Gas Capillary Inhibition to Oil Production

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Abstract

In a sand reservoir, grain-size variations and the presence of two separate phases can result in capillary barriers that impair oil production. We calculate the effect capillary barriers could have on the cumulative oil production and derive a formula for the time at which the capillary loss of production will reach 20%. Our calculations replace the traditional Darcy's law with a plastic flow law that prohibits flow unless the driving pressure gradient is larger than a specified minimum. The minimum pressure gradient depends on reservoir conditions that can be controlled to some degree.

Introduction

The purpose of this study is to estimate the effect capillary barriers can have on cumulative oil production. In a reservoir having grain-size variations, small capillary pressure barriers can develop if gas exsolves during oil production. These barriers can reduce oil recovery. We derive an expression that estimates the time it takes for the production loss to reach 20%, compared to an ideal reservoir without capillary barriers.

In a sand reservoir, changes in grain-size can occur for several reasons. Fan deposits contain abrupt vertical and lateral changes in sorting and mean grain-size (Nilsen, 1982). Tidal deposits can have mud layers as often as every 100 cm having very large grain-size contrasts. Grain-size contrasts can occur due to cross bedding or cementation. Within a layer, the interfaces of fine/coarse grains can, together with the presence of two fluid phases (oil and gas), cause capillary bubble barriers that create small pressure-drops (Cathles, 2001, this volume; Shosa and Cathles, 2001, this volume). The loss in production is primarily dependent on: (1) the frequency and magnitude of grain-size variations in the sand, (2) the permeability and porosity of the sand, and (3) reservoir pressure. Other variables of importance include the viscosity and compressibility of the fluid and the fraction of gas in the liquid. With a higher fraction of methane in the hydrocarbons, the bubble point is reached earlier and gas is exsolved sooner in the production of a reservoir.

In laboratory experiments at Cornell University, Shosa (2000) created capillary barriers that completely and for a sustained amount of time shut off any flow in a tube containing water and CO_2. These laboratory experiments show that the presence of a non-aqueous phase together with a combination of contrasting grain-sizes can form capillary barriers that act as very durable seals. The porous media experiments carried out by Shosa used a metal tube packed with sediments that had a number of fine-coarse interfaces (2 μm / 45 μm). Initially, water was driven through the tube at a pressure at which the CO_2 was dissolved in the water. When the pressure was lowered, reaching the bubble point of the water-

CO_2 mixture, a gas phase was exsolved and small additive pressure-drops where observed. These pressure-drops could be accurately predicted by the following equation:

$$\Delta p = 2\gamma\left(\frac{1}{r_{fine}} - \frac{1}{r_{coarse}}\right) \quad (1)$$

The effects of capillary barriers have been studied in various contexts. Purcell (1949) and Schowalter (1979) give an expression for the resistant force fluids must overcome to move through the pores of sediments. An analysis of capillary seals is given by Revil *et al*. (1998).

The traditional Darcy's law, $v = -(k/\mu)\nabla p$, will allow fluids to move no matter how small the pressure gradient. In this paper we simulate an environment having occasional flow barriers by introducing a minimum pressure gradient that must be overcome for any flow to take place. The flow law is a "plastic" Darcy's law,

$$v = -(k/\mu)(\nabla p - \nabla p_{min})$$

indicating that there is no flow of either fluid phase if ∇p is less than ∇p_{min}. If the pressure-drops are on average Δp and if they occur with mean distance *m*, then ∇p_{min} will be equal to $\Delta p/m$.

In our analysis, the production lost due to capillary barriers will be given by the quotient of the cumulative production, with and without production inhibition, using the traditional and the plastic Darcy laws.

