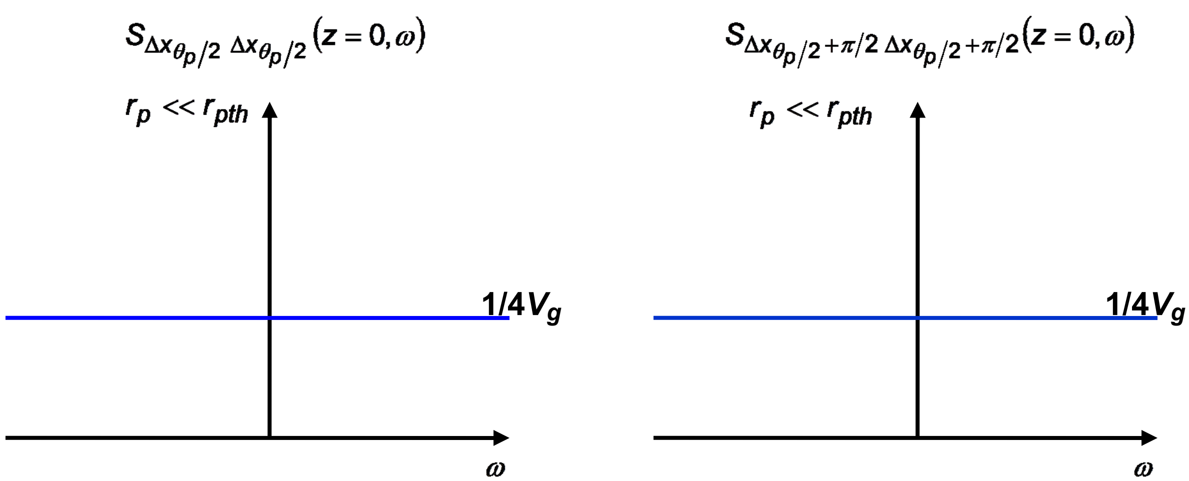
The spectral densities of the noise in the two quadratures in the field outside the cavity are,

$$\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\left\langle \Delta\hat{x}^{out}_{\theta_p/2}(z=0,\omega)\ \Delta\hat{x}^{out}_{\theta_p/2}(z=0,\omega')\right\rangle = \frac{1}{4v_g}\frac{\left[\frac{1}{2\tau_{p1}}\left(1+\sqrt{\frac{r_p}{r_{pth}}}\right)\right]^2+\omega^2}{\left[\frac{1}{2\tau_{p1}}\left(1-\sqrt{\frac{r_p}{r_{pth}}}\right)\right]^2+\omega^2}$$

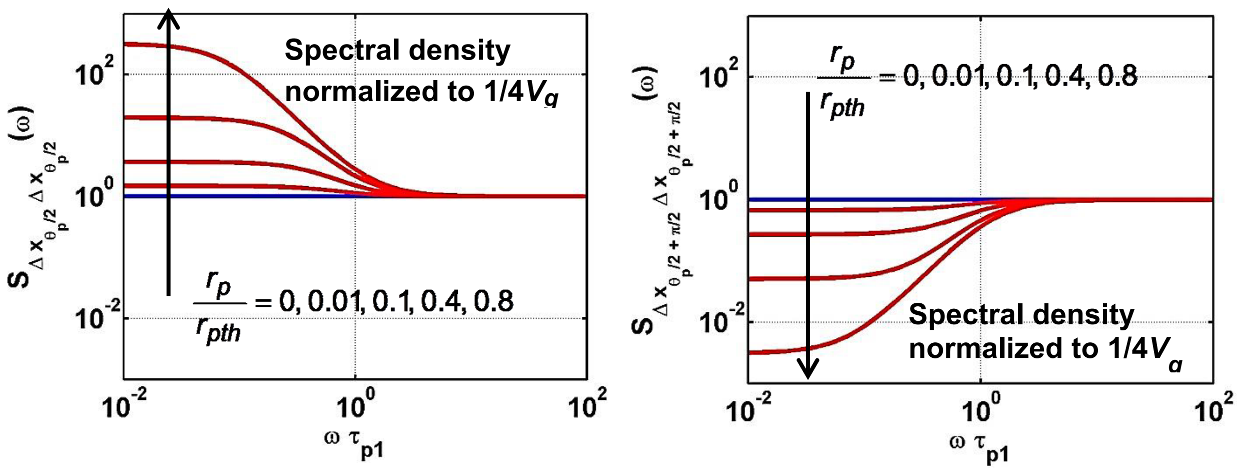
$$\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\left\langle \Delta\hat{x}^{out}_{\theta_p/2+\pi/2}(z=0,\omega)\ \Delta\hat{x}^{out}_{\theta_p/2+\pi/2}(z=0,\omega')\right\rangle = \frac{1}{4v_g}\frac{\left[\frac{1}{2\tau_{p1}}\left(1-\sqrt{\frac{r_p}{r_{pth}}}\right)\right]^2+\omega^2}{\left[\frac{1}{2\tau_{p1}}\left(1+\sqrt{\frac{r_p}{r_{pth}}}\right)\right]^2+\omega^2}$$

