The Minimax Redundancy of Lossy Compression

Adeel Mahmood Aaron Wagner

School of Electrical and Computer Engineering
Cornell University

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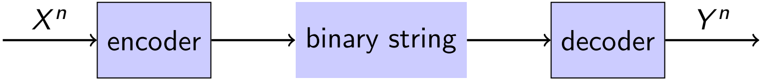
Setup



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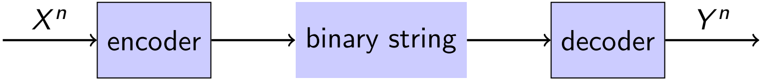

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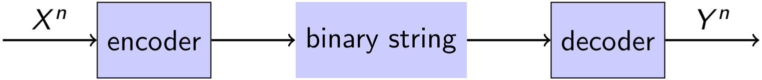

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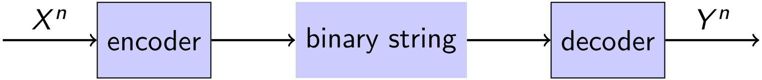

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- Distortion measure $\rho : A \times B \to [0, \rho_{\max}]$. Assume $\max_{a \in A} \min_{b \in B} \rho(a, b) = 0$.

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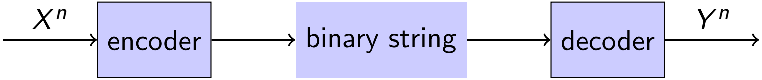

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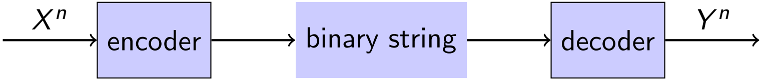

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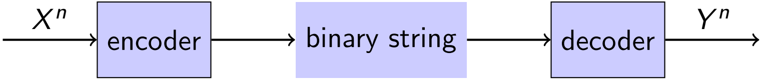
- $\rho =$ Hamming and $d = 0$ recovers the lossless case.

Performance Metric



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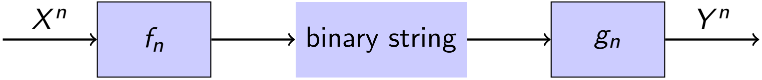

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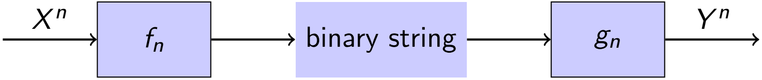

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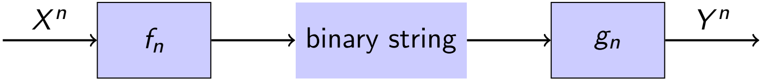

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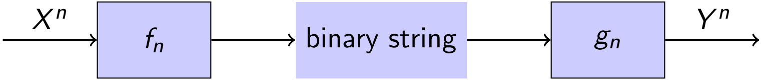

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- Figure-of-merit: the *expected rate redundancy*:

$$R(C_n, p) - R(p, d)$$

Weak and Strong Universality

For any source p, there exists $\{C_n\}$ so that

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- ... *non-universal* if $\{C_n\}$ depends on p.

Redundancy Bounds

	Lossless	Lossy
Non-universal	$\Theta\left(1/n\right)$	$\Theta(\ln n/n)$ [Zhang, Yang, and Wei '97]
Weakly universal	$\Theta\left(\ln n/n\right)$ [Rissanen '84]	$\Theta(\ln n/n)$ [Yang and Zhang (unpublished)]
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Another view says that the lossy redundancy is $\tilde{\Theta}(1/\sqrt{n})$ . . . .

A Decomposition

- Consider the *expected rate-distortion function*

$$\mathbb{E}\left[R(P_{X^n}, d)\right],$$

where P_{X^n} is the random type of the i.i.d. source sequence $X^n \sim p$

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- Conventionally, we upper bound the minimax rate redundancy as

$$\inf_{C_n} \sup_{p} \left[R(C_n, p) - R(p, d)\right] \leq \inf_{C_n} \sup_{p} \left[R(C_n, p) - \mathbb{E}\left[R(P_{X^n}, d)\right]\right] + \sup_{p} \left[\underbrace{\mathbb{E}\left[R(P_{X^n}, d)\right] - R(p, d)}_{\tilde{O}\left(\frac{1}{\sqrt{n}}\right) \text{ using Palaiyanur and Sahai '08}} \right]$$

Achievability: First Attempt

Given x^n:

1. Generate an infinite, shared codebook with codewords drawn according to the *normalized ML distribution*:

$$Q^{NML}(y^n) = \frac{\max_{q \in \mathcal{P}(B)} q^n(y^n)}{\sum_{z^n \in B^n} \max_{q \in \mathcal{P}(B)} q^n(z^n)}$$

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3. Send the (absolute) index of the first codeword in the subcodebook with distortion at most d.