Homogeneous Darcy Flow

The equation for diffusion of pressure in a porous medium is the same as the equation for conduction of heat in a solid. It is derived from the conservation of mass equation:

$$\nabla\frac{\rho k}{\mu}(\nabla p + \rho\bar{g}) + Q = \frac{\partial(\rho\phi)}{\partial t} \quad (2)$$

In our discussion we will have no source term, *Q,* and neglect the effects of gravity. The density is dependent on pressure according to the relationship:

$$\rho = \rho_o e^{\beta(p - p_o)} \quad (3)$$

where β is the compressibility, which should take into account both the expansion of the fluid and the compression of the porous matrix. The derivative of the density, ρ, with respect to time is given by:

$$\frac{d\rho}{dt} = \rho\beta\frac{dp}{dt} \quad (4)$$

Substituting (3) into (2), and assuming porosity does not change in the reservoir, we obtain:

$$\nabla\frac{\rho k}{\mu}\nabla p = \rho\phi\beta\frac{dp}{dt} \quad (5)$$

If we ignore lateral changes in fluid density, divide by density, and consider a homogeneous porous media, we obtain a simple diffusion equation:

$$\nabla^2 p = \frac{\phi\mu\beta}{k}\frac{dp}{dt} \quad (6)$$

Here we consider the 1D case of a semi-infinite reservoir bounded by the half plane $x = 0$ extending to infinity in the positive *x* direction. The pressure *p(x,t)*, is a function of distance *x* [m] from the well and time *t* [s]. The pressure in the reservoir is initially at p_r, and for $t > 0$ is kept at a constant *0* on the borehole side, so the equation takes the form

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{\kappa}\frac{\partial p}{\partial t} = 0$$

$$p(x, 0) = p(\infty, t) = p_r$$

$$p(0, t) = 0 \qquad (7)$$

Here $\kappa = k/(\phi\mu\beta)$ is the hydraulic diffusivity [m^2/s], k is the permeability, ϕ is the porosity, μ is the dynamic viscosity of the hydrocarbon fluid, and β is the compressibility reflecting both how much the fluid expands and how much the porous media compresses due to a change in pressure. The derivation and solution of (7) is given in Appendix A as

$$p(x, t) = \frac{2p_r}{\sqrt{\pi}} \int_0^{x/2\sqrt{\kappa t}} \exp(-\xi^2)d\xi \qquad (8)$$

A Plastic Darcy's Law

If a gas phase is present in the sand layer, small pressure barriers may develop at the fine-coarse grain interfaces. The gas phase can exsolve from the oil if the pressure-drops below the bubble point during production. We will assume that a mean frequency of fine-coarse interfaces and capillary pressure-drops develops in all directions in order to estimate the effect those drops will have. The pressure-drops can arise when an oil filament tries to move into a gas-filled pore of larger throat size, forming an exit pressure drop. Or the pressure-drops can also arise when a gas-phase attempts to enter finer oil-filled pores. The entry pressure for mercury into a water-wet 100 md sand is about 2 psi (Smith, 1966), for example.

In the standard Darcy's law there is always flow no matter how small the pressure gradient. An environment with regular small pressure drops, will change this behavior and will be approximated here by a minimum pressure gradient that must be exceeded for any flow to take place. Figure 1 shows a schematic sand layer with occasional pressure-drops. If the average pressure drop is Δp and the mean distance between drops is m, then a pressure gradient $\nabla p_{min} = \nabla p/m$ must be overcome for any flow to take place. The plastic Darcy's law (see also Shosa and Cathles, 2001, this issue) will change differential equation (7) to the form:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{\kappa}\frac{\partial p}{\partial t} = 0, \quad \frac{\partial p}{\partial x} > \nabla p_{min}$$

$$\frac{\partial p}{\partial t} = 0, \quad \frac{\partial p}{\partial x} \le \nabla p_{min} \qquad (9)$$

This equation is solved numerically using a finite difference method described in Appendix B. Equation (9) suggests directly how the system will behave. If the pressure gradient at a given position is less than or equal to the minimum gradient ∇p_{min} no fluids will move, no pressure drop will take place, and $\partial p/\partial t$ is zero. The pressure drawdown curve will converge to a straight line with a constant slope. As a result there is a maximum distance, $x_{max} = p_r/\nabla p_{min}$, from which fluids can be produced.

