A Study of the Integrated Parking and Ridesharing Pricing/Incentives and their Social and Environmental Impacts in Metropolitan Areas

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Abstract

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1 Introduction

Ridesharing (also referred as carpool) is considered as an important demand management tool and has been practiced for decades. In 1970, 20.4% of American workers commuted to work by carpool, according to the US Census. However, until very recently, ridesharing was not the most popular mode in people's daily travels. According to the US Census, ridesharing has declined to 10.7% in 2008 (Chan and Shaheen, 2012). Recently, a ridesharing market starts to grow thanks to the popularity of GPS-enabled smart devices and the innovative ridesharing services. The rapid growth and success of ridesourcing services, such as Uber and Lyft, suggests that paid ridesharing services can also bring a marketplace to utilize the empty seats in commuting vehicles, and the commuters can easily choose to be a driver or a passenger and be paired to each other in real-time. Unlike the traditional carpool services, recent paid ridesharing services allow travelers to be paired in real-time instead of in a pre-arranged fashion.

Traditional ridesharing studies are mainly limited to the static case. Enabled by emerging technologies and ridesharing marketplaces, recent ridesharing studies have started to explore dynamic ridesharing problems. Qian and Zhang (2011) studies the morning commute problem with transit, driving alone and carpool

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modes on a homogeneous population. The ridesharing payments were internalized in the homogeneous carpoolers, and thus neglected in the formulation. Xu et al. (2015) extended the ridesharing problem for general networks with multiple origin-destination pairs, and formulated the ridesharing problem with pick-up and drop-off choices, while the ridesharing equilibrium was simplified in static and formulated as a nonlinear complementarity problem.

Parking is a critical component of daily commuting trips. Parking charges have been suggested as an efficient alternative of managing travel demand (e.g. Button (1995); Johansson and Mattsson (1995)). Most of the parking pricing studies are limited in static cases (e.g., Spiess (1996); Vianna et al. (2004)). The cutting-edge sensing and information technology enable the time-varying parking pricing and the information broadcast to the travelers. For instance, the SFPark program establishes varying parking prices in different time-of-day at San Francisco downtown area, and the online system is able to broadcast the pricing to the travelers via smartphone apps and web pages. However, there are handful studies on leveraging dynamic parking pricing to influence travel demand (e.g., Qian et al. (2011), Qian et al. (2012) and Fosgerau and De Palma (2013). Most studies and practices on parking pricing have not considered the dynamic ridesharing demand in the traffic, and research on simultaneously applying parking pricing and ridesharing to manage traffic demand is quite rare.

The morning commute problem on a single bottleneck has been extensively studied for decades. In Vickrey (1969), a dynamic toll on a single route with a single bottleneck for a single mode of homogeneous travelers was proposed to eliminate the queuing congestion. The equilibrium of departure-time choice was shown to exist, where travelers with any departure time share the same cost, including the travel and arrival penalty costs. A large number of studies (e.g., Arnott et al. (1990); Arnott and Kraus (1998)) have been made to extend Vickrey's pioneering work, but none of theses studies, to the best of our knowledge, has included ridesharing in its analysis.

The purpose of this report is to formulate a continuous-time dynamic ridesharing problem for a single bottleneck in the morning commute scenario, investigate how travelers behave at the equilibrium of departure-time and ridesharing mode choices as heterogeneous groups with different values of time, and determine how the dynamic ridesharing payments and the parking pricing mechanism can affect travelers' departure time and mode (as a driver or passenger) choice and the commuting traffic dynamics. We will further calculate the system optimal parking pricing that eliminates the queuing congestion in the morning commute.

In the studied scenarios, travelers can either choose to become a driver or a passenger. A passenger would save the operational costs, such as fuel consumption, driving and pick-up costs, while s/he needs to pay the driver for the ride. We note here that in the literature, heterogeneous groups may have different VOTs, arrival penalties and other parameters; for instance, in Wang and Du (2016), the heterogeneity is applied to preferred arrival time, value of travel time and schedule delays. In general, heterogeneous groups may likely have different arrival penalty parameters when the VOTs are not the same; for instant, groups with higher VOTs usually have higher early/late arrival penalty parameters. However, it was also found that the ratios of such penalty parameters over the VOTs may also vary over groups, which indicates different preferences on early/late arrival with less congestion versus on-time arrival with more travel times among heterogeneous groups. Considering the variety of the VOTs and arrival penalty parameters at the same time would inevitably lead to discussions with higher dimensional problems. Therefore, the group heterogeneity in this study only focuses on the monetary value of travel time (VOT), i.e., travelers in a same group have an identical VOT; while all heterogeneous groups share the same preferred arrival time and the early/late arrival penalty parameters. Similar setup can be found in van den Berg and Verhoef (2011) with only solo-drivers.

At equilibrium, travelers in a same group, regardless their departure-time or mode choices, would share a common disutility, which incorporates travel time costs, early/late arrival penalties, and monetary costs/earnings. The travel time cost of a traveler is transformed into monetary cost first before it is included in the total cost of a traveler. In this study we assume that only travelers in the same group share rides, which is reasonable in the sense that such ridesharing is most likely to occur within socially connected groups who live close to each other. And people who live close to each other are likely to have the same level of income and value of time. In the proposed formulation, we quantify the aggregated time-varying variables, such as vehicular flow rates and passenger/driver ratios for each traveler group, instead of distinguishing or assigning an individual traveler to a vehicle as in microscopic simulations. In this way, non-integer ratios may appear in the solution. For instance, '*at time t, for group i, the passenger/driver ratio is* 2.4' means on average, among travelers that enter the bottleneck entrance at time t, the number of passengers is 2.4 times that of drivers for group i travelers. In the system optimal scenario, time-varying parking pricing can be derived for the heterogeneous ridesharing travelers. The parking price would be shared among all travelers in a vehicle, including the driver and the passengers.

In a ridesharing system, it is critical to properly design the payment policies. A good policy can encourage ridesharing, and leads to a better system performance in terms of social equity, economic costs and environmental benefits. We will explore the impacts of different policies on the ridesharing system in this study, through a number of system performance indicators that include system total time and monetary cost, vehicle-miles-traveled (VMT), and vehicle-hours-traveled (VHT).

The rest of this report is organized as follows. Section 2 lists the notation used in the following sections. Section 3 discusses the equilibrium states of the ridesharing system, and the properties of the system, which are distinct from their solo driver counterpart, are analyzed. Section 4 reformulates the equilibrium problem into a differential complementarity system (DCS), so that it can be discretized and solved numerically. Section 5 introduces the dynamic parking pricing for the heterogeneous ridesharing travelers, so that the system is at free-flow condition to achieve the system optimum. It is shown that the dynamic parking pricing is distinct from that of the solo driver scenarios. Section 6 discusses the ridesharing payment policies with numerical examples. Section 7 concludes this report.

2 Notation

$\mathcal{I}$ Heterogeneous group set $\{1, 2, \cdots, M\}$; $i \in \mathcal{I}$ is the i-th group

Parameters all positive scalars

T	The terminal time of the studied time span $[0, T]$
$\overline{n}_i$	Total demand of all travelers in group i
s	Bottleneck flow capacity
τ^0	Bottleneck link free-flow travel time
h	Average pick-up travel time for each passenger;
	average travel time from home to the entrance of the bottleneck link for each driver
$\overline{b}$	Maximum ridesharing ratio
$\overline{P}$	An upper bound of common disutilities for any group
α_i	Value of travel time for group i
β	Early arrival penalty parameter
γ	Late arrival penalty parameter; Generally, $\beta < \gamma$
ϕ	parameter for modified bottleneck trip cost of passengers, $0 < \phi < 1$
t^*	Preferred arrival time

Variables

$q(t)$	Queue on the bottleneck at time t
$\tau(t)$	Travel time going through the bottleneck link for travelers entering the bottleneck link at time t
$d(t)$	Vehicular demand rate of all groups at time t
$v(t)$	Exit flow rate at time t
$d_i(t)$	Vehicular demand rate for group i at time t
$b_i(t)$	Ridesharing ratio for group i at time t
$n_i(t)$	Cumulative demand for group i at time t
$P_i^v(t)$	Generalized cost for drivers in group i entering the bottleneck at time t
$P_i^p(t)$	Generalized cost for passengers in group i entering the bottleneck at time t
p_i	Base ridesharing payment, exogenous given
P_i	Generalized cost (or common disutility) for group i ridesharing travelers at equilibrium
$\mu(t)$	Slack variable of flow rate in the continuous-time point-queue model
$\eta_i(t)$	Slack variable of marginal ridesharing payment for group i at time t, when passenger seats are fully occupied
$\eta_i^-(t)$	Slack variable of (reversed) marginal ridesharing payment for group i at time t, when passenger seats are empty
$\hat{k}_i(t)$	parking charges for travelers in group i arriving at the destination at time t
$k_i(t)$	parking charges for travelers in group i entering the bottleneck at time t, $k_i(t) = \hat{k}_i(t + \tau(t))$

3 Equilibrium with heterogeneous travelers

This section discusses the equilibrium with M traveler groups with group-specific value of travel time α_i and no parking charges. We assume that ridesharing only occurs for travelers within the same traveler group. Such an assumption is reasonable in the sense that the type of ridesharing (also known as social carpooling) that we consider is more likely to occur in socially connected groups who live in proximity. And people who live close to each other in suburban areas are likely to have similar levels of income and hence value of time. Furthermore, ridesharing across heterogeneous groups can lead to cases where travelers in the same vehicle will arrive at the same time but have different travel costs. Under equilibrium, the departure times of heterogeneous groups of ridesharing travelers would be overlapping, which significantly complicates the problem. Our attempt here is therefore the first step in studying this complex problem, and we will leave the problems with cross-group sharing to future work. Despite this, the reader will find that our subsequent analysis in this study has already obtained some good insights about the dynamic ridesharing problem in the morning commute.

Ridesharing drivers need to pick up their passengers from the local streets before entering the main highway. After entering the bottleneck link at time t, the travel time spent on the bottleneck is $\tau(t)$ for the driver and the corresponding passengers, which is composed of the free-flow travel time τ^0 and the queuing delays. The travel time cost of the drivers in group i is denoted as $\alpha_i\tau(t)$. Since passengers do not need to drive, their travel time costs (in group i) are taken as $\alpha_i\phi\tau(t)$ $(0 < \phi < 1)$, which are lower than those of the drivers in the same group. Similar maneuver on the reduction of the travel costs for passengers in autonomous cars can be found in van den Berg and Verhoef (2016).

To compensate the driver for the pick-up and driving, each passenger pays the driver a ridesharing fee. In this study, the ridesharing payment is composed of two parts. The first part is defined as 'base payment' p_i for group i, which is a flat payment independent of passenger demands. Such base payments can be interpreted as flat government regulated ridesharing payments. If the ridesharing vehicle is not fully occupied, the transaction between a driver and a passenger should be equal to the base payment; Otherwise, if the ridesharing vehicle is fully occupied, in order to stimulate more drivers to provide ridesharing, a surcharge can be added onto the base payment. Such a surcharge is the second part of the ridesharing payments, which can be time-varying and thus defined as $\eta_i(t)$. Note that the surcharge is positive only if the ridesharing vehicle is fully occupied. As the base payment is exogenously given, we will show in this study that different policies on the base payment may lead to distinct system performance.

We define the ridesharing ratio as the average number of passengers in group i in a vehicle entering the bottleneck link at time t as $b_i(t)$. The dynamic surcharge and the boundedness of the ridesharing ratio are thus in a complementarity relation as

$$\begin{aligned} b_i(t) < \bar{b} \quad &\Rightarrow \eta_i(t) = 0, \\ \eta_i(t) > 0 \quad &\Rightarrow b_i(t) = \bar{b}. \end{aligned} \tag{1}$$

3.1 Generalized costs for ridesharing travelers

Ridesharing travelers can choose to become either a driver or a passenger. The generalized cost of a driver in group i entering the bottleneck link at time t is composed of the travel time costs (pick-up and highway traveling), the arrival penalties and the gain of ridesharing payments (as negative cost).

For $\forall i \in \mathcal{I}$ and $\forall t \in [0, T]$,

$$P_i^v(t) = \alpha_i \left[(1 + b_i(t))h + \tau(t) \right] + F(t + \tau(t)) - b_i(t) \left[p_i + \eta_i(t) \right] \tag{2}$$

where $F(t + \tau(t))$ is the arrival penalty for the arrival time $t + \tau(t)$.