The following trends should be noted in the above expressions:

i) When $r_p << r_{pth}$, the spectral densities of the noise in the two quadratures in the output are both constant (as a function of the frequency) and equal to $1/4v_g$.



ii) As $r_p \to r_{pth}$, the low frequency spectral density of the noise in the quadrature $\Delta\hat{x}^{out}_{\theta_p/2}$ becomes very large and the low frequency spectral density of the noise in the quadrature $\Delta\hat{x}^{out}_{\theta_p/2+\pi/2}$ becomes very small (much smaller than $1/4v_g$). Therefore, low frequency fluctuations in the quadrature $\Delta\hat{x}^{out}_{\theta_p/2+\pi/2}$ are squeezed at the expense of the low frequency fluctuations in the quadrature $\Delta\hat{x}^{out}_{\theta_p/2}$, which are amplified.



iii) The bandwidth over which the low frequency fluctuations are squeezed in the quadrature $\Delta\hat{x}^{out}_{\theta_p/2+\pi/2}$ roughly corresponds roughly to the inverse of the cavity photon lifetime τ_{p1}. For frequencies larger than the inverse of the cavity photon lifetime τ_{p1}, the noise in the output field is essentially due to the reflected vacuum fluctuations and therefore the spectral density of the noise for both the quadratures equals $1/4v_g$.

14.5.2 Operation above Threshold

Above threshold we have,

$$\beta_{1s} = \sqrt{\frac{2}{\kappa\sqrt{\tau_{p2}}}}\left(\sqrt{r_p}-\sqrt{r_{pth}}\right)e^{i\frac{\theta_p}{2}}$$

$$\beta_{2s} = 2\sqrt{\tau_{p2}\, r_{pth}}\, e^{i\theta_p}$$

and the parametric gain for mode 1 equals the cavity loss,

$$\kappa|\beta_{2s}| = \frac{1}{2\tau_{p1}}$$

As before, we assume,

$$\hat{a}_1(t) = \left[|\beta_{1s}| + \Delta\hat{b}_1(t)\right]e^{-i\omega_o t + i\frac{\theta_p}{2}}$$

$$\hat{a}_2(t) = \left[|\beta_{2s}| + \Delta\hat{b}_2(t)\right]e^{-i2\omega_o t + i\theta_p}$$

where,

$$\left[\Delta\hat{b}_1(t), \Delta\hat{b}_1^+(t)\right] = \left[\Delta\hat{b}_2(t), \Delta\hat{b}_2^+(t)\right] = 1$$

Substituting the above expressions in the equations for the field operators give,

$$\frac{d\,\Delta\hat{b}_1(t)}{dt} = -\frac{1}{2\tau_{p1}}\left[\Delta\hat{b}_1(t) - \Delta\hat{b}^+_1(t)\right] + \kappa\,|\,\beta_{1s}|\,\Delta\hat{b}_2(t) + \sqrt{\frac{1}{\tau_{p1}}}\,\hat{S}_{in}(t)\,e^{-i\frac{\theta_p}{2}}$$

$$\frac{d\,\Delta\hat{b}_2(t)}{dt} = -\frac{1}{2\tau_{p2}}\Delta\hat{b}_2(t) - \kappa\,|\,\beta_{1s}|\,\Delta\hat{b}_1(t) + \sqrt{\frac{1}{\tau_{p2}}}\left(\hat{F}_{in}(t)\,e^{-i\theta_p} - \sqrt{r_p}\right)$$

$$= -\frac{1}{2\tau_{p2}}\Delta\hat{b}_2(t) - \kappa\,|\,\beta_{1s}|\,\Delta\hat{b}_1(t) + \sqrt{\frac{1}{\tau_{p2}}}\left(\hat{F}_{in}(t)\,e^{-i\theta_p} - \sqrt{r_p}\right)$$