Figures 2A and B show the standard Darcy's law and the drawdown curves derived from equation (7) in the previous section. Figures 2C and D are calculated with the plastic Darcy law and equation (9), which was solved by a numerical method described in Appendix B. In the example of Figure 2, the reservoir pressure was chosen to be 100 bars. Flow barriers of 2 psi were assumed to occur every 10 meters, giving a threshold pressure gradient of 0.2 psi/m or 13.7 bar/km.

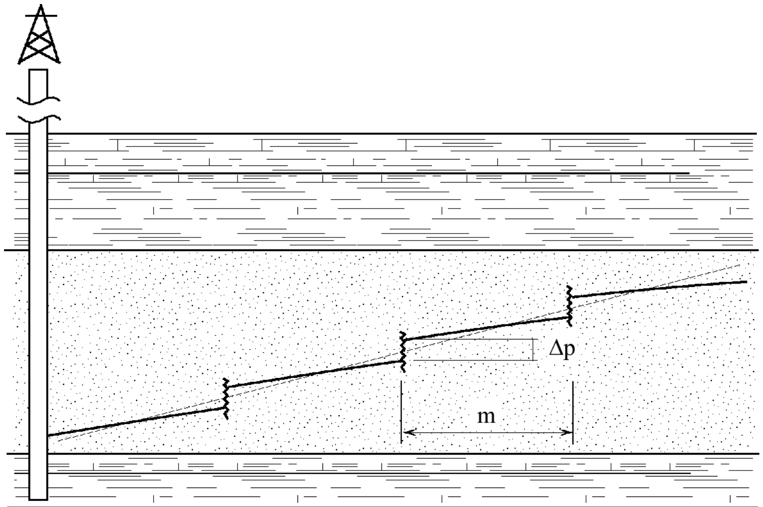

Figure 1. A schematic drawdown curve (thick line) having average pressure drops Δp and mean distance *m* between drops; pressure is the *y*-axis and distance is the *x*-axis over a sand layer.

