Asymptotic error rates for classical and quantum antidistinguishability

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Classical Antidistinguishability

Given probability densities $p_1, \ldots, p_r$ on a sample space (Ω, μ)

Uniform prior

Underlying assumption is that any observed i.i.d. data from (Ω, μ) is equally likely to follow one of the given probability distributions $p_1, \ldots, p_r$

$H_1, \ldots, H_r$ be hypotheses where H_i assumes that the observed i.i.d. data is generated by p_i

Classical Antidistinguishability

Antidistinguishability: by observing i.i.d. data from the sample space, reject a hypothesis

Example: Given coins nickel, quarter, dime with head probabilities $\alpha_1, \alpha_2, \alpha_3$. The sample space is $\Omega = \{H, T\}$ and hypotheses are nickel, quarter, or dime.

By observing coin tosses, we reject one of the hypotheses about the coin being nickel, quarter, or dime.

Antidistinguishability Error Probability

A (randomized) decision rule is a function

$$\delta : \Omega \to [0,1]^r, \quad \delta(\omega) = (\delta_1(\omega), \ldots, \delta_r(\omega))$$

$$\sum_{i=1}^{r} \delta_i(\omega) = 1$$

In other words,

On observing ω, select p_i with probability $\delta_i(\omega)$

If p_i is the true density, then antidistinguishability error probability is

$$\int \mathrm{d}\mu(\omega)\ \Pr(\omega \text{ observed}, p_i \text{ selected}) = \int \mathrm{d}\mu(\omega)\ \delta_i(\omega) p_i(\omega)$$

Total error probability:

$$\mathrm{Err}_{\mathrm{cl}}(\delta; p_1, \ldots, p_r) = \frac{1}{r} \int \mathrm{d}\mu(\omega) \sum_{i=1}^{r} \delta_i(\omega) p_i(\omega)$$

Uniform prior

Optimal Error Probability

$$\mathrm{Err}_{\mathrm{cl}}(p_1, \ldots, p_r) = \frac{1}{r}\int \mathrm{d}\mu(\omega)\ \ \min\{p_1(\omega), \ldots, p_r(\omega)\}$$

This corresponds to a *minimum likelihood decision* function:

$$\delta^*(\omega) = (0,\ldots,0,1,0,\ldots,0), \quad \forall \omega \in \Omega$$

where 1 is at i-th position and

$$p_i(\omega) = \min\{p_1(\omega), \ldots, p_r(\omega)\}$$

Asymptotic Regime

The error rate vanishes exponentially in the number of i.i.d. trials:

$$\mathrm{Err}_{\mathrm{cl}}(p_1^{\otimes n}, \ldots, p_r^{\otimes n}) \sim \exp(-n\xi), \quad \xi > 0$$

We are interested in the optimal error exponent:

$$\xi = \lim_{n\to\infty} -\frac{1}{n} \log \mathrm{Err}_{\mathrm{cl}}(p_1^{\otimes n}, \ldots, p_r^{\otimes n})$$

Classical Asymptotic Error Rate

The optimal asymptotic error rate is given by

$$\lim_{n\to\infty} -\frac{1}{n}\log \mathrm{Err}_{\mathrm{cl}}(p_1^{\otimes n}, \ldots, p_r^{\otimes n}) = -\log \mathrm{H}(p_1, \ldots, p_r)$$

Chernoff bound for $r = 2$

where the Hellinger transform $\mathbf{H}$ is given by

$$\mathbf{H}(p_1, \ldots, p_r) = \inf_{\substack{s_1, \ldots, s_r \geq 0 \\ s_1 + \cdots + s_r = 1}} \int \mathrm{d}\mu \; p_1^{s_1} \cdots p_r^{s_r}$$

Quantum Antidistinguishability

Given quantum system prepared in one of the quantum states $\rho_1, \ldots, \rho_r$ on finite dimensional $\mathscr{H}$

Uniform prior

Antidistinguishing task is to reject ρ_i for a true state of a quantum system through a measurement

This is in contrast with the state discrimination problem in quantum multiple hypothesis testing

Antidistinguishability Error Probability

A measurement is given by POVM $\mathscr{E} = \{E_1, \ldots, E_r\}$

$$E_i \geq 0, \quad \sum_{i=1}^{r} E_i = \mathrm{id}_{\mathscr{H}}$$

If ρ_i is the true state, then the Born rule gives error probability

$$\mathrm{Tr}\left[E_i \rho_i\right]$$

Total error probability:

$$\mathrm{Err}_{\mathrm{qu}}(\mathscr{E};\rho_1,\ldots,\rho_r)=\frac{1}{r}\sum_{i=1}^{r}\mathrm{Tr}\left[E_i\rho_i\right]$$