The quadratures of the cavity fields are defined as,

$$\Delta\hat{b}_1(t) = \Delta\hat{x}_{\theta_p/2}(t) + i\,\Delta\hat{x}_{\theta_p/2+\pi/2}(t)$$

$$\Delta\hat{b}_2(t) = \Delta\hat{y}_{\theta_p}(t) + i\,\Delta\hat{y}_{\theta_p+\pi/2}(t).$$

We get the following equations for the noise quadratures,

$$\frac{d\,\Delta\hat{x}_{\theta_p/2}(t)}{dt} = \kappa\,|\beta_{1s}|\,\Delta\hat{y}_{\theta_p}(t) + \sqrt{\frac{1}{\tau_{p1}}}\left[\frac{\hat{S}_{in}(t)\,e^{-i\frac{\theta_p}{2}} + \hat{S}^+_{in}(t)\,e^{+i\frac{\theta_p}{2}}}{2}\right]$$

$$\frac{d\,\Delta\hat{y}_{\theta_p}(t)}{dt} = -\frac{1}{2\tau_{p2}}\Delta\hat{y}_{\theta_p}(t) - \kappa\,|\beta_{1s}|\,\Delta\hat{x}_{\theta_p/2}(t) + \sqrt{\frac{1}{\tau_{p2}}}\left[\frac{\hat{G}_{in}(t)\,e^{-i\theta_p} + \hat{G}^+_{in}(t)\,e^{+i\theta_p}}{2}\right]$$

and

$$\frac{d\,\Delta\hat{x}_{\theta_p/2+\pi/2}(t)}{dt} = -\frac{1}{\tau_{p1}}\Delta\hat{x}_{\theta_p/2+\pi/2}(t) + \kappa\,|\beta_{1s}|\,\Delta\hat{y}_{\theta_p+\pi/2}(t)$$

$$+\sqrt{\frac{1}{\tau_{p1}}}\left[\frac{\hat{S}_{in}(t)\,e^{-i\frac{\theta_p}{2}} - \hat{S}^+_{in}(t)\,e^{+i\frac{\theta_p}{2}}}{2i}\right]$$

$$\frac{d\,\Delta\hat{y}_{\theta_p+\pi/2}(t)}{dt} = -\frac{1}{2\tau_{p2}}\Delta\hat{y}_{\theta_p+\pi/2}(t) - \kappa\,|\beta_{1s}|\,\Delta\hat{x}_{\theta_p/2+\pi/2}(t)$$

$$+\sqrt{\frac{1}{\tau_{p2}}}\left[\frac{\hat{G}_{in}(t)\,e^{-i\theta_p} - \hat{G}^+_{in}(t)\,e^{+i\theta_p}}{2i}\right]$$

where,

$$\hat{G}_{in}(t) = \hat{F}_{in}(t) - \sqrt{r_p}\,e^{i\theta_p}$$

$\hat{G}_{in}(t)$, defined as above, is a zero mean noise source with the same correlations and commutation relations as the noise source $\hat{S}_{in}(t)$. Note that above threshold, the quadrature fluctuations of the pump (at frequency $2\omega_o$) and the signal (at frequency ω_o) are coupled. We can solve the first two equations in the frequency domain to get,

$$\Delta\hat{x}_{\theta_p/2}(\omega)=\frac{\kappa|\beta_{1s}|\sqrt{\frac{1}{\tau_{p2}}}\left[\frac{\hat{G}_{in}(\omega)e^{-i\theta_p}+\hat{G}_{in}^{+}(\omega)e^{+i\theta_p}}{2}\right]+\left(-i\omega+\frac{1}{2\tau_{p2}}\right)\sqrt{\frac{1}{\tau_{p1}}}\left[\frac{\hat{S}_{in}(\omega)e^{-i\frac{\theta_p}{2}}+\hat{S}_{in}^{+}(\omega)e^{+i\frac{\theta_p}{2}}}{2}\right]}{-\omega^2-i\omega\frac{1}{2\tau_{p2}}+\kappa^2|\beta_{1s}|^2}$$