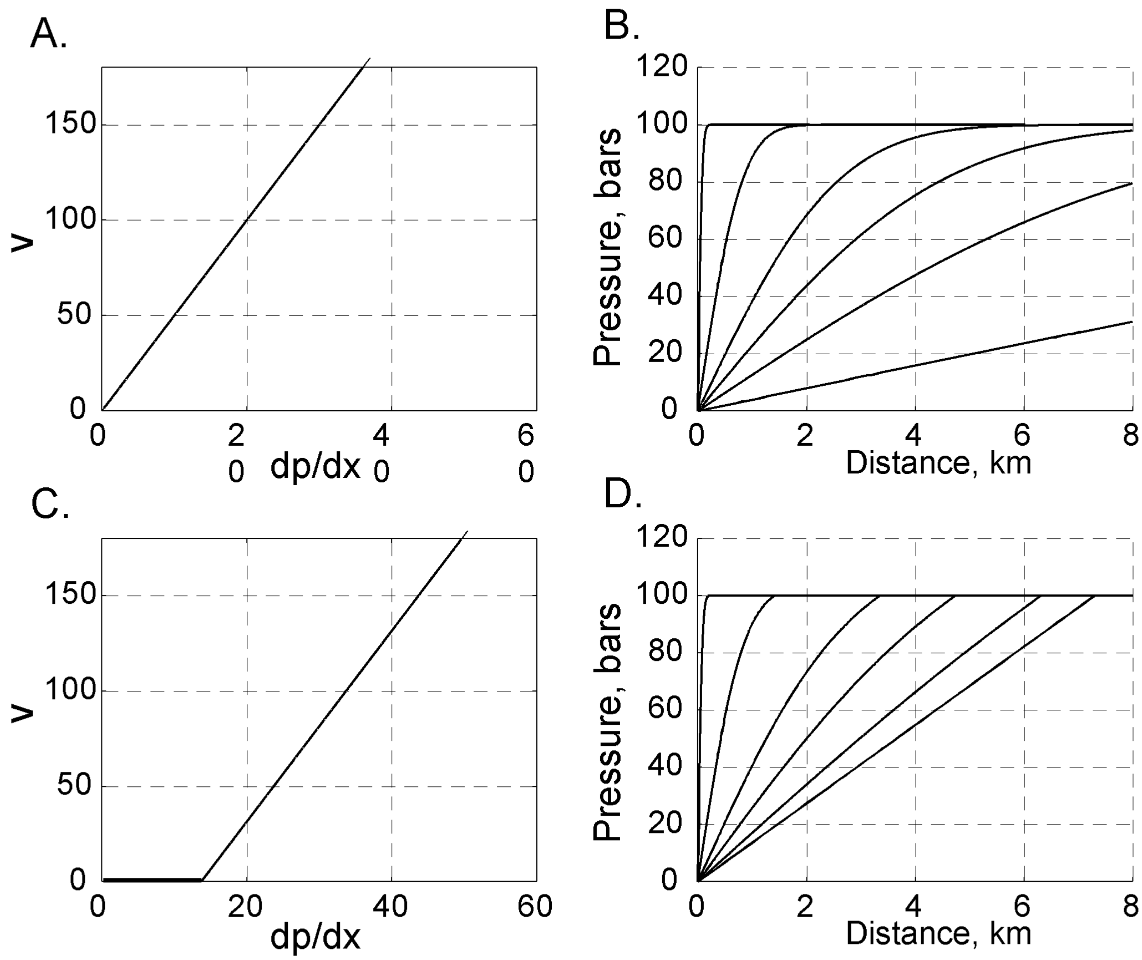

Figure 2. Figure A shows the normal Darcy's law with a flux, *v*, even at very small pressure gradients. The corresponding drawdown curve is given in (B). In the plastic Darcy case (C) no flow occurs for low-pressure gradients and the drawdown curves converge to a straight line with time (D). The times corresponding to the drawdown curves are, from top down: 0.1, 10, 100, 300, 1000, and 10000 days.

An Approximate Analytical Solution

The numerical method is time consuming for long simulations; it is therefore convenient to derive an expression in closed form that approximates the behavior of the system. In Appendix C we derive an expression for production in a reservoir that initially has a pressure that decreases linearly from p_r at the wellbore to 0 at a distance $x = p_r / \nabla p_{min}$. The pressure decay in this hypothetical reservoir is shown in Figure 3. We have added a pressure $p = x \nabla p_{min}$ to the solution so that the pressure decay is more physically realistic. The motivation for our approximation is clear if Figure 3 is compared to Figure 2D. The pressure curves in Figure 3 are given by:

$$p(x,t) = x\nabla p_{min} + \sum_{n=1}^{\infty} \frac{2p_r}{x_{max}} \sin\left(\frac{n\pi x}{x_{max}}\right) \exp(-\kappa n^2 \pi^2 t / x_{max}^2), \quad x \le x_{max}$$

$$p(x,t) = p_r x, \quad x > x_{max} \tag{10}$$

where x_{max} is given by $(p_r / \nabla p_{min})$. The horizontal pressure gradient has the form:

$$\frac{\partial p}{\partial x} = \nabla p_{min} + \frac{2p_r}{x_{max}} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{x_{max}}\right) \exp(-\kappa n^2 \pi^2 t / x_{max}^2)$$

The horizontal pressure gradients at the production well (x=0) for all models are shown in Figure 4. The upper most curve show the pressure gradient using the traditional Darcy model, while the lower two are the pressure gradients using the plastic Darcy law by numerical and analytical solutions. The analytical solution underestimates the pressure gradient, compared to the numerical solution.