Optimal error probability:

$$\mathrm{Err}_{\mathrm{qu}}(\rho_1,\ldots,\rho_r)=\min_{\mathrm{POVM}}\frac{1}{r}\sum_{i=1}^{r}\mathrm{Tr}\left[E_i\rho_i\right]$$

Optimal measurement is not known in general

We give an upper bound on the optimal error probability:

$$\mathrm{Err}_{\mathrm{qu}}(\rho_1, \ldots, \rho_r) \leq \frac{1}{2} \min_{i<j} \mathrm{Tr}\left[\frac{1}{2}(\rho_i + \rho_j - |\rho_i - \rho_j|)\right]$$

For pure states:

$$\mathrm{Err}_{\mathrm{qu}}(|\psi_1\rangle, \ldots, |\psi_r\rangle) \leq \frac{1}{2} \min_{i<j} \langle\psi_i|\psi_j\rangle$$

Asymptotic Regime

The error rate vanishes exponentially in the number of copies of states:

$$\mathrm{Err}_{\mathrm{qu}}(\rho_1^{\otimes n}, \ldots, \rho_r^{\otimes n}) \sim \exp(-n\xi), \quad \xi > 0$$

We are interested in the optimal error exponent:

$$\xi = \lim_{n\to\infty} -\frac{1}{n} \log \mathrm{Err}_{\mathrm{qu}}(\rho_1^{\otimes n}, \ldots, \rho_r^{\otimes n})$$

Quantum Asymptotic Error Rate

A lower bound on the asymptotic error rate

$$\liminf_{n\to\infty} -\frac{1}{n}\log \mathrm{Err}_{\mathrm{qu}}(\rho_1^{\otimes n}, \ldots, \rho_r^{\otimes n}) \geq \max_{i<j} \xi_{QCB}(\rho_i, \rho_j)$$

where ξ_{QCB} is the quantum Chernoff bound:

$$\xi_{QCB}(\rho, \sigma) = -\log \inf_{0\leq s\leq 1} \mathrm{Tr}\left[\rho^s \sigma^{1-s}\right]$$

An upper bound on the asymptotic error rate

$$\limsup_{n\to\infty} -\frac{1}{n}\log \mathrm{Err}_{\mathrm{qu}}(\rho_1^{\otimes n},\ldots,\rho_r^{\otimes n}) \le C^\flat(\rho_1,\ldots,\rho_r)$$

where $C^\flat(\rho_1,\ldots,\rho_r)$ is the *log-Euclidean Hellinger bound*:

$$C^\flat(\rho_1,\ldots,\rho_r) = -\log \inf_{\mathbf{s}} Q_{\mathbf{s}}^\flat(\rho_1,\ldots,\rho_r)$$

$$Q_{\mathbf{s}}^\flat(\rho_1,\ldots,\rho_r) = \lim_{\varepsilon\to 0^+} \mathrm{Tr}\left[\exp\left(\sum_{i=1}^{r} s_i \ln(\rho_i + \varepsilon\ \mathrm{id}_{\mathscr{H}})\right)\right]$$

Example: Quantum Asymptotic Error Rates

Two-qubit Werner states for $p \in [0,1]$:

$$\omega_p = \left(\frac{3-4p}{3}\right) |-\rangle\langle-| + \left(\frac{4p}{3}\right) \frac{1}{4} I_{AB}$$

Bell state $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$

Maximally mixed state

$$\omega_p = \frac{1}{6}\begin{pmatrix} 2p & 0 & 0 & 0 \\ 0 & 3-2p & -3+2p & 0 \\ 0 & -3+4p & 3-2p & 0 \\ 0 & 0 & 0 & 2p \end{pmatrix}$$

# states	Werner states (ω_p)	Antidistinguishability error exponent	Discrimination error exponent
2	$\{\omega_{1/4}, \omega_{3/4}\}$		
3	$\{\omega_{1/4}, \omega_{1/2}, \omega_{3/4}\}$		
4	$\{\omega_{1/8}, \omega_{1/4}, \omega_{1/2}, \omega_{3/4}, \omega_{7/8}\}$		
5	$\{\omega_{1/16}, \omega_{1/8}, \ldots, \omega_{7/8}, \omega_{15/16}\}$		
6	$\{\omega_{1/32}, \omega_{1/16}, \ldots, \omega_{15/16}, \omega_{31/32}\}$		
7	$\{\omega_{1/64}, \omega_{1/32}, \ldots, \omega_{31/32}, \omega_{63/64}\}$		
8	$\{\omega_{1/128}, \omega_{1/64}, \ldots, \omega_{63/64}, \omega_{127/128}\}$		
9	$\{\omega_{1/512}, \omega_{1/128}, \ldots, \omega_{127/128}, \omega_{511/512}\}$		
10	$\{\omega_{1/1024}, \omega_{1/512}, \ldots, \omega_{511/512}, \omega_{1023/1024}\}$		
11	$\{\omega_{1/2048}, \omega_{1/1024}, \ldots, \omega_{1023/1024}, \omega_{2047/2048}\}$		