The quadrature of the output field is,

$$\hat{x}^{out}_{\theta_p/2}(z=0,\omega)=\sqrt{\frac{1}{v_g\tau_{p1}}}\Delta\hat{x}_{\theta_p/2}(\omega)-\sqrt{\frac{1}{v_g}}\left[\frac{\hat{S}_{in}(\omega)e^{-i\frac{\theta_p}{2}}+\hat{S}_{in}^{+}(\omega)e^{+i\frac{\theta_p}{2}}}{2}\right]$$

$$=\frac{\kappa|\beta_{1s}|\sqrt{\frac{1}{v_g\tau_{p1}\tau_{p2}}}}{-\omega^2-i\omega\frac{1}{2\tau_{p2}}+\kappa^2|\beta_{1s}|^2}\left[\frac{\hat{G}_{in}(\omega)e^{-i\theta_p}+\hat{G}_{in}^{+}(\omega)e^{+i\theta_p}}{2}\right]$$

$$+\left[\frac{\left(-i\omega+\frac{1}{2\tau_{p2}}\right)\frac{1}{\tau_{p1}}}{-\omega^2-i\omega\frac{1}{2\tau_{p2}}+\kappa^2|\beta_{1s}|^2}-1\right]\sqrt{\frac{1}{v_g}}\left[\frac{\hat{S}_{in}(\omega)e^{-i\frac{\theta_p}{2}}+\hat{S}_{in}^{+}(\omega)e^{+i\frac{\theta_p}{2}}}{2}\right]$$

Similarly the other quadrature of the output field can also be found,

$$\hat{x}^{out}_{\theta_p/2+\pi/2}(z=0,\omega)=\frac{\kappa|\beta_{1s}|\sqrt{\frac{1}{v_g\tau_{p1}\tau_{p2}}}}{\left(-\omega^2-i\omega\left[\frac{1}{\tau_{p1}}+\frac{1}{2\tau_{p2}}\right]+\kappa^2|\beta_{1s}|^2+\frac{1}{2\tau_{p1}\tau_{p2}}\right)}\left[\frac{\hat{G}_{in}(\omega)e^{-i\theta_p}-\hat{G}_{in}^{+}(\omega)e^{+i\theta_p}}{2i}\right]$$

$$+\left[\frac{\left(-i\omega+\frac{1}{2\tau_{p2}}\right)\frac{1}{\tau_{p1}}}{-\omega^2-i\omega\left[\frac{1}{\tau_{p1}}+\frac{1}{2\tau_{p2}}\right]+\kappa^2|\beta_{1s}|^2+\frac{1}{2\tau_{p1}\tau_{p2}}}-1\right]\sqrt{\frac{1}{v_g}}\left[\frac{\hat{S}_{in}(\omega)e^{-i\frac{\theta_p}{2}}-\hat{S}_{in}^{+}(\omega)e^{+i\frac{\theta_p}{2}}}{2i}\right]$$

The spectral densities of the quadrature noise can be found from the above expressions. Needless to say, the resulting expressions are complicated. The coupling between the pump (at frequency $2\omega_o$) and the signal (at frequency ω_o) quadratures above threshold, and the resulting second order nature of the frequency response function, results in resonance in the quadrature noise spectral densities at frequencies given by,

$$\sqrt{\kappa^2|\beta_{1s}|^2}\quad\text{and}\quad\sqrt{\kappa^2|\beta_{1s}|^2+\frac{1}{2\tau_{p1}\tau_{p2}}}$$

In the amplified quadrature, the low frequency (~DC) noise, which was much larger than $1/4v_g$ near threshold, approaches $1/4v_g$ in the limit of strong pumping when $|\beta_{1s}| \to \infty$. Similarly in the attenuated quadrature, the low frequency (~DC) noise, which was much smaller than $1/4v_g$ near threshold, also approaches $1/4v_g$ in the limit $|\beta_{1s}| \to \infty$,

$$S_{\Delta x_{\theta_p/2}\,\Delta x_{\theta_p/2}}(z=0,\omega \approx 0) = \frac{1}{4v_g}\left(1+\left(\frac{1}{2\tau_{p2}\tau_{p1}\kappa^2|\beta_{1s}|^2}\right)^2\right)$$

$$S_{\Delta x_{\theta_p/2+\pi/2}\,\Delta x_{\theta_p/2+\pi/2}}(z=0,\omega \approx 0) = \frac{1}{4v_g}\left(\frac{\kappa^2|\beta_{1s}|^2\left(\kappa^2|\beta_{1s}|^2+\frac{1}{\tau_{p2}\tau_{p1}}\right)}{\left(\kappa^2|\beta_{1s}|^2+\frac{1}{2\tau_{p2}\tau_{p1}}\right)^2}\right)$$