We stipulate that all travelers desire to arrive within the window $[\, t^* - \Delta, t^* + \Delta \,]$, where t^* is the preferred arrival time; $\Delta \geq 0$ is the half length of the preferred arrival time window. A penalty function F is employed to penalize early or late arrival outside of the this arrival time window. Similar definitions of the arrival penalty can be found in Ran and Boyce (1996); Friesz et al. (2010); Ma et al. (2017), which are composed of a driver's early, late, or on-time arrival.

For $\forall t \in [0, T]$,

$$\begin{aligned} F(t + \tau(t)) \quad &\triangleq \quad \begin{cases} \beta \,(\, t^* - t - \tau(t) - \Delta \,) & \text{if } t + \tau(t) - t^* < -\Delta \\ 0 & \text{if } -\Delta \leq t + \tau(t) - t^* \leq \Delta \\ \gamma \,(\, t + \tau(t) - t^* - \Delta \,) & \text{if } t + \tau(t) - t^* > \Delta \end{cases} \\ &= \quad \max \left\{ \beta \,(\, t^* - t - \tau(t) - \Delta \,) \,,\, 0,\, \gamma \,(\, t + \tau(t) - t^* - \Delta \,) \right\} \end{aligned} \tag{3}$$

Following the morning commute literature, where the length of the preferred arrival time window is usually treated as zero, we take $\Delta = 0$ in this study which simplifies the subsequent derivations but to not change the fundamentals of the problem. The arrival penalty is then simplified as

$$F(t + \tau(t)) = \begin{cases} \beta \, (t^* - (t + \tau(t))) & t + \tau(t) \leq t^*, \\ \gamma \, (t + \tau(t) - t^*) & t + \tau(t) \geq t^*. \end{cases} \tag{4}$$

The generalized cost of a passenger in group i entering the bottleneck at time t is composed of a reduced travel time cost and the monetary cost of ridesharing payment.

For $\forall i \in \mathcal{I}$ and $\forall t \in [0, T]$,

$$P_i^p(t) = \alpha_i \phi \tau(t) + F(t + \tau(t)) + (p_i + \eta_i(t)) \,. \tag{5}$$

3.2 Equilibrium solution and travel time curves

At the equilibrium, the generalized cost of travelers in group i should be equal to each other. More specifically, it is equivalent to the following two conditions.

a) At time t, if the demand rate of the passengers is positive, the generalized cost of a passenger should be equal to that of a driver in the same group departing at the same time, i.e.,

$$b_i(t) d_i(t) > 0 \Rightarrow P_i^p(t) = P_i^v(t); \tag{6}$$

If the generalized cost of a passenger is higher than that of a driver in the same group departing at the same time, then the demand rate of the passengers is zero, i.e.,

$$P_i^p(t) > P_i^v(t) \Rightarrow b_i(t)d_i(t) = 0; \tag{7}$$

b) the generalized cost of a driver departing at any time should be equal to each other.

$$d_i(t) > 0 \Rightarrow \dot{P}_i^v(t) = 0, \tag{8}$$

where $\dot{P}_i^v(t) \triangleq \frac{dP_i^v(t)}{dt}$.

From the generalized cost functions (2) and (5), we can observe that the travel time function $\tau(t)$ is essential for the calculation of the equilibrium.

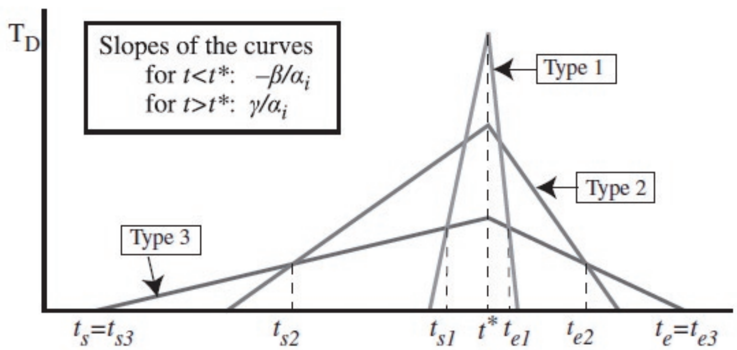


Figure 1: The isocost curves for three types of travelers (Adopted from Fig.1 in van den Berg and Verhoef (2011))

As discussed in van den Berg and Verhoef (2011), the arrival sequences of the solo drivers are determined by β/α_i. On the early arrival side, the groups are ordered in time by increasing values of β/α_i; On the late arrival side, the groups are ordered by decreasing values of γ/α_i. Since the early and late arrival penalty parameters β and γ are assumed to be identical for all groups in this study, such orders are determined by α_i. Without loss of generality, it is assumed that the group labels are ordered by increasing values of α_i, i.e., group 1 has the lowest α_i, and group M has the highest α_i. Thus, the group with higher α_i is with larger index, and arrives further from the preferred arrive time t^*. Figure 1 shows the isocost curves, which represents the time-varying travel time for three groups of solo drivers.

Similarly, in this study on ridesharing travelers, the arrival sequences of traveler groups follow the similar pattern. However, as we show in Theorem 1, the travel time is a more complicated piecewise linear function that that in the solo driver case.

Theorem 1 The travel time (isocost curve) is piecewise linear for each group's early arrival (or late arrival) time period for ridesharing travelers with the same β, γ and heterogeneous α's.

Proof. As discussed, the group labels are ordered by increasing values of α_i, and group 1 has the lowest α_i. We denote the beginning time of the early arrival time period for group i as t_{ib}, and the ending time of the late arrival time period for group i as t_{ie}. There is no time gap between sequential groups i and $i+1$, because travelers in group $i+1$ could always switch their departure time closer to that of travelers in group i, if any time gap was presented. We further note $t_{0b} + \tau(t_{0b}) = t^*$ and $t_{0e} = t_{0b}$.

We define time $t_{ib}^0 \in [t_{ib}, t_{i-1,b}]$ as the last time instant that the ridesharing ratio of group i remains at zero while the vehicular demand rate at that time $d_i(t_{ib}^0)$ is positive; if the ridesharing ratio of group i is always higher than zero, t_{ib}^0 is set as t_{ib}. We further define time $t_{ib}^{\bar{b}} \in [t_{ib}^0, t_{i-1,b}]$ as the first time instant that the ridesharing ratio of group i reaches its capacity $\bar{b}$ while the demand rate at that time $d_i(t_{ib}^{\bar{b}})$ is positive; if the ridesharing ratio of group i never reaches its capacity, $t_{ib}^{\bar{b}}$ is set as $t_{i-1,b}$. The early arrival time period of group i, i.e., $(t_{ib}, t_{i-1,b})$, can be divided into three time segments.

1) Solo-driver time segment (t_{ib}, t_{ib}^0);

2) Ridesharing-with-vacancy time segment $(t_{ib}^0, t_{ib}^{\bar{b}})$; and

3) Full-ridesharing time segment $(t_{ib}^{\bar{b}}, t_{i-1,b})$.

The ridesharing ratio in each time segment is

$$b_i(t) \begin{cases} = 0, & t \in (t_{ib}, t_{ib}^0); \\ \in (0, \bar{b}), & t \in (t_{ib}^0, t_{ib}^{\bar{b}}); \\ = \bar{b}, & t \in (t_{ib}^{\bar{b}}, t_{i-1,b}). \end{cases}$$

For the time t when the passenger cost is higher than that of a driver, the passenger demand of group i is zero $b_i(t)d_i(t) = 0$, i.e., $b_i(t) = 0$. And the generalized cost of a driver should be equal at any time, i.e., $\dot{P}_i^v(t) = 0$. Then for the early arrival solo-driver time segment $t \in (t_{ib}, t_{ib}^0)$, $t+\tau(t) < t^*$, and $\dot{F}(t+\tau(t)) = -\beta$. We then have the derivative of travel time during this time segment as

$$\dot{P}_i^v(t) = \alpha_i \dot{\tau}(t) + (1 + \dot{\tau}(t))(-\beta) = 0 \Rightarrow \dot{\tau}(t) = \frac{\beta}{\alpha_i - \beta}. \tag{9}$$

For the time t when the passenger demand of group i is positive $b_i(t)d_i(t) > 0$, we know the generalized cost of a passenger equals to that of the corresponding driver, i.e., $P_i^p(t) = P_i^v(t)$. Further, since $b_i(t) \geq 0$, we have $d_i(t) > 0$, so that $\dot{P}_i^v(t) = 0$. Then $\dot{P}_i^p(t) = 0 \Rightarrow \alpha_i \phi \dot{\tau}(t) + \dot{F}(t + \tau(t))(1 + \dot{\tau}(t)) + \dot{\eta}_i(t) = 0$.

For the early arrival time segment $t \in (t_{ib}^0, t_{i-1,b})$, $t + \tau(t) < t^*$, so that $\dot{F}(t + \tau(t)) = -\beta$. As $P_i^p(t) = P_i^v(t)$, we have the derivative of travel time during this time segment as

$$\dot{\tau}(t) = \frac{\beta - \dot{\eta}_i(t)}{\alpha_i \phi - \beta}. \tag{10}$$

From $P_i^p(t) = P_i^v(t)$, we can also have

$$\tau(t) = \frac{(1 + b_i(t))(p_i - \alpha_i h + \eta_i(t))}{\alpha_i(1 - \phi)} \tag{11}$$

For different states of the ridesharing ratio $b_i(t)$, the travel time curves also vary.

• When $t \in (t_{ib}, t_{ib}^0)$, $b_i(t) = 0$, and $\eta_i(t) = 0$. Since the derivative of the travel time is a constant during such a time segment, the travel time is an increasing linear function as

$$\tau(t) = \tau(t_{ib}) + \frac{\beta}{\alpha_i - \beta}(t - t_{ib}). \tag{12}$$

To guarantee the travel time is realistic and always no less than the free-flow travel time τ^0, it implies that $\alpha_i - \beta > 0$.

• When $t \in (t_{ib}^0, t_{ib}^{\bar{b}})$, $0 < b_i(t) < \bar{b}$, $\eta_i(t) = 0$ and $\dot{\eta}_i(t) = 0$,

$$\begin{aligned} \tau(t) &= \frac{(1 + b_i(t))(p_i - \alpha_i h)}{\alpha_i(1 - \phi)}, \\ \dot{\tau}(t) &= \frac{\beta}{\alpha_i \phi - \beta}. \end{aligned} \tag{13}$$

To guarantee the travel time is realistic and always no less than the free-flow travel time τ^0, it implies that $\alpha_i \phi - \beta > 0$.

It is straightforward from (13) that the travel time is an increasing linear function of time t during time $(t_{ib}^0, t_{ib}^{\bar{b}})$,

$$\tau(t) = \tau(t_{ib}^0) + \frac{\beta}{\alpha_i \phi - \beta}\left(t - t_{ib}^0\right). \tag{14}$$

As long as the base payment p_i is higher than the (monetary) pick-up cost $\alpha_i h$, i.e., $p_i > \alpha_i h$, from (13), it also shows that the ridesharing ratio $b_i(t)$ is also an increasing linear function of time t during time $(t_{ib}^0, t_{ib}^{\bar{b}})$. (Section 3.4 will show that if $p_i \leq \alpha_i h$, the the ridesharing ratio $b_i(t) = \bar{b}$, which means the length of the time period $(t_{ib}^0, t_{ib}^{\bar{b}})$ is zero.) Therefore, the derivative of $b_i(t)$ is a constant, calculated as

$$\dot{b}_i(t) = \frac{\alpha_i(1 - \phi)\dot{\tau}(t_{ib})}{p_i - \alpha_i h} = \frac{\alpha_i(1 - \phi)\beta}{(\alpha_i \phi - \beta)(p_i - \alpha_i h)}. \tag{15}$$

and the initial value of $b_i(t)$ is $b_i(t^0_{ib}) = \frac{\alpha_i(1-\phi)\tau(t^0_{ib})}{p_i-\alpha_i h} - 1$. Note that if $t^0_{ib} \neq t_{ib}$, it means that the length of the solo-driver time segment is positive, so that the ridesharing ratio starts from zero at time t^0_{ib}, i.e., $b_i(t^0_{ib}) = 0$.

- When $t \in (t^{\bar{b}}_{ib}, t_{i-1,b})$, $b_i(t) = \bar{b}$,

$$\begin{aligned} \tau(t) &= \frac{\left(1+\bar{b}\right)\left(p_i - \alpha_i h + \eta_i(t)\right)}{\alpha_i(1-\phi)}, \\ \dot{\tau}(t) &= \frac{\beta - \dot{\eta}_i(t)}{\alpha_i\phi - \beta}. \end{aligned} \tag{16}$$

Solving (16) for $\tau(t)$, we have

$$\tau(t) = \tau(t^{\bar{b}}_{ib}) + \frac{\beta(1+\bar{b})}{(1+\bar{b})(\alpha_i\phi - \beta) + \alpha_i(1-\phi)}\left(t - t^{\bar{b}}_{ib}\right), \tag{17}$$

Since $\alpha_i\phi - \beta > 0$, (17) shows that the travel time is also a linear increasing function of time t during time $(t^{\bar{b}}_{ib}, t_{i-1,b})$.

