Linear Systems – Review Notes

adapted from notes by Michael Braun

Typically in electrical engineering, one is concerned with functions of time, such as a voltage waveform. System description is therefore defined in the domains of time and temporal frequency. In these notes, a more general approach is taken. The generalized independent variable x may be time, but it may also represent spatial position in one dimension (1D). If treated as a vector, x may also stand for multidimensional quantities such as position in 3D space.

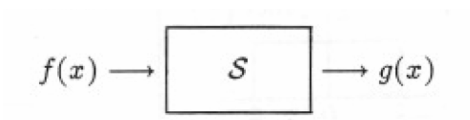
Let $f(x)$ be a function of the generalized variable x. We will assume that the function satisfies the existence conditions for the Fourier transform. We will denote the Fourier transform of $f(x)$ by $F(s)$,

$$F(s) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{i2\pi s x}\, dx\,,$$

where $i = \sqrt{-1}$. Note that if x is time, then s is the temporal frequency measured in cycles per second. If, on the other hand, x is the spatial position, then s is the "spatial frequency" measured in cycles per unit length.

System

A system is anything we may care to examine that can be characterized by a black box. When the system receives a stimulus (input function $f(x)$), it produces a response (output function $g(x)$),



The system can be represented mathematically by a system operator S which maps the input functions to output functions,

$$g(x) = S\{f(x)\}$$

Linear System

A system is said to be linear if the response to a sum of two different inputs is a sum of the responses produced separately by each input. It also follows that scaling the input scales the output by the same factor. Thus a system is linear if S is a linear operator, that is,

$$S\{\alpha\, f_1(x) + \beta\, f_2(x)\} = \alpha\, S\{f_1(x)\} + \beta S\{f_2(x)\}$$

where α, β are constants. This naturally extends to any finite sum of weighted functions,

$$S\left\{\sum_{i=1}^{n} \alpha_i\, f_i(x)\right\} = \sum_{i=1}^{n} \alpha_i\, S\{f_i(x)\}.$$

A system is said to possess *extended linearity* if the above holds when *n* is infinite and when the summation is replaced by integration. For the latter case, we have

$$S\left\{\int_a^b \alpha(x')\, f(x,x')\,dx'\right\} = \int_a^b \alpha(x')\, S\{f(x,x')\}\,dx'$$

Physical systems are never strictly linear. Nevertheless, many physical systems can be approximated, at least in part, by a linear system. There are also many systems that are deliberately nonlinear. A logarithmic amplifier is clearly an example of a nonlinear system since

$$\log\left[\alpha f_1(x) + \beta f_2(x)\right] \neq \alpha \log f_1(x) + \beta \log f_2(x).$$

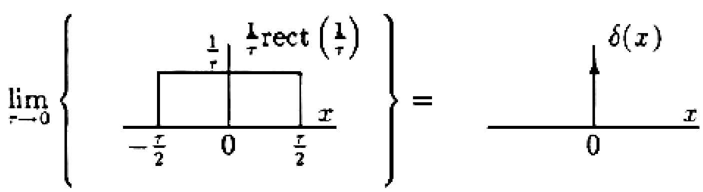
Delta Function

The Dirac delta function $\delta(x)$ is used to define an impulse. It is zero everywhere except at $x = 0$ where it is infinite and its integral is unity,

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{and } \int_{-\infty}^{\infty} \delta(x)dx = 1$$

By convention, the delta function is interpreted as a limit of a function whose width vanishes as its height rises to infinity while the area under the function is unity. Taking the rectangle function, for example, we can have the following definition:



The delta function may also be defined by its *sifting* property, whereby an operation on $f(x)$ sifts out a single value $f(0)$,

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0)$$

Shifting the delta function to $x = a$ in the expression above will have the effect of sifting the function f at that point,

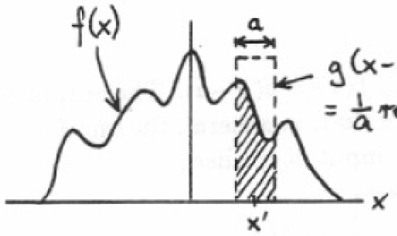
$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a).$$

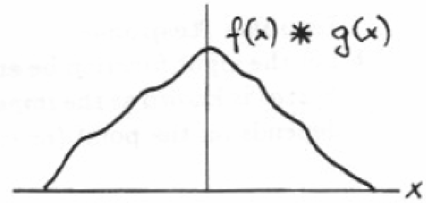
Convolution

The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x')g(x-x')\,dx'.$$

A useful way of interpreting convolution is to consider it as a smoothing operation. The simplest case of smoothing is known as the *moving average* where the smoothing function is a rectangular window.