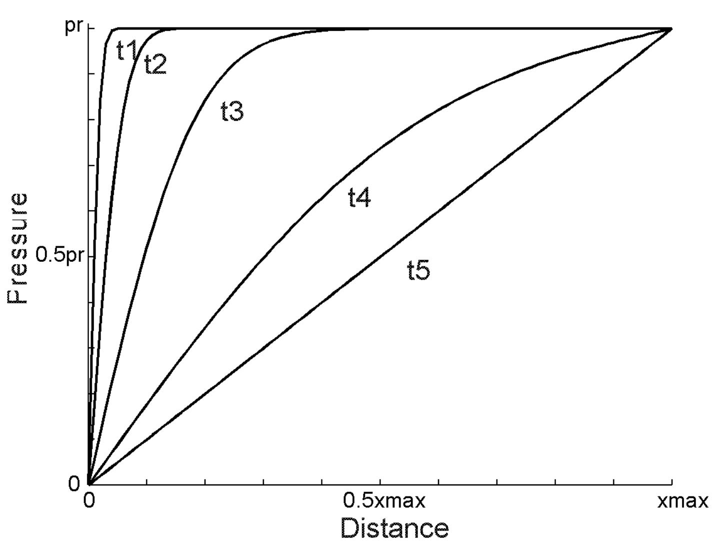

Figure 3. This drawdown curve is for the approximative solution to the plastic Darcy's law pressure equation as derived in Appendix C. It serves as an analytical approximation to the solution in Figure 2D.

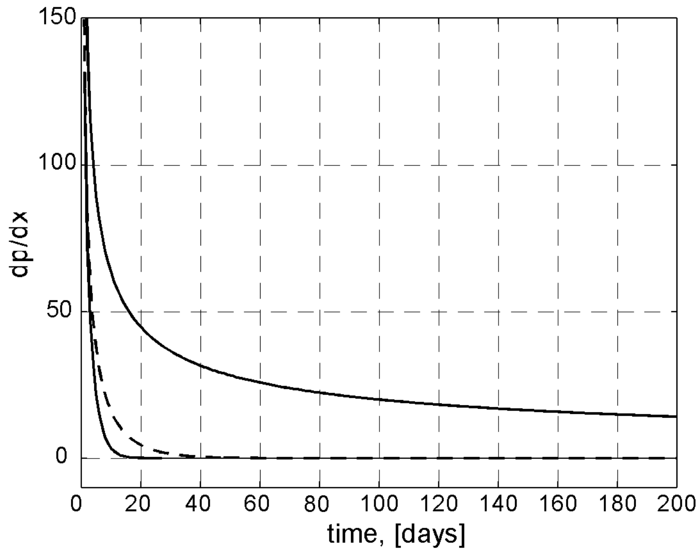

Figure 4. The pressure gradients at the borehole as a function of time. The uppermost gradient is the standard Darcy's law and the middle line is the numerical solution to the plastic Darcy's law case. It is seen that the approximation to the plastic Darcy's law (lowermost curve) overestimates the damage due to capillary blockage but has approximately the right form.

Loss of Production due to Capillary Pressure-Drops

To estimate how the capillary seals damage (inhibit) production, we calculate the cumulative volume produced with and without capillary seals. The flux into the borehole as given by Darcy's law at $x = 0$, is:

$$v = -\frac{k}{\mu}\frac{\partial p}{\partial x}\bigg|_{x=0} \qquad [m/s] \tag{12}$$

From (5) we see that the pressure gradient at, $x = 0$, becomes:

$$\frac{\partial p}{\partial x}\bigg|_{x=0} = \frac{p_r}{\sqrt{\kappa\pi t}} \tag{13}$$

and the cumulative production (volume per area), is calculated as the integral of the flux, v, over time is:

$$V = \int_0^t v\,dt \qquad [m] \tag{14}$$

Hence, for the traditional Darcy's law we have:

$$V_{darcy} = \frac{2kp_r}{\mu\sqrt{\kappa\pi}}\sqrt{t}, \qquad 0 \le t \le \infty \tag{15}$$

In the case of a plastic Darcy's law, the gradient from Expression (9) is given by:

$$\frac{\partial p}{\partial x}\bigg|_{x=0} = \nabla p_{min} + \frac{2p_r}{x_{max}}\sum_{n=1}^{\infty}\exp(-\kappa n^2\pi^2 t/x_{max}^2), \quad x \le x_{max} \tag{16}$$