(16) suggests that the monotonicity of $\tau(t)$ and $\eta_i(t)$ are the same, given $\frac{1+\bar{b}}{\alpha_i(1-\phi)} > 0$. Thus, the surcharge $\eta_i(t)$ is also an increasing linear function of time t during time $(t^{\bar{b}}_{ib}, t_{i-1,b})$.

To conclude the above analysis,

(1) During time (t_{ib}, t^0_{ib}), $b_i(t) = 0$ and $\tau(t)$ is a linearly increasing function; $\eta_i(t) = 0$.

(2) During time $(t^0_{ib}, t^{\bar{b}}_{ib})$, $\tau(t)$ and $b_i(t)$ are linearly increasing functions; $b_i(t) < \bar{b}$ and $\eta_i(t) = 0$.

(3) During time $(t^{\bar{b}}_{ib}, t_{i-1,b})$, $\tau(t)$ and $\eta_i(t)$ are linearly increasing functions; $b_i(t) = \bar{b}$.

Similarly, we can define time $t^0_{ie} \in [t_{i-1,e}, t_{ie}]$ as the first time instant that the ridesharing ratio of group i drops back to zero while the vehicular demand rate at that time $d_i(t^0_{ie})$ is positive; if the ridesharing ratio of group i is always higher than zero, t^0_{ie} is set as t_{ie}. And further define time $t^{\bar{b}}_{ie} \in [t_{i-1,e}, t^0_{ie}]$ as the last time instant that the ridesharing ratio of group i drops off its capacity $\bar{b}$ while the demand rate at that time $d_i(t^{\bar{b}}_{ie})$ is positive; if the ridesharing ratio of group i never reaches its capacity, $t^{\bar{b}}_{ie}$ is set as $t_{i-1,e}$.

The late arrival time period of group i, i.e., $(t_{i-1,e}, t_{ie})$, can be divided into three time segments.

4) Full-ridesharing time segment $(t_{i-1,e}, t^{\bar{b}}_{ie})$.

5) Ridesharing-with-vacancy time segment $(t^{\bar{b}}_{ie}, t^0_{ie})$; and

6) Solo-driver time segment (t^0_{ie}, t_{ie});

In fact, the travel time during the late arrival time segment is monotonic decreasing, and piecewise linear. More specifically,

(4) During time $(t_{i-1,e}, t^{\bar{b}}_{ie})$, $\tau(t)$ and $\eta_i(t)$ are linearly decreasing functions; $b_i(t) = \bar{b}$.

(5) During time $(t^{\bar{b}}_{ie}, t^0_{ie})$, $\tau(t)$ and $b_i(t)$ are linearly decreasing functions; $0 < b_i(t) < \bar{b}$ and $\eta_i(t) = 0$.

(6) During time (t^0_{ie}, t_{ie}), $b_i(t) = 0$ and $\tau(t)$ is a linearly increasing function; $\eta_i(t) = 0$.

Despite the different slope of the arrival penalty function $\dot{F}(t+\tau(t)) = \gamma$ in the late arrival time segments and the reverse sequence of the time segments, the analysis would be very similar to the above analysis on the early arrival time segments. The details are thus omitted here.

□

Note that for each group, all the above-mentioned six time segments may not necessarily be present in full. It depends on the travel times in the specific time segments, as well as the parameters such as p_i and α_i. Six possibilities are listed here.

- There only exists solo-driver time segments ($t_{ib} < t^0_{ib} = t^{\bar{b}}_{ib} = t_{i-1,b}$), due to a high p_i and low $\tau(t)$;
- There only exists ridesharing with vacancy time segments ($t_{ib} = t^0_{ib} < t^{\bar{b}}_{ib} = t_{i-1,b}$), due to a median p_i and a narrow ranged $\tau(t)$;
- There only exists full ridesharing time segments ($t_{ib} = t^0_{ib} = t^{\bar{b}}_{ib} < t_{i-1,b}$), due to a low p_i and high $\tau(t)$;
- Only solo-driver time segments are missing ($t_{ib} = t^0_{ib} < t^{\bar{b}}_{ib} < t_{i-1,b}$), due to a low p_i and a wide ranged $\tau(t)$;
- Only full ridesharing time segments are missing ($t_{ib} < t^0_{ib} < t^{\bar{b}}_{ib} = t_{i-1,b}$), due to a high p_i and a wide ranged $\tau(t)$;
- All types of time segments are presented ($t_{ib} < t^0_{ib} < t^{\bar{b}}_{ib} < t_{i-1,b}$).

3.3 Vehicular demand rates

Once the travel time curve is calculated, the vehicular demand rate (i.e., the demand rate of drivers) can be determined from the (point-queue) bottleneck model.

The point-queue link travel time is calculated as

$$\tau(t) = \tau^0 + \frac{q(t+\tau^0)}{s} = \tau^0 + \int_{-\infty}^{t} \frac{\sum_{i\in\mathcal{I}} d_i(\xi) - v(\xi+\tau^0)}{s} d\xi, \tag{18}$$

where $d_i(t)$ is the demand rate of drivers in group i. We further note here that at any given time t, there is no more than one group of drivers departing; otherwise, they have to share the same generalized cost and essentially become a single group. In other words, the departure times of different groups do not overlap with each other.

For the early departure time segment $t \in (t_{ib}, t_{i-1,b})$, the corresponding exit flow always equals to the bottleneck capacity, i.e., $v(t+\tau^0) = s$, and the travel time is recalculated according to Eq (18) as discussed in Section 3.2.

$$\tau(t) = \tau^0 + \sum_{j=M-1}^{i} \int_{t_{j+1,b}}^{t_{jb}} \frac{d_j(\xi) - s}{s} d\xi + \int_{t_{ib}}^{t} \frac{d_i(\xi) - s}{s} d\xi \tag{19}$$

Note that the group with higher α_i is with larger index i, as .

In each departure time segment, the travel time is differentiable almost everywhere. Kink points, where the travel time is continuous but the derivative is not defined, only exist at these following time instants.

- The junction time instants between departure time segments of sequential groups, i.e., t_{ib} and t_{ie}, $i \in \mathcal{I}$;
- The time instants that the ridesharing ratio starts to increase from zero for the first time in the early arrival part for each group, i.e., t_{ib}^{0}, $i \in \mathcal{I}$;
- The time instants that the ridesharing ratio reaches its capacity for the first time in the early arrival part for each group, i.e., $t_{ib}^{\bar{b}}$, $i \in \mathcal{I}$;
- The peak time when the travel time reaches its maximum, i.e., t_{0b};
- The time instants that the ridesharing ratio is no longer at its capacity for the first time in the late arrival part for each group, i.e., $t_{ie}^{\bar{b}}$, $i \in \mathcal{I}$.
- The time instants that the ridesharing ratio reduces back to zero for the first time in the late arrival part for each group, i.e., t_{ie}^{0}, $i \in \mathcal{I}$;

The number of such kink points is at most $6M + 1$. Note that it is possible for some group, all the above-mentioned six time segments may not be present in full. In such cases, the number of kink points is even less. For each time segment that is divided by the above-mentioned kink points, the travel time is differentiable. The derivative is expressed as

$$\dot{\tau}(t) = \frac{d_i(t) - s}{s}. \tag{20}$$

So that the vehicular demand rate is expressed by the derivative of travel time as

$$d_i(t) = s\left(1 + \dot{\tau}(t)\right). \tag{21}$$

As shown in (14) and (17), the travel time is a piecewise function, so that in each time segment, the derivative of travel time and the vehicular demand rate of each group are constants. The vehicular demand rates for the early arrival time segments of group i are calculated as

$$d_i(t) = \begin{cases} \left(1 + \dfrac{\beta}{\alpha_i - \beta}\right) s & \triangleq d_i^1, \quad t \in \left(t_{ib}, t_{ib}^{0}\right); \\ \left(1 + \dfrac{\beta}{\alpha_i \phi - \beta}\right) s & \triangleq d_i^2, \quad t \in \left(t_{ib}^{0}, t_{ib}^{\bar{b}}\right); \\ \left(1 + \dfrac{\beta(1+\bar{b})}{(1+\bar{b})(\alpha_i\phi - \beta) + \alpha_i(1-\phi)}\right) s & \triangleq d_i^3. \quad t \in \left(t_{ib}^{\bar{b}}, t_{i-1,b}\right). \end{cases} \tag{22}$$

The vehicular demand rates for the late arrival time segments of group i are calculated as

$$d_i(t) = \begin{cases} \left(1 + \dfrac{-\gamma(1+\bar{b})}{(1+\bar{b})(\alpha_i\phi+\gamma)+\alpha_i(1-\phi)}\right)s & \triangleq d_i^4. \quad t \in \left(t_{i-1,e}, t_{ie}^{\bar{b}}\right); \\ \left(1 + \dfrac{-\gamma}{\alpha_i\phi+\gamma}\right)s & \triangleq d_i^5, \quad t \in \left(t_{ie}^{\bar{b}}, t_{ie}^0\right). \\ \left(1 + \dfrac{-\gamma}{\alpha_i+\gamma}\right)s & \triangleq d_i^6, \quad t \in \left(t_{ie}^0, t_{ie}\right). \end{cases} \tag{23}$$

3.4 Ridesharing ratios and payments

The ridesharing ratios, i.e., the ratio of the number of ridesharing passengers over that of drivers for each group, are time-varying for most cases. In some special cases, the ridesharing ratio could be a constant. We here discuss how the values of fixed base payment p_i and the travel time at the beginning time $\tau(t_{ib})$ for each early arrival time segment can affect the ridesharing ratios $b_i(t)$.

We first determine the value of $b_i(t_{ib})$. If $P_i^p(t_{ib}) = P_i^v(t_{ib})$, by (11) we have

$$(1 + b_i(t_{ib}))\left(p_i - \alpha_i h + \eta_i(t_{ib})\right) = \alpha_i(1-\phi)\tau(t_{ib}). \tag{24}$$

If $P_i^p(t_{ib}) > P_i^v(t_{ib})$, by (7) and (9), $b_i(t_{ib}) = 0 = \eta_i(t_{ib})$, and we have

$$p_i - \alpha_i h > \alpha_i(1-\phi)\tau(t_{ib}). \tag{25}$$

Case 1: Low base payment. $p_i < \dfrac{\alpha_i(1-\phi)\tau(t_{ib})}{1+\bar{b}} + \alpha_i h.$

From (24) and $b(t_{ib}) \le \bar{b}$, we have $p_i - \alpha_i h + \eta(t_{ib}) \ge \dfrac{\alpha_i(1-\phi)\tau(t_{ib})}{1+\bar{b}}$. On the other hand, in this case, $p_i - \alpha_i h < \dfrac{\alpha_i(1-\phi)\tau(t_{ib})}{1+\bar{b}}$, so that $p_i - \alpha_i h < p_i - \alpha_i h + \eta(t_{ib})$. Then we have $\eta(t_{ib}) > 0$ and by the complementarity (1), the ridesharing ratio must reach its capacity, i.e., $b_i(t_{ib}) = \bar{b}$. The starting surcharge payment is $\eta_i(t_{ib}) = \dfrac{\alpha_i(1-\phi)\tau(t_{ib})}{1+\bar{b}} - (p_i - \alpha_i h) > 0$.

Case 2: Critical base payment. $p_i = \dfrac{\alpha_i(1-\phi)\tau(t_{ib})}{1+\bar{b}} + \alpha_i h.$

In this case, we first suppose $b_i(t_{ib}) < \bar{b}$, then by (1), $\eta(t_{ib}) = 0$. From (24) we have

$$\begin{aligned} & 1 + b_i(t_{ib}) = \frac{\alpha_i(1-\phi)\tau(t_{ib})}{p_i - \alpha_i h} = 1 + \bar{b} \\ \Rightarrow \quad & b_i(t_{ib}) = \bar{b}, \end{aligned}$$

which is a contradiction. Thus $b_i(t_{ib}) = \bar{b}$. It still holds that $\eta(t_{ib}) = 0$ in this case.