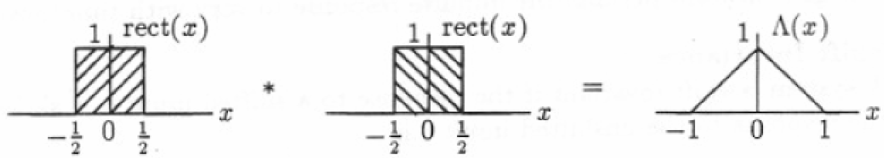


The moving average is obtained by placing the window $g(x) = (1/a)\text{rect}(x/a)$ at a point $x = x'$, then computing the average within the window. The process is repeated as the window is moved to each new value of x'. The result of the moving average operation is a smoother and more spread out function. If the window function is allowed to take any form, then the moving average will generalise to a convolution.

The *graphical algorithm* for performing convolution is as follows:

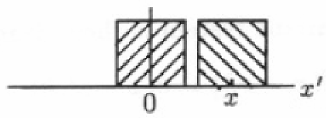
1. take $g(x')$ and flip to get $g(-x')$;
2. shift to right by x to get $g(x - x')$;
3. multiply by $f(x)$;
4. integrate the product;
5. repeat above steps for every point x.

Example: $h(x) = \text{rect}(x) * \text{rect}(x)$. The result of convolving a rectangle function with itself is a triangle function:



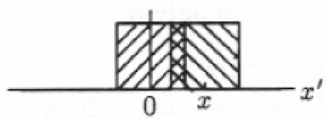
To ascertain that the above is true, consider the three cases:

$|x| > 1$



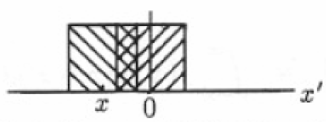
$\Longrightarrow h(x) = 0$

$0 < x < 1$



$\Longrightarrow h(x) = 1 - x$

$-1 < x < 0$



$\Longrightarrow h(x) = x + 1$

It can be shown that the convolution has the following properties:

commutation	$f * g = g * f$
association	$f * (g * h) = (f * g) * h$
distribution	$f * (g + h) = f * g + f * h$

Impulse Response

Let the input function be an *impulse* at $x = a$, $f(x) = \delta(x-a)$. The output function $h(x;a)$ is known as the *impulse response*. Note that, in general, the impulse response depends on the point (or time) at which the input is applied.

$$\delta(x-a) \longrightarrow \boxed{\quad \mathcal{S} \quad} \longrightarrow h(x;a)$$

Superposition Integral

The importance of the impulse response stems from the following theorem:

A linear system is completely characterised by its impulse response.

Proof:
Using the sifting property of δ, we get

$$f(x) = \int_{-\infty}^{\infty} \delta(x-x')f(x')dx'.$$

Now the extended linearity of $\mathcal{S}$ gives

$$\mathcal{S}\{f(x)\} = \int_{-\infty}^{\infty} f(x')\mathcal{S}\{\delta(x-x')\}\, dx'.$$

Therefore,

$$\mathcal{S}\{f(x)\} = \int_{-\infty}^{\infty} f(x')h(x;x')dx'.$$

This is known as the *superposition integral* because the output function can be seen to be composed of superimposed weighted impulse response functions. Note that this general form permits the impulse response to vary with time/position.

Shift Invariance

A system is shift invariant if the response to a shifted input is a shifted replica of the response to the unshifted input, i.e.

$$\text{if } \mathcal{S}\{f(x)\} = g(x), \text{ then } \mathcal{S}\{f(x-x')\} = g(x-x').$$

Impulse response of a shift invariant system is then given by

$$\mathcal{S}\{\delta(x-x')\} = h(x-x';0).$$

Thus the impulse response is a function only of the the difference $x - x'$.

Linear Shift Invariant (LSI) System

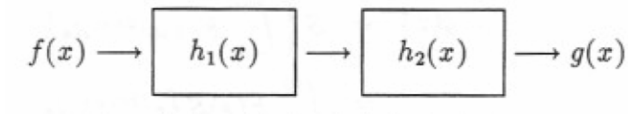
Making use of the properties of linearity and shift invariance, we can rewrite the superposition integral as

$$\begin{aligned} \mathcal{S}\{f(x)\} &= \int_{-\infty}^{\infty} f(x')h(x-x')dx' \\ &= f(x) * h(x) \end{aligned}$$

The shift invariant impulse response thus completely characterizes an LSI system.