Integrating the flux, v, using this pressure gradient, the cumulative production for the plastic Darcy's law can be written:

$$V_{plastic} = \frac{1}{3}\frac{k}{\mu}\nabla p_{min}\frac{x_{max}^2}{\kappa}\left(1-\frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-\kappa n^2\pi^2 t/x_{max}^2)\right) \tag{17}$$

Here we used the fact that $\Sigma_n 1/n = \pi^2/6$, and substituted $\nabla p_{min} x_{max}$ for p_r. To obtain an expression for the loss in production we form the ratio $V_{plastic}/V_{darcy}$:

$$\frac{V_{darcy}}{V_{plastic}} = \frac{1}{6}\frac{\sqrt{\pi}}{\sqrt{\kappa t}}\frac{\nabla p_{min}\, x_{max}^2}{p_r}\left(1-\frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-\kappa n^2\pi^2 t/x_{max}^2)\right)$$

By introducing the dimensionless time, $T = t\kappa/x_{max}^2$, and $x_{max}^2 = (p_r/\nabla p_{min})^2$, Equation (18) can be written:

$$\frac{V_{darcy}}{V_{plastic}}(T) = \frac{1}{6}\frac{\sqrt{\pi}}{\sqrt{T}}\left(1-\frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-n^2\pi^2 T)\right) \tag{19}$$

The loss in production as a function of dimensionless time, T, is shown in Figure 5. By inspecting this graph, we find that the loss in production reaches 20% when $T = 0.05$, that is, $V_{plastic}/V_{darcy} = 0.8$. From the expressions for T and x^2_{max}, and $\nabla p_{min} = \Delta p/m$, we see that

$$t = \frac{p_r}{20(\nabla p_{min})^2\kappa} \qquad [days] \tag{20}$$

This is the time, in days, it takes for the production loss to become 20%. The hydraulic diffusivity is given by $\kappa = k/(\phi\mu\beta)$ with permeability in millidarcies, viscosity in poise, and compressibility in 1/psi. The pressure is given in bars and the mean distance between pressure-drops, m, in kilometers, so ∇p_{min} is measured in bars/km.

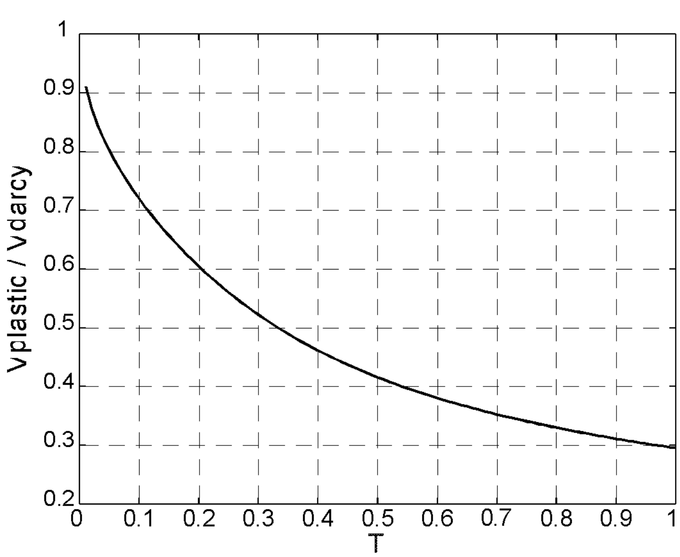

Figure 5. The loss in production as a function of the dimensionless time *T*. Dimensionless time *T = 0.05* when the production loss is 20% (Vplastic/Vdarcy = 0.8).

A Real World Example

Drawing from a well-test study by Earlougher (1977), we entered the following typical reservoir system values in our equation (20), as a test case:

Δp = 2 psi/interface = 0.137 bar/interface

p_r = 100 bar = 20×10^{6} Pa

k = 100 md = 100×10^{-15} m^2

ϕ = 0.2 = 0.2

μ = 0.01 poise = 0.001 Pas

β = 1.2×10^{-5} 1/psi = 1.45×10^{-4} 1/Pa

m = 0.10 km

This data gives a $\kappa = k/(\phi\mu\beta) = 0.0199$ km^2 /day. Under these conditions, the time it takes for the loss in production to become 20% is calculated from Expression (20) to be 183 days.

