I Maxwell's Equations

Reverses – W drift
[illegible]
@ 1°/yr

Ok, so now let's see if we can understand the earth's magnetic field using the electromagnetic theory expressed in Maxwell's equations. First step is to remember Maxwell's equations and what they mean.

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \qquad (11-1)$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad (11-2)$$

$$c^2\, \underline{\nabla} \times \underline{B} = \frac{\partial \underline{E}}{\partial t} + \frac{\underline{j}}{\epsilon_0} \qquad (11-3)$$

$$\nabla \cdot B = 0 \qquad (11-4)$$

The equations are based on concept of a force field. Remember the gravity field, for example.

$$\underline{F}_{gravity} = -G \frac{Mm\,\hat{r}}{r^2}$$

where M is a large mass like the earth, m is a small mass, r is the distance from the centers of mass of the two, and G is the gravitational constant ($G = 6.673 \times 10^{-11}$ nt-m^2/kg). We can define a ==gravitational field strength== as the force per unit mass:

mass-independent

force for an infinitesimal mass (so as not to perturb field)

$$\underline{g} = \lim_{m \to 0} \frac{\underline{F}}{m} = -\frac{GM\hat{r}}{r^2} \stackrel{m}{=} m/\text{sec}^2$$

= "test particle"

Note gravity is an acceleration, and in fact gravitational mass is the same as the inertial mass, which is one of the fundamental connections in physics. $\underline{g}$ is the ==gravitational force field.==

Dicke + [illegible] expts!

By the same logic we can define an ==electric field $\underline{E}$==, and a ==magnetic field $\underline{B}$== that will exert ==force== on a unit of charge. From Coulomb's law

$$\underline{F}_{\text{electrostatic}} = \frac{-1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}\hat{r}$$

where q_1 and q_2 are two charges, r is the separation between them, and ϵ_0 is the permittivity of free space $\left(\epsilon_0 = 8.85415 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2\right)$.

see $\leftarrow$ $E = -\nabla V_E$ ↑ Voltage

$$\frac{F}{q_1} \equiv \boxed{\underset{\text{electric field}}{\underline{E}} \overset{\frac{nt}{coul} \text{ or } \frac{Volts}{m}}{\equiv} \lim_{q_1 \to 0} \frac{1}{4\pi\epsilon_0} \frac{q_2}{r^2}} \qquad (11\text{-}5)$$

"test charge"

Now we can generalize q_2 on a charge distribution, ρ [coul/m³] and express (11-5) equivalently:

"sources of $\underline{E}$ is charge" $\rightarrow$

$$\underline{\nabla} \cdot \underline{E} = \rho/\epsilon_0$$

which is the first of the Maxwell equations. Since

gauss' Theorem

$$\int_V \underline{\nabla} \cdot \underline{E}\, dV = \int_S \underline{E} \cdot d\underline{a} = \int_V \rho/\epsilon_0\, dV$$

$$4\pi r^2 E = \frac{q}{\epsilon_0}$$

This is same as (11-5).

For the magnetic field, we earlier defined B in terms of force it exerts on a moving charge:

$$\underline{F} = q\, \underline{u} \times \underline{B} \,,$$

So

(11-6) $$\boxed{\underline{B} = \lim_{q \to 0} \frac{F}{q u}}$$ ← "test particle"

$$\overset{M}{\equiv} \frac{nt}{\text{coul} \cdot \text{m/sec}}$$

coul/sec ≡ 1 ampere

$$= \frac{nt}{\text{ampere-m}} = \frac{\text{weber}}{m^2}$$

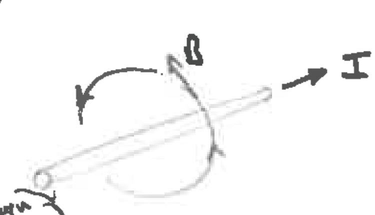
In fact an ampere is defined as the current w/ produces force of 1 nt for a 1 m length of wire in a magnetic field of 1 weber/m²

There is no such thing as a magnetic monopole so:

(11-7) $$\boxed{\underline{\nabla} \cdot \underline{B} = 0}$$

How is $\underline{B}$ generated?

A current in a wire generates a magnetic field



$$B = \frac{1}{4\pi\epsilon_0 c^2} \frac{2I}{r}$$

coul/m²-sec

(maxwell's eqn) (stokes theorem)

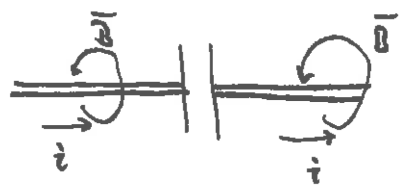
$$\oint (\nabla \times B) \cdot da = \int_S \frac{j}{\epsilon_0 c^2} da = \frac{I}{\epsilon_0 c^2}$$

$$B = \frac{I}{2\pi r \epsilon_0 c^2} \leftarrow \oint_\Gamma B \cdot dl = 2\pi r B = \frac{I}{\epsilon_0 c^2}$$

(11-8) $\therefore$ $$\boxed{c^2 \, \nabla \times B = j/\epsilon_0}$$

line integral of tangential component of B equals surface integral of normal component of $\nabla \times B$