Case 3: High base payment. $\dfrac{\alpha_i(1-\phi)\tau(t_{ib})}{1+\bar{b}} + \alpha_i h < p_i \le \alpha_i(1-\phi)\tau(t_{ib}) + \alpha_i h.$

From (24), we have

$$1 + b(t_{ib}) = \frac{\alpha_i(1-\phi)\tau(t_{ib})}{p_i - \alpha_i h + \eta(t_{ib})} \le \frac{\alpha_i(1-\phi)\tau(t_{ib})}{p_i - \alpha_i h} < 1 + \bar{b}.$$

Then for this case, $b(t_{ib}) < \bar{b}$ and $\eta(t_{ib}) = 0$. The starting ridesharing ratio can be calculated as $b_i(t_{ib}) = \dfrac{\alpha_i(1-\phi)\tau(t_{ib})}{p_i - \alpha_i h} - 1 < \bar{b}$.

Case 4: Very high base payment. $p_i > \alpha_i(1-\phi)\tau(t_{ib}) + \alpha_i h$.

In this case, the generalized cost for a passenger is higher than that of a driver, so there are only solo-drivers at time t_{ib}. $b(t_{ib}) = 0 = \eta_i(t_{ib})$.

The aforementioned cases for $t = t_{ib}$ helps to determine the type of the time segment (solo-driver, ridesharing with vacancy or full ridesharing) for the beginning time segment of group i. In fact, it can be easily extended to any time instant t other than t_{ib}, so that one can track any time instant and find the type of the corresponding time segment.

3.5 Demand conservation

In this study, we assume the total demand of each group of ridesharing travelers is exogenously given and fixed. The total demand of each group is split into the early and late arrival parts, while in each part, there may be both drivers and passengers.

As discussed before, for group i, the departure time is divided into up to six time segments, and within each segment, the expression of demand rates (see (22) and (23)) and ridesharing ratios (see the concluding remarks in Section 3.2) are different. The total demand of group i is the summation of all travelers demand of group i in these six time segments, i.e.,

$$\begin{aligned}
\overline{n}_i \quad &= (t_{ib}^0 - t_{ib})d_i^1 \\
&+\frac{1}{2}\left(1 + b(t_{ib}) + 1 + b(t_{ib}^{\overline{b}})\right)(t_{ib}^{\overline{b}} - t_{ib}^0)d_i^2 \\
&+(1+\overline{b})(t_{i-1,b} - t_{ib}^{\overline{b}})d_i^3 \\
&+(1+\overline{b})(t_{ie}^{\overline{b}} - t_{i-1,e})d_i^4 \\
&+\frac{1}{2}\left(1 + b(t_{ie}) + 1 + b(t_{ie}^{\overline{b}})\right)(t_{ie}^0 - t_{ie}^{\overline{b}})d_i^5 \\
&+(t_{ie} - t_{ie}^0)d_i^6
\end{aligned} \tag{26}$$

Analytically, by solving all the beginning and ending time instants of the time segments from the above constraints listed in this Section, we can obtain the exact departure times, as well as all dynamics, including travel time $\tau(t)$, ridesharing ratios $b_i(t)$ and actual ridesharing payments $p_i + \eta(t)$ at any specific time instant t. However, solving such a problem analytically involves dealing with the recursive representation of the time instants (e.g., the travel time of the beginning time of the early arrival part of group i is determined by the travel time of the ending time of the early arrival part of group $i+1$), and nonlinear terms may occur in the demand conservation constraints. Furthermore, there are multiple 'if-then' conditions imposed by the complementarities and the division of the time segment due to different states of the ridesharing ratios that further complicate the problem. Due to the above mentioned reasons, although in theory it is possible to obtain an analytical solution to the problem from those constraints, for multiple heterogeneous ridesharing traveler groups, analytical solution would acquire significant derivation efforts.

To simplify the solution procedure, we can instead solve the problem numerically. In the next Section, we reformulate the heterogeneous ridesharing problem in a differential complementarity system (DCS), and then discretize the system to a complementarity system for numerical solutions.

4 Equivalent DCS Formulation and Its Discretization

In this Section, we first briefly introduce the DCS, then reformulate the problem into a DCS, where the bottleneck queue dynamics is differentiable almost everywhere, and all the other constraints are formulated as complementarities. The DCS is then discretized to a nonlinear compplementarity system, which can be solved numerically by commercial solvers such as PATH in GAMS (Ferris and Munson, 1998).

4.1 A brief introduction of DCS

Differential complementarity systems (DCSs) are constrained dynamical systems, which is different from the traditional unconstrained ordinary differential equations (ODEs). DCSs were introduced in Pang and Stewart (2009) as a mathematical paradigm that combines the explicit form of an ODE augmented by a finite-dimensional complementarity condition on an algebraic variable, as shown in the following form:

$$\begin{aligned}
\dot{q}(t) \quad &= \quad f(t, q(t), u(t)) \\
0 \le u(t) \quad &\perp \quad g(t, q(t), u(t)) \ge 0,
\end{aligned} \tag{27}$$

where $a \perp b$ means $ab = 0$. The $\perp$ notation is a shorthand for the complementarity relation between two nonnegative scalars. The introduction of the DCS enables the application of the advances in the field of mathematical programming, such as the subjects of complementarity problems and variational inequalities Cottle et al. (2009); Facchinei and Pang (2003), to effectively deal with constrained time-evolutionary physical and economic systems as well as optimal control problems.

4.2 DCS formulation

1. Demand conservation

For each group $\forall i \in \mathcal{I}$ and any time $\forall t \in [0, T]$,

$$n_i(t) = \int_0^t d_i(t)\left[1 + b_i(t)\right] dt \tag{28}$$

2. Demand and common disutility

Any vehicular demand in the same group should share the same generalized cost (the common disutility) at equilibrium. Note that the common disutility for each group is not predefined and should be solved by the system endogenously. At equilibrium, if there is positive demand at time t for group i, the generalized cost of travelers entering the bottleneck link at time t in group i should be equal to the general cost of any travelers entering the bottleneck link at any time in the same group i, which is exactly the common disutility. If the generalized cost is higher at some time t than the common disutility, then there would be zero demand. The following complementarity presents the demand equilibrium for each group. For $\forall i \in \mathcal{I}$ and $\forall t \in [0, T]$,

$$0 \le d_i(t) \perp P_i^v(t) - P_i \ge 0 \tag{29}$$

The cumulative demand at the terminal time should meet the total demand. Suppose violation of such constraints on cumulative demand occurs, an unrealistic large common disutility would appear. So we use the following constraint to enforce the cumulative demand to be equal to the total demand for each group.

For $\forall i \in \mathcal{I}$,

$$0 \le \overline{P} - P_i \perp \overline{n}_i - n_i(T) \ge 0 \tag{30}$$

3. Ridesharing

For each group, the cost of a ridesharing passenger is no less than that of a driver departing at the same time. When the cost of a ridesharing passenger is strictly higher than that of a driver, there is no ridesharing passenger at that time. When there are ridesharing passengers, at the same time the costs of a passenger and a driver should be equal to each other.

For $\forall i \in \mathcal{I}$ and $\forall t \in [0, T]$,

$$0 \le b_i(t) \perp P_i^p(t) - P_i^v(t) \ge 0 \tag{31}$$

When the passenger seats are full, i.e., the ridesharing ratio reaches its maximum, a potential raise of the rideshaing payment is presented as the slack variable $\eta_i(t)$. For $\forall i \in \mathcal{I}$ and $\forall t \in [0, T]$,

$$0 \le \eta_i(t) \perp \overline{b} - b_i(t) \ge 0 \tag{32}$$

To construct the nonnegativity of the ridesharing ratio into a complementarity form, we assume that there is a 'reverse' payment from a driver to a would-be passenger, when there is no passenger (i.e., the ridesharing ratio is zero). The reverse payment is presented as the slack variable $\eta_i^-(t)$. For $\forall i \in \mathcal{I}$ and $\forall t \in [0, T]$,

$$0 \le \eta_i^-(t) \perp b_i(t) \ge 0 \tag{33}$$

4. Costs of drivers and passengers

As defined in (2), the cost of a driver in group i entering the bottleneck at time t is calculated as for $\forall i \in \mathcal{I}$ and $\forall t \in [0, T]$,

$$P_i^v(t) = \alpha_i\left[(1 + b_i(t))h + \tau(t)\right] + F(t + \tau(t)) - b_i(t)\left[p_i(t) + \eta_i(t) - \eta_i^-(t)\right]. \tag{34}$$

The cost of a passenger in group i entering the bottleneck at time t is calculated according to (5): for $\forall i \in \mathcal{I}$ and $\forall t \in [0, T]$,

$$P_i^p(t) = \alpha_i\left[\phi\tau(t)\right] + F(t + \tau(t)) + \left[p_i(t) + \eta_i(t) - \eta_i^-(t)\right]. \tag{35}$$

The only differences between the above definitions in (34) and (35) and those in (2) and (5) are the introduction of $\eta_i^-(t)$, which is a mathematical construct to ensure the nonnegativity of $b_i(t)$. Note that $\eta_i^-(t) = 0$ as long as $b_i(t) > 0$. Since $\eta_i^-(t) > 0$ only if $b_i(t) = 0$ (no passenger), we can see that $\eta_i^-(t)$ is nominal and does not affect the actual payments or the costs of travelers. Such a mathematical construct, along with the complementarity (31), equalize the passenger and driver cost, and if would-be passenger cost is higher than driver cost, then no ridesharing occurs.

5. Arrival penalties

Here we reformulate the arrival penalties $F(t+\tau(t))$ to be a complementarity expression for the case that $\Delta = 0$ by following a procedure simplified from Ma et al. (2017).

First we rewrite $t+\tau(t)-t^* \triangleq \zeta^-(t)+\zeta^+(t)$, where the two new variables $\zeta^{-,+}(t)$ satisfy the following complementarity condition

$$0 \;\le\; -\zeta^-(t) \perp \zeta^+(t) \;\ge\; 0.$$

Since $\zeta^-(t) = t+\tau(t)-t^*-\zeta^+(t)$, the arrival penalty can be written as a function of the arrival time $t+\tau(t)$ and the added slack variables $\zeta^+(t)$ and $\zeta^+(t)$. For $\forall t \in [0,T]$,

$$F(t+\tau(t)) = -\beta\left(t+\tau(t)-t^*-\zeta^+(t)\right)+\gamma\zeta^+(t) \tag{36}$$

The added complementarity condition is rewritten to eliminate $\zeta^-(t)$. For $\forall t \in [0,T]$,

$$0 \;\le\; t^*-t-\tau(t)+\zeta^+(t) \perp \zeta^+(t) \;\ge\; 0. \tag{37}$$

6. Bottleneck queue and travel times

The bottleneck queue is modeled as a continuous-time point queue. The inflow rate of the point-queue is the demand rate of drivers in all groups, which is the sum of the demand rates of all individual groups,

$$d(t) = \sum_{i\in\mathcal{I}} d_i(t) \tag{38}$$

According to the DCS point-queue in continuous-time as introduced in Ban et al. (2012a), the point-queue dynamics are given as

$$\begin{array}{llll} & \dot{q}(t) & = & d(t-\tau^0)-v(t) & \\ 0 \le & \mu(t) & \perp & q(t) & \ge 0 \\ & v(t) & = & s-\mu(t) & \end{array} \tag{39}$$

where $\mu(t)$ is a slack variable for the exit flow, and $v(t)$ is the exit flow rate at time t.

The point-queue link travel time is calculated as in (18).

The complete differential complementarity system should include the queue dynamics (39), and all complementarity conditions (28)-(33) and (37), with the link travel time (18), reformulated arrival penalty (36) and cost functions (34) (35) substituted.

4.3 Time discretization of the DCS

The DCS is with infinite dimensions so that discretization is needed for numerical calculation. The discretization for the control and slack variables, including the ridesharing ratios, flow rates and etc., is straight forward. On the other hand, for the state variable, i.e., the queue length, the candidate discretization schemes are either implicit or explicit; see the discussion on the discretization schemes for point queue models in Ma et al. (2015). In this study, we adopt the implicit discretization scheme by applying the Euler backward difference for the queue dynamics. More specifically, the discrete queue length at time step $r+r^0$ is

$$q^{r+r^0} = \sum_{0\le r'\le r}\sum_{i\in\mathcal{I}}\left(d_i^{r'}-s+\mu^{r'+r^0}\right)\Delta t$$

where Δt is the time step length, $r^0 = \frac{\tau^0}{\Delta t}$.

Note that the desired time step length should be properly selected, so that the free-flow travel time of the highway τ^0 should be integer multiples of the time step length. Furthermore, the first time step $r=0$ should correspond to proper time instant in the continuous time, so that the entire study time period $[t_{Mb}, t_{Me}]$ is included in the discrete time steps. Since similar discretization schemes can be referred in Ma et al. (2017) and Ban et al. (2012b), and detailed discrete model is thus omitted in this report.