Discussion and Conclusions

Overpressures in sedimentary basins have been observed, in some cases, to be fully encapsulated. This means that a sealing mechanism must be present in all directions around the overpressured area. A situation like this can be difficult to explain using common theories of low permeability zones are caused by lithology. The phenomenon, of capillary seals, has been described in several papers in this volume and confirmed by simple laboratory experiments.

The notion of capillary seals is surrounded with controversy, and the work in this paper is designed to help understand how the segregation of phases in layered media inhibits flow. The process is not a relative permeability seal, which can only lower the total permeability of a phase to about 0.5. But it is a relative permeability effect, which when repeated over several layers of contrasting grain-size, will cause the whole system to have close to zero permeability. This is because when a fine-grained layer is saturated with the wetting phase, the non-wetting phase cannot go through this region. The wetting phase is depleted near the fine layers and this inhibits the flow of the wetting phase into the fine layer. The result is that the flow of both wetting and non-wetting phase is inhibited. With hundreds of such small regions, large overpressures could be retained for protracted periods of time (Cathles, 2001, this volume; Shosa and Cathles, 2001, this volume; Revil and Cathles, 2001, this volume).

In this paper we derive a simple expression (Equation 20) to estimate the time required for a well to suffer a 20% production loss due to capillary barriers. The expression shows that for a given frequency of capillary barriers, ∇p_{min}, production damage is greatest in the most permeable reservoirs. This is because the drawdown cone will extend outward farther and faster (and encounter more capilary barriers) in permeable reservoirs. It is important whether production impairment is capillary. If it is capillary, productioin may recover if the reservoir repressures or if surfactant is introduced to erase the capillary barriers.

Acknowledgments

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Appendix A: Analytical Solution to Pressure Drawdown

We derived the following diffusion equation in the test:

$$\frac{\partial^2 p}{\partial x^2} - \frac{\phi\beta\mu}{k}\frac{\partial p}{\partial t} = 0 \tag{A-1}$$

where the $\phi\beta\mu/k$ is the reciprocal of the hydraulic diffusivity κ.

In order to solve (A-1) with initial and boundary conditions $p(x,0) = p_r$ and $p(0,t) = 0$, it is assumed that the pressure is a function of the dimensionless ratio $\eta = x/2\sqrt{\kappa t}$, known as the Boltzmann transformation (Turcotte, 1982). With this transformation, Equation (A-1) becomes:

$$\frac{\partial^2 p}{\partial \eta^2} + 2\eta\frac{\partial p}{\partial \eta} = 0 \tag{A-2}$$

which is a nonlinear ordinary differential equation. When $\eta \to 0$ and $\eta \to \infty$, we have that $x \to 0$ and $t \to 0$. The initial and boundary conditions become, $p(\eta) = p_r$ as $\eta \to \infty$ and $p(\eta) = 0$ as $\eta \to 0$. By the variable substitution $d\phi/d\eta = \phi$, we have:

$$\frac{d\phi}{d\eta} + 2\eta\phi = 0 \tag{A-3}$$

This equation is solved by multiplying with an integrating factor, resulting in:

$$\frac{d\phi}{d\eta}\exp(\eta^2) + 2\eta\phi\exp(\eta^2) = \frac{d}{d\eta}(\phi\exp(\eta^2)) \tag{A-4}$$

Applying the boundary conditions when $\eta \to 0$ and $\eta \to \infty$, and using $\int_0^\infty \exp(-x^2)dx = \sqrt{(\pi)}/2$ our final solution becomes, with $\eta = x/2\sqrt{\kappa t}$,

$$p(x,t) = \frac{2p_r}{\sqrt{\pi}}\int_0^{x/2\sqrt{\kappa t}} \exp(-\xi^2)\, d\xi \tag{A-5}$$