# states	Werner states (ω_p)	Antidistinguishability error exponent	Discrimination error exponent
2	$\{\omega_{1/4}, \omega_{3/4}\}$	0.144	
3	$\{\omega_{1/4}, \omega_{1/2}, \omega_{3/4}\}$	0.144	
4	$\{\omega_{1/8}, \omega_{1/4}, \omega_{1/2}, \omega_{3/4}, \omega_{7/8}\}$	0.413	
5	$\{\omega_{1/16}, \omega_{1/8}, \ldots, \omega_{7/8}, \omega_{15/16}\}$	0.725	
6	$\{\omega_{1/32}, \omega_{1/16}, \ldots, \omega_{15/16}, \omega_{31/32}\}$	1.056	
7	$\{\omega_{1/64}, \omega_{1/32}, \ldots, \omega_{31/32}, \omega_{63/64}\}$	1.394	
8	$\{\omega_{1/128}, \omega_{1/64}, \ldots, \omega_{63/64}, \omega_{127/128}\}$	1.737	
9	$\{\omega_{1/512}, \omega_{1/128}, \ldots, \omega_{127/128}, \omega_{511/512}\}$	2.427	
10	$\{\omega_{1/1024}, \omega_{1/512}, \ldots, \omega_{511/512}, \omega_{1023/1024}\}$	2.773	
11	$\{\omega_{1/2048}, \omega_{1/1024}, \ldots, \omega_{1023/1024}, \omega_{2047/2048}\}$	3.119	

# states	Werner states (ω_p)	Antidistinguishability error exponent	Discrimination error exponent
2	$\{\omega_{1/4}, \omega_{3/4}\}$	0.144	0.144
3	$\{\omega_{1/4}, \omega_{1/2}, \omega_{3/4}\}$	0.144	0.0347
4	$\{\omega_{1/8}, \omega_{1/4}, \omega_{1/2}, \omega_{3/4}, \omega_{7/8}\}$	0.413	0.0132
5	$\{\omega_{1/16}, \omega_{1/8}, \ldots, \omega_{7/8}, \omega_{15/16}\}$	0.725	0.00593
6	$\{\omega_{1/32}, \omega_{1/16}, \ldots, \omega_{15/16}, \omega_{31/32}\}$	1.056	0.00282
7	$\{\omega_{1/64}, \omega_{1/32}, \ldots, \omega_{31/32}, \omega_{63/64}\}$	1.394	0.00138
8	$\{\omega_{1/128}, \omega_{1/64}, \ldots, \omega_{63/64}, \omega_{127/128}\}$	1.737	0.000680
9	$\{\omega_{1/512}, \omega_{1/128}, \ldots, \omega_{127/128}, \omega_{511/512}\}$	2.427	0.000169
10	$\{\omega_{1/1024}, \omega_{1/512}, \ldots, \omega_{511/512}, \omega_{1023/1024}\}$	2.773	0.0000842
11	$\{\omega_{1/2048}, \omega_{1/1024}, \ldots, \omega_{1023/1024}, \omega_{2047/2048}\}$	3.119	0.0000420

# states	Antidistinguishability error exponent	Discrimination error exponent	Ratio (approximate)
2	0.144	0.144	1
3	0.144	0.0347	4
4	0.413	0.0132	31
5	0.725	0.00593	122
6	1.056	0.00282	374
7	1.394	0.00138	1012
8	1.737	0.000680	2552
9	2.427	0.000169	14395
10	2.773	0.0000842	32943
11	3.119	0.0000420	74169

Summary

- Antidistinguishability of multiple classical probability densities
- Asymptotic error exponent: *negative log-Hellinger bound*
- Antidistinguishability of multiple quantum states
- Lower bound on the asymptotic error exponent: pairwise max *Chernoff bound*
- Upper bound on the asymptotic error exponent: *log-Euclidean quantum Hellinger transform*

Thank You