5 System Optimal Shared Parking Cost for Ridesharing Travelers

Parking cost can be a significant part of the travel cost for morning commuters who work in downtowns. Sharing the cost of parking with the driver can reduce the overall commuting cost for passengers and therefore their choice of travel mode (drive alone or sharing a ride). In this Section, we explore how the dynamic parking charges can shift heterogeneous ridesharing travelers to reach the system optimum (SO). It is found that the time-varying parking charges can be piecewise linear in early and late arrival time segments, respectively, which is different from the linear pattern for the solo drivers.

5.1 Generalized costs in the SO scenario

In this report, the parking charge $\hat{k}(t)$ is shared equally among the travelers in one vehicle arriving at time t. Given the FIFO condition, we define the parking charge for the travelers departing at time t as $k(t)$, so that $k(t) = \hat{k}(t + \tau(t))$. For the SO case, $\tau(t) = \tau^0$, so that $k(t)$ is with a pure time shift from $\hat{k}(t)$, $k(t) = \hat{k}(t + \tau^0)$. For travelers departing at time t, the parking charge is shared among travelers, so that each traveler in group i pays $\dfrac{k(t)}{1 + b_i(t)}$ more than the costs defined in (2) and (5). The costs of passengers and drivers departing at time t are defined as $P_i^{vk}(t)$ and $P_i^{pk}(t)$, respectively, for the SO scenario as

$$\begin{aligned} P_i^{vk}(t) &= \alpha_i \left[(1 + b_i(t))h + \tau^0\right] + F(t + \tau^0) - b_i(t)\left[p_i + \eta_i(t)\right] + \tfrac{k(t)}{1+b_i(t)} \\ P_i^{pk}(t) &= \alpha_i \phi \tau^0 + F(t + \tau^0) + (p_i + \eta_i(t)) + \tfrac{k(t)}{1+b_i(t)}. \end{aligned} \tag{40}$$

5.2 Ridesharing ratios and surcharge payments

With the shared parking charges, the costs of passengers and the corresponding driver should be equal, i.e., $P_i^{vk}(t) = P_i^{pk}(t)$, when the passengers' demand is positive. Further, since $\eta(t) \geq 0$, we have

$$1 + b_i(t) = \frac{\alpha_i(1-\phi)\tau^0}{p_i - h + \eta_i(t)} \leq \frac{\alpha_i(1-\phi)\tau^0}{p_i - \alpha_i h}.$$

Similar to the cases in the equilibrium without parking charges, different selections of fixed base payment p_i can lead to different ridesharing ratio $b_i(t)$.

Case 1: If $p_i < \dfrac{\alpha_i(1-\phi)\tau^0}{1+\bar{b}} + \alpha_i h$, then $\eta_i(t) = \dfrac{\alpha_i(1-\phi)\tau^0}{1+\bar{b}} - p_i + \alpha_i h > 0$, $b_i(t) = \bar{b}$.

Case 2: If $p_i = \dfrac{\alpha_i(1-\phi)\tau^0}{1+\bar{b}} + \alpha_i h$, then $\eta_i(t) = 0$, $b_i(t) = \bar{b}$.

Case 3: If $\dfrac{\alpha_i(1-\phi)\tau^0}{1+\bar{b}} + \alpha_i h < p_i \leq \alpha_i(1-\phi)\tau^0 + \alpha_i h$, then $\eta_i(t) = 0$, $b_i(t) = \dfrac{\alpha_i(1-\phi)\tau^0}{p_i - \alpha_i h} - 1 < \bar{b}$.

Case 4: If $p_i > \alpha_i(1-\phi)\tau^0 + \alpha_i h$, then $\eta_i(t) = 0 = b_i(t)$.

Detailed proofs for these four cases are omitted since they are similar to the cases without parking charges.

It is interesting that all three cases indicate that the ridesharing ratio $b_i(t)$ and the surcharges $\eta_i(t)$ for each group are always constants in the SO scenario. A special case is that when the value of p_i is proportional to α_i, the ridesharing ratios of all groups would equal; In other words, if

$$\frac{\alpha_i}{p_i} = \frac{\alpha_j}{p_j}, \forall i, j \in \mathcal{I},$$

then $\forall t \in [0, T]$ and $\forall i, j \in \mathcal{I}$, $b_i(t) = b_j(t) = b$ (constant). In this case, the vehicular flow in the SO scenario with a demand of $\overline{n}_i$ for group i is reduced into that of solo drivers with a demand of $\overline{n}_i/b$ for group i. The time-varying parking charge is a proportion of the dynamic tolling that was discussed in van den Berg and Verhoef (2011). Specifically, the dynamic parking charge for each vehicle is with a slope of $(1 + b)\beta$ in the early arrival time segment, and a slope of $-(1 + b)\gamma$ in the late arrival time segment.

If p_i is not proportional to α_i, the ridesharing ratios of heterogeneous groups would be different, and the departure order of heterogeneous groups would depend on their corresponding ridesharing ratios, which is discussed in the following subsection.

5.3 Dynamic parking charges and the order of departure

We here discuss the case where p_i is not proportional to α_i. Since the ridesharing ratio and the surcharge are both constants for each group, without loss of generality, we set the ratios for group i $b_i(t)$ as $b_i \in [0, \bar{b}]$ and the surcharge $\eta_i(t)$ as η_i. Note that b_i and η_i are determined by p_i as shown above.

For the time t when the vehicular demand is positive, $\dot{P}_i^{vk}(t) = 0$ for $i \in \mathcal{I}$, which is, for $i \in \mathcal{I}$,

$$\frac{dF(t+\tau^0)}{dt} + \frac{1}{1+b_i}\frac{dk(t)}{dt} = 0.$$

Since

$$\frac{dF(t+\tau^0)}{dt} = \begin{cases} -\beta, & \text{if } t+\tau^0 < t^*; \\ \gamma. & \text{if } t+\tau^0 > t^*. \end{cases}$$

The derivative of parking charges $dk(t)/dt$ for the departure time segment of group i would be

$$\frac{dk(t)}{dt} = \begin{cases} (1+b_i)\beta, & \text{if } t+\tau^0 < t^*; \\ -(1+b_i)\gamma. & \text{if } t+\tau^0 > t^*. \end{cases} \tag{41}$$

For the bottleneck with ridesharing travelers, the dynamic parking charge would be no longer linear for either the entire early or entire late arrival time segment. We note the beginning time of early arrival time segment for group i as t_{bi}^k, and the ending time of late arrival time segment for group i as t_{ei}^k. We further define the critical time $t^k = t^* - \tau^0$.

It is possible that such an order would be different from the arrival order without parking charges (i.e., the decreasing order of α_i) for early arrivals. In fact, the groups would be ordered by increasing values of b_i for early arrivals. In the following, we take two groups of ridesharing travelers as an example to show this.

As noted, $\alpha_1 < \alpha_2$, so that group 2 departs earlier than group 1 for early arrival time segment in the equilibrium case without dynamic parking charges. If p_i is not proportional to α_i, $b_1 \neq b_2$, we then have the following two possible cases, $b_1 > b_2$ or $b_1 < b_2$. We mainly discuss the first case in detail. For the second case, simply switching groups 1 and 2 and all analysis still holds.

Case 1: $b_1 > b_2$. Under a successful dynamic parking charge, any traveler would not save her generalized cost by switching her departure time. We will show that by contradiction, group 2 arrives first in the early arrival.

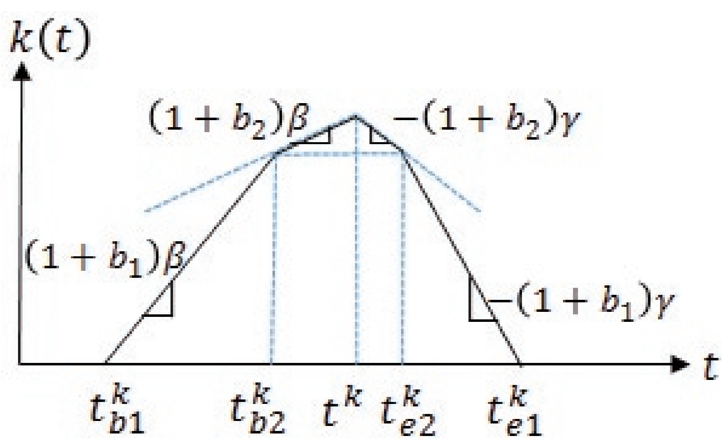


Figure 2: A hypothetical dynamic parking charges profile for SO with two heterogeneous groups, $b_1 > b_2$

Suppose group 1 arrives first in the early arrival, and thus group 2 arrives closer to the preferred arrival time t^*. From (41), the corresponding parking charges can be derived, which are illustrated in Figure 2. Since for the time segment $[t_{b2}^k, t^k)$, the vehicular demand of group 2 is $d_2(t) = s > 0$, we have $\dot{P}_2^{vk}(t) = 0$ for $t \in [t_{b2}^k, t^k)$, i.e, $P_2^{vk}(t_1) = P_2^{vk}(t_2)$ for $t_1, t_2 \in [t_{b2}^k, t^k)$.

For the vehicular demand departing at time t_{b2}^k, switching their departure time from t_{b2}^k to $t_{b2}^k - \Delta t$, $\Delta t > 0$, the generalized cost (including shared parking charge) should not decrease, which makes the current parking charge stable for SO.

$$\begin{aligned} P_2^{vk}(t_{b2}^k - \Delta t) - P_2^{vk}(t_{b2}^k) &= F(t - \Delta + \tau^0) - F(t + \tau^0) + \frac{1}{1+b_2}\left(k(t-\Delta t) - k(t)\right) \\ &= \beta \Delta t + \frac{-(1+b_1)\beta \Delta t}{1+b_2} \\ &= \left(1 - \frac{1+b_1}{1+b_2}\right)\beta \Delta t < 0. \end{aligned}$$

The above calculation shows the generalized cost after the departure time shift is lower than the original one, which means that the supposed order of arrival and the corresponding parking charges are not stable for SO.

So for the case $b_1 > b_2$, group 2 arrives first in the early arrival, and group 1 arrives closer to the preferred arrival time t^*. The correct parking charges for a stable SO can be derived from (41) and are illustrated in Figure 3.

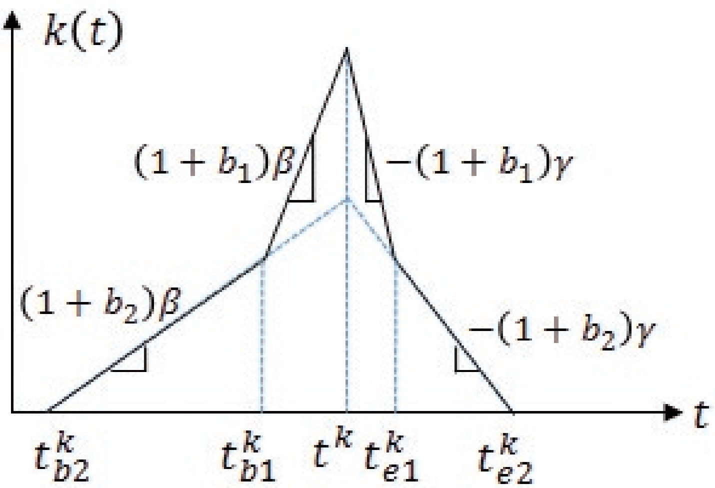


Figure 3: The stable dynamic parking charges profile for SO with two heterogeneous groups, $b_1 > b_2$

Case 2: $b_1 < b_2$. In this case, group 1 arrives first in the early arrival.

We can conclude from the two-group example that the order of departure in the early arrival time segment follows the increasing order of b_i. Such conclusion can be extended to multi-group scenarios, which is, the departure time of the group with smaller b_i is further from the center. Assume $b_{l_1} < b_{l_2} < \cdots < b_{l_M}$, where $l_j \in \mathcal{I}$, $j = 1, 2, \cdots, M$, then the beginning time of each group's early arrival time segment is

$$t_{bl_1}^k < t_{bl_2}^k < \cdots < t_{bl_M}^k < t^k,$$

and the ending time of each group's late arrival time segment is

$$t^k < t_{el_M}^k < \cdots < t_{el_2}^k < t_{el_1}^k.$$

5.4 Calculation of departure times

Compared with the equilibrium state without parking charges, it is relatively easier to calculate the critical departure times for the SO scenario, where the free-flow travel times for all groups are equal to the constant free flow travel time. Furthermore, for each group the vehicular demand rate is always equal to the constant flow capacity during its departure time periods, and the ridesharing ratio of each group is a constant as well.