Cascaded LSI system

If two or more LSI systems are cascaded, their combined response is given by



$$g(x) = f(x) * h_1(x) * h_2(x)$$

Thus the two systems can be considered as a single system with impulse response $h = h_1 * h_2$.

Response to a Complex Exponential

Response of an LSI system to a complex exponential $f(x) = \exp(-i2\pi sx)$ is a scaled version of the input, i.e.

$$S\left\{e^{-i2\pi sx}\right\} = a\, e^{-i2\pi sx}$$

where a is a complex constant. This is sometimes referred to as "harmonic response to harmonic input" because the output has the same frequency as the input (amplitude and phase may, however, differ). In other words, no frequencies can appear at the output that are not present at the input.

To prove this assertion, write $g(x)$ as the superposition integral

$$g(x) = \int_{-\infty}^{\infty} e^{-i2\pi sx'} h(x - x')dx'$$

Introducing a new variable $x'' = x - x'$, we obtain

$$g(x) = -e^{-i2\pi sx} \int_{-\infty}^{\infty} e^{i2\pi sx''} h(x'')dx'.$$

Since the integrand is independent of x, the whole integral is simply a complex constant,

$$g(x) = a\, e^{-i2\pi sx} = a\, f(x).$$

A complex exponential is said to be an *eigenfunction* of a LSA system, and the complex constant a is its *eigenvalue*.

Transfer function

The complex eigenvalue a is known as the *transfer function* of the system and is usually denoted as a function of frequency (temporal or spatial) $H(s)$.

$$S\left\{e^{-i2\pi sx}\right\} = H(s)\, e^{-i2\pi sx}$$

We will now show that, if the input and output functions have Fourier transforms $F(s) = \mathcal{F}\{f(x)\}$ and $G(s) = \mathcal{F}\{g(x)\}$, then for a linear system with transfer function $H(s)$,

$$G(s) = H(s)F(s).$$

Writing the Fourier transform in full,

$$f(x) = \int_{-\infty}^{\infty} F(s)\, e^{-i2\pi sx}\, ds$$

we see that this is equivalent to expressing a function as a superposition of complex exponentials. Since in a linear system each exponential is passed through independently, we can write

$$\begin{aligned} g(x) &= S\left\{\int_{-\infty}^{\infty} F(s)\, e^{-i2\pi s x}\, ds\right\} \\ &= \int_{-\infty}^{\infty} F(s)\, S\left\{ e^{-i2\pi s x} \right\} ds \\ &= \int_{-\infty}^{\infty} F(s) H(s)\, e^{-i2\pi s x}\, ds \\ &= \mathcal{F}\left\{F(s)\, H(s)\right\}. \end{aligned}$$

Therefore, $G(s) = H(s)F(s)$ as stated above.

It follows directly that the transfer function is the Fourier transform of the impulse response,

$$H(s) = \mathcal{F}\left\{h(x)\right\}.$$

For a cascade of *n* LSI systems, if the overall impulse response is

$$h(x) = h_1(x) * h_2(x) * \ldots * h_n(x),$$

then the overall transfer function is simply the product of individual transfer functions,

$$H(s) = \prod_{i=1}^{n} H_i(s).$$

Response of Physical Systems to a Sinusoid

In physical systems, real-valued inputs lead to real-valued responses. The impulse response will also be real-valued.

Consider a sinusoid input function of frequency s_0,

$$f(x) = \cos 2\pi s_0 x = \mathrm{Re}\left\{e^{-i2\pi s_0 x}\right\}$$

The response to this input can be found as follows. Let the transfer function,

$$H(s) = A(s)\, e^{-i\phi(s)},$$

be given by amplitude and phase functions $A(s)$ and $\phi(s)$, respectively. Then the response to the sinusoid input is

$$\begin{aligned} g(x) &= \mathrm{Re}\left\{A(s_0)\, e^{-i\phi(s_0)}\; e^{-i2\pi s_0 x}\right\} \\ &= A\cos(2\pi s_0 x + \phi) \end{aligned}$$

Thus the response is a sinusoid of the same frequency as the input but with possibly different amplitude and phase.