Appendix B: Numerical Solution to Plastic Darcy's Law

The pressure equation using the plastic Darcy's law (9) are solved by discretizing the time and distance separately. If the x-interval is discretized with a step size of k and the time-step is h then $p^n{}_i$ represents the numerical approximation of pressure at $x = ik$ and at time nh. The approximate pressure gradient is $\nabla p_i^n \approx (p_{i+1}^n - p_i^n)/k$ and the second derivative is approximated by $\partial^2 p/\partial x^2 \approx (p_{i+1}^n - 2p_I^n + p_{i-1}^n)/k^2$. The explicit finite difference scheme becomes

$$p_i^{n+1} = p_i^n + \kappa \frac{h}{k^2}(p_{i+1}^n - 2p_i^n + p_{i-1}^n),\ \nabla p_i^n > \nabla p_{min}$$

$$p_i^{n+1} = p_i^n, \quad \nabla p_i^n \le \nabla p_{min} \tag{B-1}$$

The method is stable given that the time-step, h, is chosen as

$$h \le \frac{1}{2\kappa} k^2 \tag{B-2}$$

A small spatial step, k, will require an even smaller time-step, h, and the method will become time consuming for long simulations. Figure 2D shows the pressure drawdown calculated using this finite difference method.

Appendix C: Approximating Drawdown History

We want to find a solution to the differential equation (9) that also satisfies the initial and boundary conditions $p(0,t=0^+) = p(x_{max},t) = p_r$, that is, it is initially everywhere at reservoir pressure, p_r, and will remain at reservoir pressure at $x = x_{max}$.

It can be verified that

$$p(x, t) = \sin\left(\frac{n\pi x}{x_{max}}\right)\exp(-\kappa n^2\pi^2 t/x_{max}^2) \tag{C-1}$$

is a solution to the differential equation

$$\kappa\frac{\partial^2 p}{\partial x^2} = \frac{\partial p}{\partial t} \tag{C-2}$$

Since the differential equation is linear, the solution will also hold if it is expressed as an linear combination of solutions

$$p(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{x_{max}}\right)\exp(-\kappa n^2\pi^2 t/x_{max}^2) \tag{C-3}$$

To satisfy the initial conditions, at $t = 0$, we see that the exponential is unity and (C-3) reduces to a sine series. A sine series can be made to converge to functions, f, that are twice differentiable. This means that we can find the coefficients, a_n, that will make $p(x,0) = p(x) = f(x)$. The definition of the sine series is

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{x_{max}}\right) = f(x)$$

where

$$a_n = \frac{2}{x_{max}}\int_0^{x_{max}} f(x)\sin\left(\frac{n\pi x}{x_{max}}\right)dx \tag{C-4}$$

By selecting $f(x) = p_r - x\nabla p_{min}$, we find that

$$a_n = \frac{2}{x_{max}}\int_0^{x_{max}} (p_r - x\nabla p_{min})\sin\left(\frac{n\pi x}{x_{max}}\right)dx = \frac{2p_r}{n\pi} \tag{C-5}$$

Hence a solution, $p(x,t)$, that satisfies both the differential equation and the initial condition, $f(x)$, is given by

$$p(x, t) = \sum_{n=1}^{\infty} \frac{2p_r}{n\pi}\sin\left(\frac{n\pi x}{x_{max}}\right)\exp(-\kappa n^2\pi^2 t/x_{max}^2) \tag{C-6}$$

By adding the term $x\nabla p_{min}$, remembering that $x_{max} = p_r/\nabla p_{min}$, to the solution we get a function that has the properties, $p(0,t=0^+) = p(x_{max},t) = p_r$. This change is in a sense cosmetic. It is not the expression that satisfies the differential equations. But it approximates the behavior of the pressure gradient in the plastic Darcy's law case. The solution with $x\nabla p_{min}$ added is shown in Figure 3. It serves as an analytical approximation to the solution in Figure 2D.