From the demand conservation for group i, we have

$$\overline{n}_i = (1+b_i)\, s \left[(t^k_{b,i+1} - t^k_{bi}) + (t^k_{ei} - t^k_{e,i+1}).\right]$$

From the slopes of the dynamic parking charges in the early and late arrival time segments for each group, it interprets that for group i,

$$(t^k_{b,i+1} - t^k_{bi})\beta = (t^k_{ei} - t^k_{e,i+1})\gamma.$$

Substituting it into the demand conservation, we have

$$t^k_{b,i+1} - t^k_{bi} = \frac{\overline{n}_i}{(1+b_i)\, s(1+\frac{\beta}{\gamma})},$$

where b_i is derived from a given p_i.

Without loss of generality, we assume that the beginning time of each group's early arrival time segment is $t^k_{bl_1} < t^k_{bl_2} < \cdots < t^k_{bl_M} < t^k$, where $l_j \in \mathcal{I}$, $j = 1, 2, \cdots, M$. Then

$$\begin{aligned} t^k_{bl_j} &= t^k_{bl_{j+1}} - \frac{\overline{n}_{l_j}}{\left(1+b_{l_j}\right) s(1+\frac{\beta}{\gamma})} \\ t^k_{el_j} &= t^k_{el_{j+1}} + \frac{\overline{n}_{l_j}}{\left(1+b_{l_j}\right) s(1+\frac{\beta}{\gamma})} \cdot \frac{\beta}{\gamma} \end{aligned}$$

where we define $t^k_{bl_{M+1}} = t^k_{el_{M+1}} \triangleq t^k$.

5.5 Performance indicators

The vehicle-miles-traveled (VMT) and vehicle-hours-traveled (VHT) are referred as two performance indicators for the vehicles in a traffic system. Here we list a general calculation of the VMT and VHT for both the equilibrium and the system optimal cases.

$$\mathrm{VMT} = \int_{-\infty}^{+\infty} \sum_{i\in\mathcal{I}} v_h h(1+b_i(t))d_i(t) + v_\tau \tau^0 d_i(t)dt.$$

where v_h is the constant travel speed of the pick-up trips on local roads; v_τ is the free-flow speed (i.e., the maximum speed) of the bottleneck trips on the highway. Usually $v_h < v_\tau$. The VMT does not require the time-varying travel time function, since the distance traveled by each vehicle in the bottleneck trips are a constant, which is $v_\tau \tau^0$.

$$\mathrm{VHT} = \int_{-\infty}^{+\infty} \sum_{i\in\mathcal{I}} h(1+b_i(t))d_i(t) + \tau(t)d_i(t)dt.$$

The time-varying travel time on the bottleneck trips as a part of the VHT so that it is required to calculate the VHT.

For the SO scenario, the vehicular demand is constantly at flow capacity s. Assume $b_{l_1} < b_{l_2} < \cdots < b_{l_M}$, where $l_j \in \mathcal{I}$, $j = 1, 2, \cdots, M$, then we have

$$\mathrm{VMT} = \sum_{j=1,2,\cdots,M} \left(t^k_{bl_{j+1}} - t^k_{bl_j} + t^k_{el_j} - t^k_{el_{j+1}}\right)\left(v_h h(1+b_{l_j}) + v_\tau \tau^0\right)s.$$

Since the travel time is always the free-flow travel time in the SO scenario, we have the VHT simplified as

$$\mathrm{VHT} = \sum_{j=1,2,\cdots,M} \left(t^k_{bl_{j+1}} - t^k_{bl_j} + t^k_{el_j} - t^k_{el_{j+1}}\right)\left(h(1+b_{l_j}) + \tau^0\right)s.$$

6 Numerical results

In this Section, we solve the equivalent DCS formulation proposed in Section 3 with different settings on the base payments p_i.

We first introduce the parameter settings for the numerical examples. The study time span is $[0, T]$, and $T = 150$ minutes. The time step of the discretization is 1 minute. There are three groups with heterogeneous values of travel time α_i as 0.25, 1.25, 6.25 dollar per minute for $i = 1, 2, 3$, respectively. Each group has 5,000 travelers. The bottleneck flow capacity s is 90 vehicles per minute. The free-flow travel time on the bottleneck link is 5 minutes. The pick-up time for each passenger is selected as 0.5 minutes. The maximum ridesharing ratio, i.e., the maximum number of passengers in a vehicle is set as $\bar{b} = 4$, which is common for a sedan car. The arrival penalty parameters $\beta = 0.125$ and $\gamma = 0.375$, and the preferred arrival time $t^* = 110$ minute. The passengers can save 25% travel cost on the highway portion compared to driving, i.e., $\phi = 0.75$. The upper bound of a traveler's total commuting cost is set as $\overline{P} = 3000$ for all travelers, sufficiently large that it is not binding in the equilibrium solution.

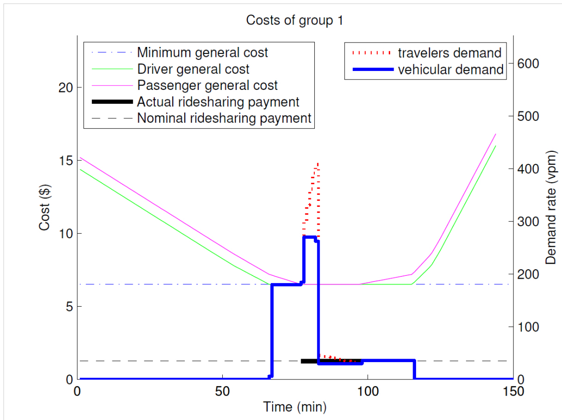
We will first show the solution details for the scenario where a positive universal base payment is imposed among all three groups. We then compare the group-specific generalized costs, the total time and parking costs, as well as the VMT and VHT for multiple scenarios with different base payment policies.

6.1 A universal base payment

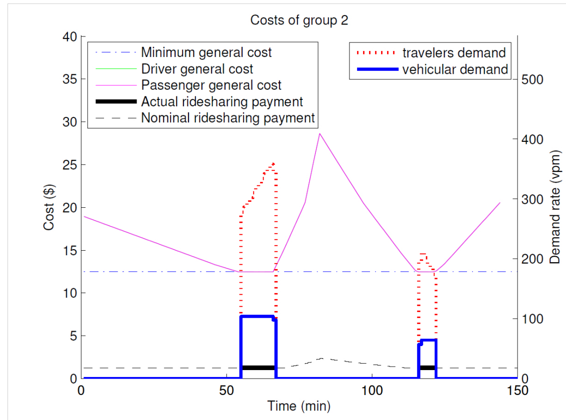
In this example we assume the base payment p_i is identical for all groups. Specifically, here we set $p_i = \$1.25$ for any group i. The total demand of each group is uniformly distributed, so that in each group there are 5,000 ridesharing travelers. We first discuss the equilibrium scenario without the SO parking charge.

Figure 4 shows the group-specific time-dependent demand rates, the ridesharing drivers' and passengers' costs, and the ridesharing payments among the ridesharing travelers. It shows that before time 82 minutes, the departing travelers arrive at the destination earlier than the preferred arrival time 110 minutes, while after 82 minutes, they arrive later than the preferred arrival time. It is found that at equilibrium, the departure time of group 1 is at the center, while that of group 3 is the furthest away from the center. Such a departure sequence follows the same order in the analysis in van den Berg and Verhoef (2011). Since in this report, we assume that groups share the same β and γ, respectively, the value-of-time is only applicable to the pick-up and travel times, and is not for arrival penalties. In this way, the group with a higher value-of-time tends to avoid queuing delays rather than arrival penalties; while the group with a lower value-of-time tends to avoid arrival penalties rather than queuing delays. This leads to heterogeneous preferences of the departure-time choices among groups with different value-of-time.

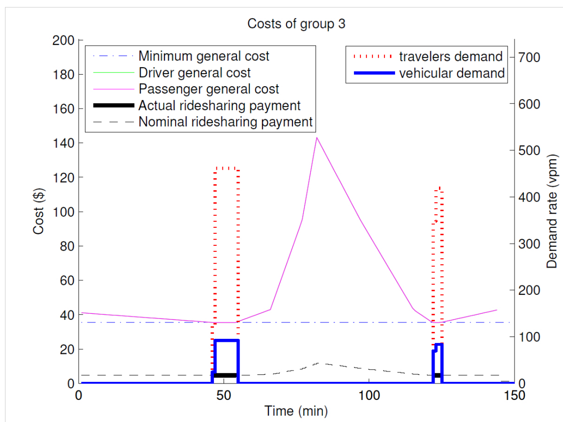
In Figure 4(a), the demand rates and the generalized costs are shown for group 1. At time 66 minutes, the vehicular demand rate increases to 5.913; and at time 66 minutes, it increases to 180. The demand rate could not directly increase to 180 from zero within one time step, because the actual starting time of the departure time segment of group 1 is not exactly on any discrete time step, and it is in fact between the discrete time 66 and 67 minutes. The exact actual starting time $t_{1b} = 67 - \frac{5.9133}{180} = 66.967$ minutes. Since the actual time instants are rarely coincidentally equal to an integer minute, i.e., a discrete time step, we can observe that between two different time segments, the demand rates may not finish the 'jump' within a single time step, and the demands of sequential groups can overlap at the boundary time step due to the similar reason. Such phenomena can be observed in any case due to the discretization 'rounding' error of the non-integral time instants. The time segment from 67 to 76 minutes is a solo-driver time segment, and the vehicular demand of group 1 is constantly 180 per minute, which is consistent with the analytical results in (22), as $d_1^1 = \left(1 + \frac{0.125}{0.15-0.125}\right) \times 90 = 180$, while there is no ridesharing passenger during this time period. This is because the passengers' generalized cost is always higher than that of the drivers', as shown by the purple and green lines (passengers' and drivers' generalized cost, respectively). Although during this time segment, there is a constant nominal ridesharing payment (as $p_1 = 1.25$), there is no actual payment since there are only solo-drivers in the system. The time segment from time 78 to 81 minutes is a ridesharing-with-vacancy time segment, and the vehicular demand is 270 per minute, which is consistent with (22), as $d_1^2 = \left(1 + \frac{0.125}{0.15\times0.75-0.125}\right) \times 90 = 270$, while the passengers' demand is no longer zero. The generalized costs of the passengers and the drivers are the same and equal to the generalized cost for group 1 at equilibrium



(a) Group 1, $\alpha_1 = 0.25$



(b) Group 2, $\alpha_2 = 1.25$



(c) Group 3, $\alpha_3 = 6.25$

Figure 4: Group-specific demands and costs

(P_1) during this time segment, when the demand is positive. At time 78 minutes, the travelers' demand (including passengers and drivers) is 301.5 per minute, and it increases linearly up to 407.7 per minute at time 82 minutes. The actual ridesharing payment is equal to the base payment p_1 during this time segment.

After 82 minutes, the departing travelers would arrive later than the preferred arrival time 110 minutes. During the time segment from time 83 to 97 minutes, the vehicular demand is 30 per minute, which is consistent with (23), as $d_1^5 = \left(1 - \frac{0.375}{0.15\times 0.75+0.375}\right) \times 90 = 30$, and is significantly lower than that of the early arrival counterpart (time 78 to 81 minutes). The travelers' demand decreases from 45.6 to 30.0 during the time segment from time 83 to 97 minutes. There are both passengers' and drivers' demand during this time segment, since the generalized costs of theirs are equal to each other. The ridesharing payment is still equal to the base payment $p_1 = 1.25$. From time 98 to 115 minutes, the passengers' generalized cost increase and is higher than its minimum P_1, so that there is no passengers' demand. The drivers' generalized cost, on the other hand, is equal to its minimum as P_1, and the vehicular demand is 36 per minute, which is consistent with (23), as $d_1^6 = \left(1 - \frac{0.375}{0.15+0.375}\right) \times 90 = 36$. Similar to its early arrival counterpart of the time segment from 67 to 76 minutes, since there are only solo-drivers in the system and thus no actual ridesharing payment is made.

The demand rates and the generalized costs for group 2 is shown in Figure 4(b). During the entire time span, the drivers' and passengers' generalized costs for group 2 are always equal to each other. During the ridesharing-with-vacancy time segments 55 to 66 minutes and 116 to 122 minutes, the generalized costs reach their minimum at equilibrium (P_2), so that the demand rates of both drivers and passengers are positive. The demand rate during time 55 to 66 minutes is 103.8 per minute, and it is 64.3 per minute during time 55 to 66 minutes. The demand rates are consistent with d_2^2 in (22) and d_2^5 in (23), respectively. The travelers' demand rates increase in the first time segment, and decrease in the second one.