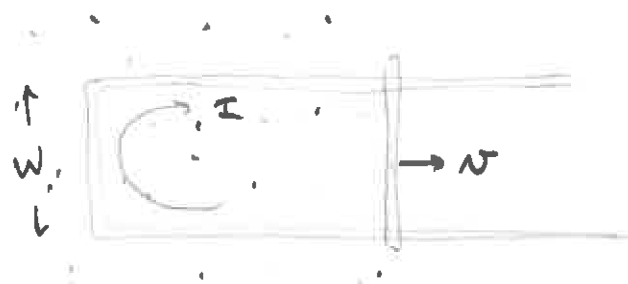
Now a term should be added to (11-8) because of where a capacitor in the current loop



it makes no sense for $\bar{B}$ to end. So we add:

(11-8') $$\boxed{c^2(\nabla \times B) = \frac{\partial E}{\partial t} + \frac{j}{\epsilon_0}}$$

As everyone knows a wire moving thru a magnetic field induces a current to flow in the wire.



This is the principle of every electric generator

Force/unit charge or EMF around loop = rate of change of total magnetic flux thru the loop: $\text{EMF} = W v B$

This can be expressed:

(11-9) $$\boxed{\nabla \times E = -\frac{\partial B}{\partial t}}$$

$$\oint_\Gamma E \cdot ds = \int_S \nabla \times E \cdot d\underline{a} = -\int_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{a}$$

Γ = voltage difference

$$= -\frac{\partial}{\partial t} \int_S B \cdot \hat{n} \, da$$

Notice symmetry of B and E: $\frac{\partial E}{\partial t}$ drives B, $\frac{\partial B}{\partial t}$ drives E

Finally consider how current, j, is induced in a fluid. Let σ = conductivity

volume and remember electric field $E = iR$ or $\sigma E = j$.

Remember also that moving thru a field B can generate E.

Combining all we can infer that:

(11-10) $$\boxed{\underline{j} = \sigma(\underline{E} + \underline{u} \times \underline{B})}$$

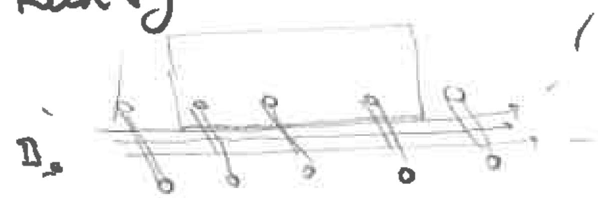
Siemens m^{-1}

$ohm^{-1}\ m^{-1}$

ohm = volt amp^{-1}

Intrinsic prop of material

One quick example: Magnetic field in solenoid. Field very low outside solenoid:

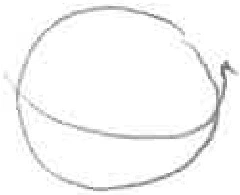


From 11.8 $$\nabla \times B = \frac{j}{c^2 \epsilon_0}$$

(Stokes theorem) $$\oint B \cdot dl = B_0 L = \frac{NI}{\epsilon_0 c^2}$$

$$\boxed{B = \frac{N/L\ I}{\epsilon_0 c^2}}$$

One more example: what current and electrical power output would be required to generate the earth's magnetic field?



I

$$B = 0.5 \text{ gauss} = 0.5 \times 10^{-4} \text{ Weber}/m^2 = \frac{2 \cdot I}{\underbrace{4\pi \epsilon_0 c^2}_{=10^7} r}$$

$$I = \frac{(4\pi\epsilon_0 c^2 = 10^7)(0.5 \times 10^{-5} \text{ W}/m^2)(r = 6370 \times 10^3 \text{ m})}{2}$$

$$\boxed{I = 1.7 \times 10^9 \text{ amps}}$$

Power = volts · amps, so we need to know the voltage required to drive this current, and that depends on the conductivity of the Cu wire we wrap around the earth.

Recall from (11-10):

$$j\,[\text{amp}/m^2] = \sigma\,[\text{ohm}^{-1}\,m^{-1}]\;E\,[\text{volts}/m]$$

$$I = j A = \sigma A \frac{V}{L}$$

Rearrange

resistivity

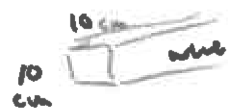
$$V = \left(\frac{L}{\sigma A}\right) I = \left(\frac{\rho L}{A}\right) I$$

conductivity

resistivity of Cu

circumference of earth

$$\frac{(1.7\times10^{-8}\ \text{ohm-m})(40{,}000\times10^{3}\ \text{m})}{.01\ \text{m}^2}$$



Wrapped around Earth

$$= 61\ \text{ohms}$$

$$V = (61\ \text{ohms})(1.7\times10^{9}\ \text{amps})$$

$$\boxed{V = 1.0\times10^{11}\ \text{volts}}$$

$$\text{Power} = \text{Volts}\cdot\text{amps} = 1.7\times10^{9}\ \text{amp} \times 1\times10^{11}\ \text{volts}$$

$$\boxed{\text{Electric Power} = 1.7\times10^{20}\ \text{watts}}$$ To generate E's mag field

Humans use 15 TW = 15×10^{12} watts today

E's magnetic field would require

$$\frac{1.7\times10^{20}}{15\times10^{12}} = 11\times10^{6}$$

or eleven million times total human power consumption!