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$$-\frac{1}{n} \log Q^{P_{x^n},d,\rho}(\rho(x^n, Y^n) \leq d) \approx R(P_{x^n}, d)$$

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Unfortunate Fact (Mahmood and Wagner '22)

For $|A| = 2$, $|B| = 3$, $\rho_{max} = 3$, $d = 1$ and any even n,

$$\sup_{x^n,\rho}\left[-\frac{1}{n}\log Q^{P_{x^n},d,\rho}(\rho(x^n, Y^n) \le d) - R(P_{x^n}, d, \rho)\right] = \infty.$$

Mitigating Fact (Mahmood and Wagner '22)

Fix $d > 0$. Then

$$\sup_{x^n,\rho}\left[-\frac{1}{n}\log Q^{P_{x^n},d,\rho}\left(\rho(x^n,Y^n)\le d+\frac{\mathrm{const}}{n^{5/8}}\right)-R(P_{x^n},d,\rho)\right]$$
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1. Change of measure $Q^{P_{x^n},d,\rho}\to Q^*_{Y|X}$ (optimal test channel) using exponential tilting.

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   - High-variance: Berry-Esseen Theorem
   - Low-variance: Chebyshev's Inequality

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Post-Correction

$$\sup_{x^n,\rho}\left[-\frac{1}{n}\log Q^{P_{x^n},d,\rho}\left(\rho(x^n,Y^n)\le d+\frac{\mathrm{const}}{n^{5/8}}\right)-R(P_{x^n},d,\rho)\right]$$
$$\le \frac{\mathrm{const}}{n^{5/8}}+O\left(\frac{\log n}{n}\right)$$

Yields $O(1/n^{5/8})$ redundancy w.r.t. $\mathbb{E}[R(P_{X^n},d)]$. But this is only $d+\frac{\text{const}}{n^{5/8}}$ semifaithful

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- We replace
$$M\propto n^{3/8}$$
symbols in y^n, say $y_{l_1},\ldots,y_{l_M}$, with $\hat{y}_{l_1},\ldots,\hat{y}_{l_M}$ so that $\rho(x_{l_i},\hat{y}_{l_i})=0$ for all $i=1,\ldots,M$.

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- Requires sending $n^{3/8}\left[\log n+\log|B|\right]$ bits. Also yields $\approx\frac{1}{n^{5/8}}$ redundancy.
- Meets d constraint a.s.

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Converse

- For the achievability, we used

$$\inf_{C_n} \sup_p [R(C_n, p) - R(p, d)] \le \inf_{C_n} \sup_p \Big[\overbrace{R(C_n, p) - \mathbb{E}[R(P_{X^n}, d)]}^{\tilde{O}\left(\frac{1}{n^{5/8}}\right)} \Big] + \sup_p \Big[\underbrace{\mathbb{E}[R(P_{X^n}, d)] - R(p, d)}_{\tilde{O}\left(\frac{1}{\sqrt{n}}\right) \text{ using Palaiyanur and Sahai '08}} \Big]$$

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- Consider the binary-Hamming case:

$$R(p, d) = (H(p) - H_b(d))^+ \qquad 0 \le d \le 1/2,$$

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- End-to-end, for the binary-Hamming case, ignoring log terms:

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Redundancy Bounds

	Lossless	Lossy
Non-universal	$\Theta\left(1/n\right)$	$\Theta(\ln n/n)$
Weakly universal	$\Theta\left(\ln n/n\right)$	$\Theta(\ln n/n)$
Strongly universal	$\Theta\left(\ln n/n\right)$	$\tilde{O}(1/\sqrt{n})$

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		with regularity cond.	without regularity cond.
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is nonzero for some $1 \leq j_1 < j_2 < \cdots < j_{|B|} \leq |A|$, where F is the vector-valued function

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Conclusion

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- Such an approach yields different redundancies and gives rise to new schemes.