The demand rates and the generalized costs for group 3 is shown in Figure 4(c). Similarly, during the entire time span, the drivers' and passengers' generalized costs for group 3 are always equal to each other. During the full-ridesharing time segments 47 to 54 minutes and 123 to 124 minutes, the generalized costs reach their minimum at equilibrium (P_3), so that the demand rates of both drivers and passengers are positive. The demand rate during time 47 to 54 minutes is 92.3 per minute, and it is 83.7 per minute during time 123 to 124 minutes. During these two time segments, the ridesharing ratio reaches its maximum $\bar{b} = 4$, and the surcharge is positive. According to Case 1 in 3.4, the surcharge $\eta_3(t)$ is a linear function with the respect to time t, ranging from \$3.44 to \$3.49 during time 47 to 54 minutes (as the travel time ranges from 5 to 5.18 minutes during this time segment). Note that the actual payments in Figure 4(c) are increasing from \$4.69 to \$4.74 (the slope is insignificant due to the scales) in the numerical results of the equivalent DCS are consistent with the above calculations. The demand rates are consistent with d_3^3 in (22) and d_3^4 in (23), respectively. The travelers' demand rates are constants in the early and late arrival time segments (with some rounding difference at the boundaries of time segments due to numerical approximation), respectively.

Note that for the generalized cost at equilibrium for each group is unique, which is $P_i \neq P_j, i \neq j$ in general. The generalized costs for group 2 in the time period between (67 to 115 minutes) are much higher, which is different from those for group 1 shown in Figure 4(a). This is also true for group 3, as shown in Figure 4(c). Due to such patterns of generalized costs, the departure times for group 1 are the closest to the center, and the group 3's departure times are the farthest from the center, which is consistent with the departure sequences of the analytical results.

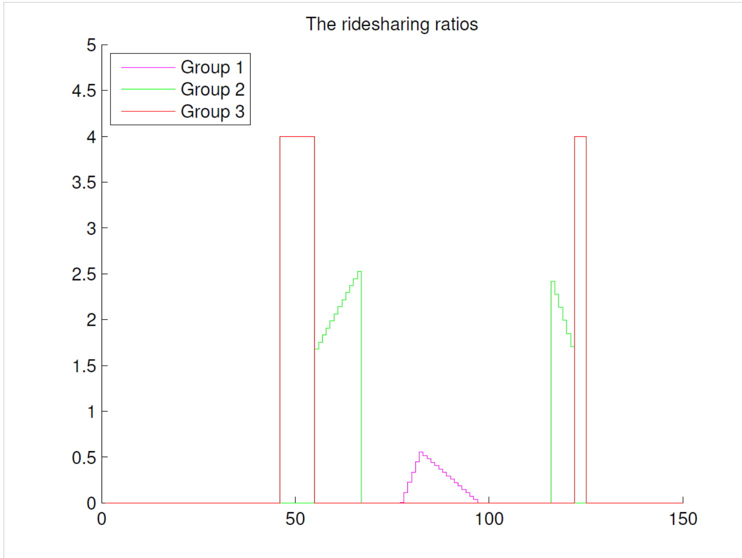
Figure 5 shows the ridesharing ratios of each group and their demand rates, as well as the travel time function (the isocost curve). Figure 5(a) tells that the ridesharing ratio is always at the maximum for group 3. The ridesharing ratio is increasing but never reaches the maximum during the early arrival time segment, and is decreasing during the late arrival time segment for group 2. Group 1 has a zero ridesharing ratio first, then it increases at the peak time, then decreases after that. Overall it is observed that the ridesharing participation of group 1 is the least, while group 3 fully participates ridesharing with fully occupancy in all vehicles. The travel time function shown in Figure 5(b) is a piecewise linear function, as proved in the analysis. In this figure, Group i-D refers to the drivers' demand rates in group i, and Group i-P refers to the passengers' demand rates in group i. The dash line is the travel time function.

We also calculate the exact solution for the scenario with SO parking charges, which is shown in Figure 6. The base payment is unchanged, according its equilibrium scenario without SO parking charges. The vehicular demand keeps at the bottleneck flow capacity of 90 vehicles per minute. Similar with the equilibrium without SO parking charges, group 3 has the highest ridesharing participation. An interesting observation is that the sequence of departing is altered in the SO case, where group 1's departure is the furthest from the center, while group 3's departure is at the center. The reason why group 1 is pushed away from the center is because it has the least ridesharing ratio in the SO scenario. As aforementioned, when the heterogeneous groups have different ridesharing ratio in the SO scenario, the group with a higher ratio is closer to the center (the peak time t^k). Only in this way, the parking charges are stable, which is also shown in Figure 6. The parking charges for group 1 differ from 0 to \$3.13 depend on the departure time. The parking charges for group 2 are higher, ranging from \$3.13 to \$13.54. The parking charges for group 3 is even higher, which starts at \$13.54 up to \$15.63. As observed, the slope of the parking charges increases as the time is closer to the peak time $t^k = t^* - \tau^0 = 105$ minutes.

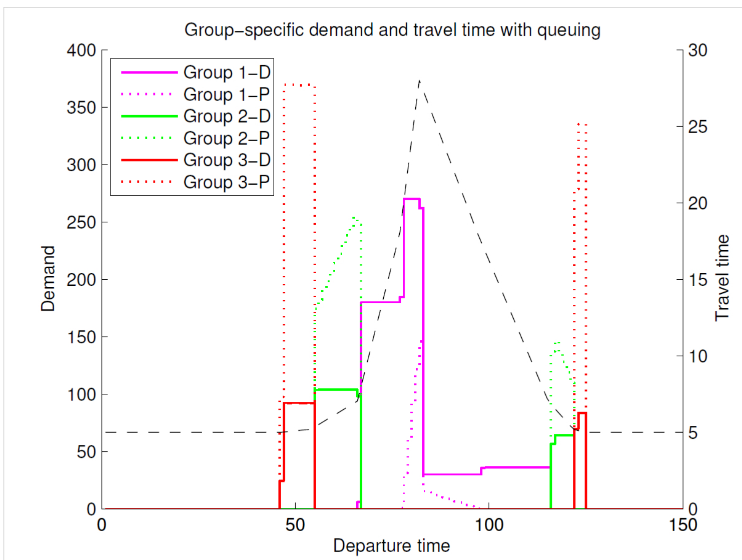
6.2 Comparison among different policies on the base payment

Here we show under different policies of the base payment, how the system performance and the costs vary. We first list six policies in Table 1.

In Policies 1, 4, 5 and 6, the base payments of all groups are different, which are proportional to some functions of the value of travel times. The functions vary from reciprocal, sub-linear, linear to quadratic forms. Policies 2 and 3 apply constant base payments, which are invariant to the change of the value of travel times. Policies 2 and 3 seem to be similar at the first glance; however, Policy 2 is quite different from all other policies, since zero base payment suggests the system has to be at the full ridesharing state for all groups, otherwise the generalized cost of a driver would be always higher than that of a passenger, and no



(a) Ridesharing ratios



(b) Demands and travel time

Figure 5: Ridesharing ratios, demands and travel times

travelers would be willing to drive. The results with Policy 2 are used as a benchmark, where all travelers are in fully occupied ridesharing vehicles, and the usage of vehicles are minimum.

Figure 7 shows the group-specific generalized cost for each traveler, where the generalized cost is the monetary costs including travel time, arrival penalties and the ridesharing payment (or gain for a driver).

In Figure 7(a) for the equilibrium scenarios without parking charges, we can find that under different base payment policies, the generalized costs of a traveler in Group 3 is always the highest due to their highest VOT; on the other hand, the generalized costs of each group can vary significantly due to the base payment policies.

It shows that some policies are relatively more preferable to certain groups than others. For instance, the quadratic proportional base payment (Policy 6) leads to a relatively lower generalized costs for Group 1, while the costs for Group 3 is penalized significantly by the same policy. Compared with Policy 6, the inverse proportional base payment (Policy 1) and the invariant base payment (Policy 3) lead to much higher

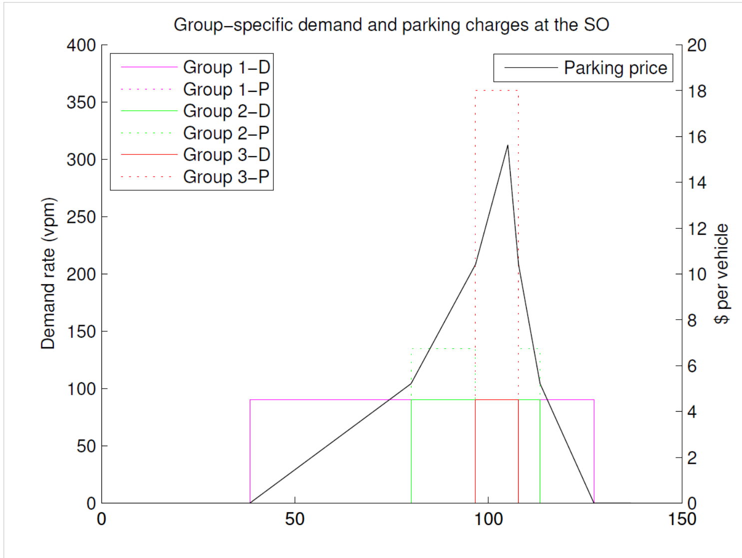


Figure 6: Group demands and dynamic SO parking charges

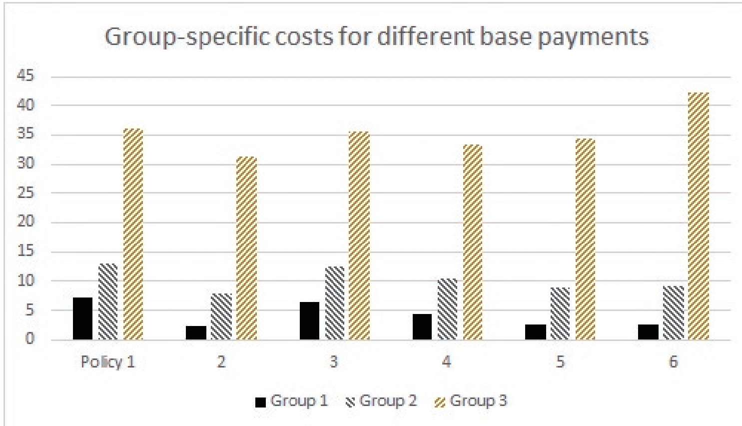
Table 1: Base payment policies

Policy	Relation to α_i	Expression, $i = 1, 2, 3$	Practical interpretation for the base payment
1	$p_i \propto {\alpha_i}^{-1}$	$p_i = 1.25\left(\frac{\alpha_2}{\alpha_i}\right)$	Travelers with higher VOT have less base payment.
2	$p_i = 0$, invariant to α_i	$p_i = 0$	No base payment.
3	$p_i > 0$, invariant to α_i	$p_i = 1.25$	Not vary over groups
4	$p_i \propto {\alpha_i}^{\frac{1}{2}}$	$p_i = 1.25\sqrt{\frac{\alpha_i}{\alpha_2}}$	Vary sub-linearly to the group VOT
5	$p_i \propto \alpha_i$	$p_i = 1.25\left(\frac{\alpha_i}{\alpha_2}\right)$	Vary linearly to the group VOT
6	$p_i \propto {\alpha_i}^2$	$p_i = 1.25\left(\frac{\alpha_i}{\alpha_2}\right)^2$	Vary quadratically to the group VOT

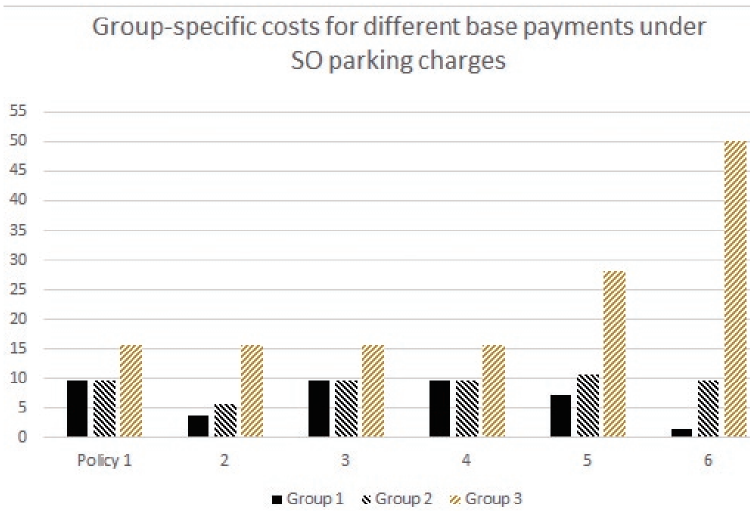
generalized costs for Groups 1 & 2, while the costs for Group 3 are reduced. So it suggests that Policy 6 is preferred by the ridesharing travelers with less VOT, while it is less preferred by those with higher VOT.

Another interesting observation is that among all tested policies, Policy 2 can lead to the least generalized costs for any group. This is because with Policy 2, all travelers are practically enforced to packed into the fully occupied ridesharing vehicles, which utilizes all available seats and thus decreases the vehicle usage and congestion.

It is shown in Figure 7(b) that when the system optimal parking charges are implemented in the system, the generalized costs of groups would change. It is found that for Policies 1-5, the SO parking charges would decrease the generalized costs for Group 3 significantly, and increase the generalized costs for Group 1. Under Policies 1-4, the SO parking charges also decrease the generalized costs for Group 2; while under Policy 5, it slightly increases the generalized costs for Group 2. It suggests that the travelers with higher VOT are beneficial from the SO parking charges under the base payments that are invariant, inverse proportional, sub-linearly or linearly proportional to the VOT. Such observation is coherent to the conclusion for the solo-drivers scenario in the literature, where the high VOT drivers gain, while the low VOT drivers loss benefits from the SO tolls or charges. On the other hand, we should not ignore that under Policy 6 where the base payment is quadratic proportional to the VOT, the SO parking charges decreases the generalized costs for Group 1 while increase those for Groups 2 and 3. Such observation is not seen from the solo-driver



(a) Group-specific costs for the UE scenario without parking charges



(b) Group-specific costs for the SO scenario with the SO parking charges

Figure 7: Group-specific generalized cost for each traveler with different base payment policies

scenarios in the literature.

The reason is that the SO parking charges in the ridesharing scenario is different from that in the solo-driver scenario. For the solo drivers, the SO tolls or charges are not shared among travelers, so that for the entire early /late arrival time segment, the tolls or charges are a linear function of the arrival time. If the parameters of arrival penalties (β and γ) are identical for all groups, respectively, then the departure time of any solo driver (regardless their VOT) can be arbitrary, as long as it is within the proper time span. Such arbitrary departure times benefits the high VOT solo-drivers by significantly reducing their arrival penalties, because without the SO charges, their departure times have to be the furthest from the center and the arrival penalties are higher. For the ridesharing travelers, the SO parking charges are shared among all travelers within a vehicle. As discussed in Section 5.3, due to the cost-sharing nature, the charges is piecewise linear (instead of linear) within the entire early or late arrival time segment. In this case, the departure time choices are not arbitrary,. The group with the highest ridesharing ratio (i.e., highest seat occupancy) would depart near the center, and arrive closer to the preferred arrival time. As the ridesharing ratio with the SO charges is determined by not only the VOT but also the base payments, the high VOT ridesharing travelers may not necesasarily switch their departure time choices closer to the center under the SO charges. In fact, under Policy 6, Group 3 is with the least ridesharing ratio under the SO charges, so they cannot benefit from the SO charges by shifting their departure times closer to the center.

From the perspective of the total cost of all travelers, Figure 8 shows the total time costs (including arrival

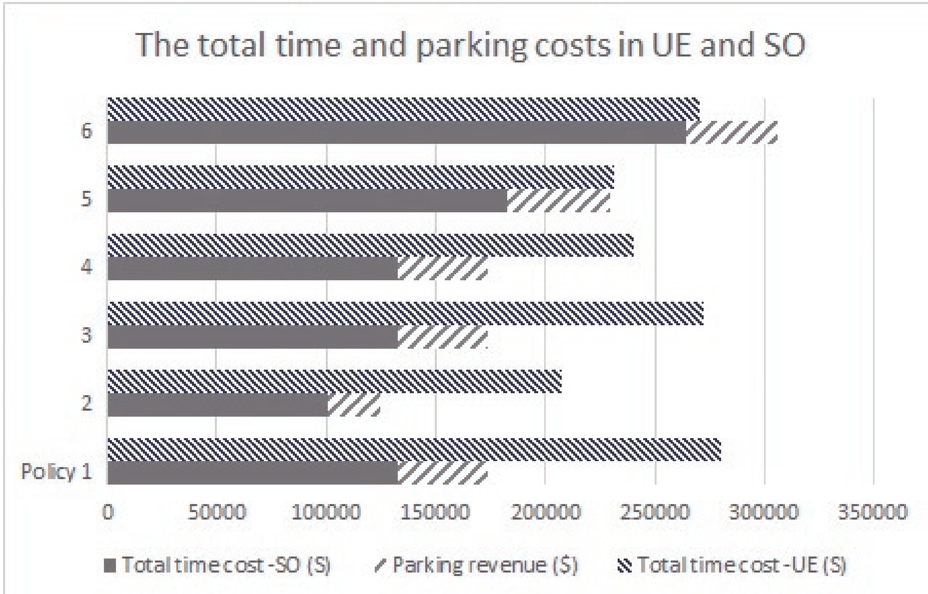


Figure 8: The total time costs of the UE and SO scenarios and the parking revenue with different base payment policies

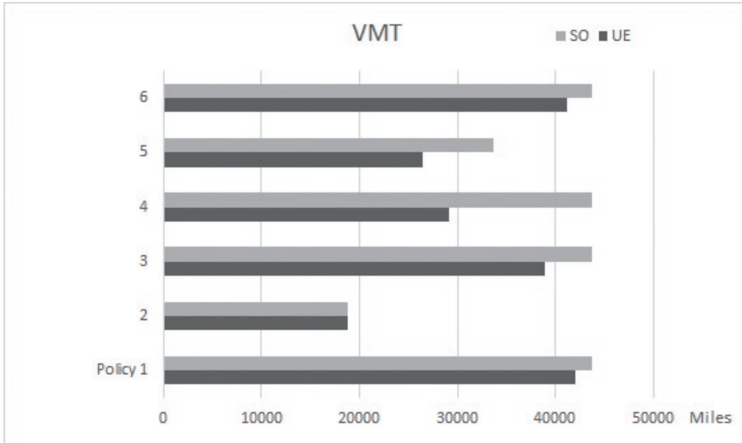
penalties and the travel time costs) at the equilibrium without parking charges, the total time costs when the SO parking charges are imposed, and the parking revenue (i.e., the total parking costs of the travelers) under different base payment policies. It shows that under Policies 1-5, the SO charges can decrease the system total cost, even considering the parking charges are costs of the system. As observed, the total time cost in the SO scenario plus the parking revenue is still less than the total time cost in the UE scenario for Policies 1-5. On the other hand, the parking charges under Policy 6 barely decreases the total time costs, while the total parking revenue is far more than the total time savings in this case. This suggests that the SO parking charges may not always save the total costs of the ridesharing travelers if the base payment policy and the parking charges are not well designed in a coherent manner.

Figure 9 shows the VMT and VHT for both UE and SO scenarios for all tested base payment policies. It is found from Figure 9(a) that except for the zero base payment policy (Policy 2), in general, the VMT increases when the SO charges are imposed. As the VMT indicates the usage of vehicles, it suggests that the SO charges in fact encourage less ridesharing. The SO charges eliminate the queuing congestion, and the travel time reaches the minimum (the free flow travel time) at all times. As discussed in Sections 3.4 and 5.2, the lower travel times in the SO scenario lead to a relatively lower ridesharing ratio, given the same base payment for each group.

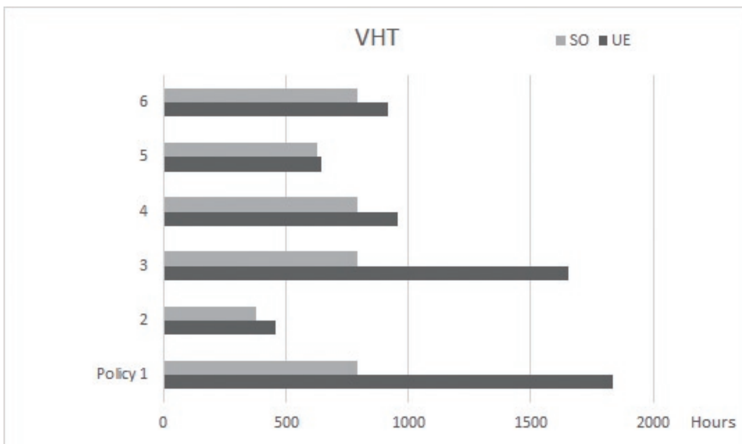
Figure 9(b) shows that under all tested base payment policies, the SO parking charges can always decrease the VHT, which is an indicator on the vehicular travel times. That is because the the queuing congestion is eliminated so that the travel times on the bottleneck highway is significantly reduced in general.

7 Conclusions

This report studied a continuous-time ridesharing problem for a single bottleneck in the morning commute scenario with heterogeneous traveler groups. The problem is analyzed in continuous-time form, and an equivalent differential complementarity system (DCS) is formulated and solved numerically. Through the analysis on the continuous-time formulations, we found that the travel time functions may not be linear within the early or late arrival time segment for each heterogeneous group. In fact, with the introduction of dynamic ridesharing ratios, the travel time function becomes piecewise linear for each group, and the kink points partitioning the piecewise functions are determined by the states of the ridesharing ratios. The states of the ridesharing ratios also influence the ridesharing payments and the demand rates of travelers in



(a) The VMT



(b) The VHT

Figure 9: The VMT and VHT of the UE and SO scenarios with different base payment policies

a similar manner. For instance, the demand rate is not a constant for each group. Instead, it is a piecewise step function partitioned by the states of the ridesharing ratios. Numerical solution of the equivalent DCS is consistent to the analytical results.

The ridesharing payment mechanism discussed in this report involves two components, namely, the base payment (exogenously given) and the surcharges. We found that the relationship between the base payment and the travel time function can determine the states of the ridesharing ratios (solo-driver, ridesharing with vacancy, and full ridesharing) and the corresponding partitions of the time segments. The connection of the base payment policy and the ridesharing participation merits more discussions. In general, our analysis shows that a lower base payment encourages ridesharing. However, a lower base payment is not necessarily referring to a lower actual ridesharing payment. In fact, with low base payments, the ridesharing drivers prefer to have their vehicles fully occupied, so that it is possible for them to receive surcharges to compensate their driving and pick-up costs. When the ridesharing ratio is high, the vehicle demand is reduced, as more travelers tend to become passengers so that ridesharing ratio can reach its maximum. An extreme case is that the base payment is set to zero. This is equivalent to enforce all vehicles to be fully occupied by ridesharing travelers, since if the ridesharing ratio is not at its maximum, the cost of a driver must be higher than that of a passenger and thus they are not at equilibrium. Another special case is that with a relatively high base payment and low travel time for all groups, ridesharing would not be encouraged. Then the problem is reduced to the solo driver morning commute problem. Apparently, traditional morning commute problem with heterogeneous solo drivers is a very simplified and special case of the proposed problem with ridesharing discussed in this report.

When the shared dynamic parking costs are introduced to eliminate the queuing delays, we found that the sequence of the departure order of heterogeneous groups is determined by the ridesharing ratios in the system optimum (SO), rather than the value-of-time at equilibrium (UE). Such departure orders are related to the sharing of parking charges among all ridesharing participators in a same vehicle. For a group with higher ridesharing participation, each ridesharing traveler pays relatively less parking charges if the parking charges would have constant slopes. As discussed in Section 5.3, a stable parking charge curve should have steeper slope for groups departing closer to the center, which results the aforementioned departure orders. We need to note that such order does not apply and was not discussed in the literature on traditional solo driver cases, since there is no shared parking costs among solo-drivers. Numerical results further confirmed such analysis, where the group with higher ridesharing participation departs closer to the center.

From the system performance indicators incluing VMT, VHT, the group-specific and total generalized costs, we found that the low VOT group do not always loss due to the SO parking charges, when certain base payment policy was adopted. This is quite different from the solo-driver cases, where the low VOT loss from the SO charges. Specifically, it is found that compared to a positive fixed base payment invariant to group VOT, fixing the base payment to zero and varying the base payment quadratically to the group VOT are two policies that benefit the groups with low VOT in both SO and UE scenarios. While on the other hand, the base payment varied quadratically to VOT can lead to the highest costs for the high VOT group in both SO and UE scenarios. The different group-specific costs performances are due to the ridesharing and the corresponding departure order of heterogeneous groups. The results suggested that the base payment policy and the parking charges should be carefully designed in a coherent manner so that the SO parking charges can effectively reduce the total costs of the ridesharing travelers.

In the future research, heterogeneous groups with different VOT and arrival penalty parameters with more realistic consideration will be analyzed. Given extra dimensions of parameters, the analysis can be more complex, but the methodologies in this study can still apply.

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