Lecture 12. The Elsasser Dynamo

A. Theory

Now lets apply maxwell's equations to magnetic dynamo. The key, as we discussed earlier, is the magnification of the magnetic field by the fluid shear that occurs in a turbulently convecting fluid. To work, magnetic lines of force should be frozen into fluid and carried with it when it moves. This should occur naturally in conductive fluids of large dimensions. All this should be obvious from the equations.

B Idea

To make these notions clear we should seek an equation for the magnetic field alone. As the magnetic field decays it should generate an electric field vorticity which will drive current loops which sustain the original field. A magnetic field may ~~them~~ take some time to decay. If the fluid moves a considerable distance in the time it takes the magnetic field to decay, we can make a magnetic dynamo that will convert fluid motion into a magnetic field.

The force on a charge from $\underline{E}$ and/or $\underline{B}$ produces a current j in proportion to the force. The proportionality constant is the electrical conductivity of the material

$$\underline{j} = \sigma \left(\underline{E} + \underline{u} \times \underline{B} \right) \qquad (11-10)$$

again

Neglecting displacement currents (capacitor effects are negligible),

$$\underline{\nabla} \times \underline{B} = \frac{\underline{j}}{c^2 \epsilon_0} = \frac{\sigma}{c^2 \epsilon_0} \left(\underline{E} + \underline{u} \times \underline{B} \right)$$

The equation we seek results from taking the curl of

This equation and using $\nabla \times E = -\dfrac{\partial B}{\partial t}$

$$\nabla \times \nabla \times B = \frac{\sigma}{c^2 \epsilon_0} \left(\nabla \times E + \nabla \times (\underline{u} \times \underline{B}) \right)$$

$$\underbrace{\nabla(\nabla \cdot B)}_{0} - \nabla^2 B \qquad\qquad -\frac{\partial B}{\partial t}$$

Thus

$$\boxed{\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \nu_m \nabla^2 B} \qquad (11-8)$$

where ν_m is the magnetic diffusivity with units m^2/s.

$$\nu_m = \frac{c^2 \epsilon_0}{\sigma} .$$

Now we can define a magnetic Peclet number

equal to the ratio of the last two terms :

$$R_m = \frac{\nabla \times (u \times B)}{\nu_m \nabla^2 B} = \frac{\text{advection}}{\text{diffusion}}$$

Let $\nabla = \dfrac{1}{\lambda}$ where λ is the scale over which B varies

$$R_m = \frac{\lambda u}{\nu_m} \qquad (11-9)$$

Values of $R_m > 1$ mean the magnetic field will decay very little. ∴ R_m can be made arbitrarily large as σ becomes large or λ (the dimension of the system) becomes large. Thus large, convecting, rotating conducting ~~may~~ systems generate magnetic fields.

$R_m = \frac{\lambda u}{\eta_m}$

$\eta_m = \frac{c^2 \epsilon_0}{\sigma}$

$R_m = \frac{\sigma \lambda u}{c^2 \epsilon_0}$

Put some numbers on it

$\lambda \sim 3000 \text{ km} = 3 \times 10^6 \text{ m}$

$\sigma = 10^5 \ (\text{ohm m})^{-1}$ ← $\sigma_{Fe} = 10^7 \ (\text{ohm m}^{-1})$, $\sigma_{Cu} = 5.9 \times 10^7$ see p 14

$\epsilon = 8.85 \times 10^{-12}$

$c = 3 \times 10^8 \text{ m/s}$

$1^\circ/\text{yr}$ drift of magnetic isopors → $u = 100 \text{ km/yr} = \frac{10^5 \text{ m}}{3.15 \times 10^7} = 3 \times 10^{-3} \text{ m/s}$

$$R_m = \frac{\lambda u}{\eta_m} = \frac{\lambda u \sigma}{c^2 \epsilon_0} = \frac{(3 \times 10^6)(3 \times 10^{-3} \text{ m/s})(10^5)}{(3 \times 10^8)^2 (8.85 \times 10^{-12})}$$

$\eta_m = \frac{c^2 \epsilon_0}{\sigma}$

$\cong 1130$

The magnetic field should advect!

→ G-7 into flow

This really is the gist of it, but there are some additional details of great interest — for our understanding.

The first is intuitively apparent from large and conductivity:

$$\boxed{\frac{D}{Dt}\int_a B\cdot da = 0} \qquad (11\text{-}10)$$

Cowling's integral

Says the fluid carries magnetic field with it!

Another matter is how electromagnetic phenomena affect the fluid through body forces and a kind of magnetic pressure (repulsion of lines of magnetic force).

Once we gain insight here, the rest of the dynamo theory is a matter of "simple" logic.

The body (or ponderomotive) force is the force on the wire carrying 1 ampere current that we used to define B.

E.g.,

$$\underline{F}_{Em} = \underline{j} \times \underline{B}$$

and this can be immediately added to the Navier-Stokes equation in a rotating coordinate system

$$(11\text{-}11) \qquad \frac{d\underline{u}}{dt} = -\frac{1}{\rho}\nabla p - \nabla U_{\text{grav}} + \nu\nabla^2 u - \frac{2\,\Omega\times u}{\rho} + \underline{j}\times\underline{B}$$

We can understand $\underline{j}\times\underline{B}$ better if we expand it by substituting $\underline{j} = \epsilon_0 c^2\, \underline{\nabla}\times\underline{B}$

Jlini

So that

$$j \times B = \epsilon_0 c^2 (\nabla \times B) \times B$$

$$= \epsilon_{mkl}\, \epsilon_{kij}\, \partial_i B_j B_l$$

shift order: ch. sym

$\epsilon_{ijk} \partial_i B_j$ skip three so ch. sym

$$= -\epsilon_{mlk}\, \epsilon_{kij}\, \partial_i B_j B_l$$

rule

$$= -\left(\delta_{mi}\delta_{lj} - \delta_{mj}\delta_{il}\right) \partial_i B_j B_l$$

$$= -\partial_m B_l B_l + \partial_l B_m B_l$$

$$= -\tfrac{1}{2} \nabla B^2 + \underline{B} \cdot \nabla \underline{B}$$

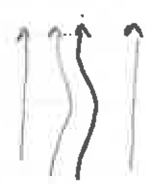
Tension

$$= -\tfrac{1}{2} \nabla B^2 + \tfrac{1}{2} \nabla \cdot \underline{B}\,\underline{B}$$

pressure term

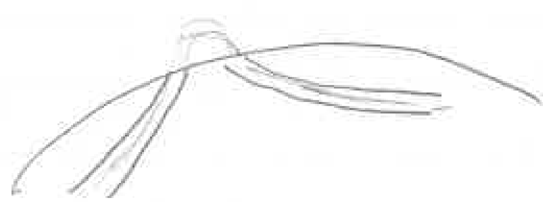
~ stress tensor adds stiffness

Can think of lines of force of B as elastic bands.

If the band is displaced  , the "tension" (actually current reduced) will tend to straighten.

Thus waves are possible, like waves in a string. These are called **Alfvén waves**. Their significance is that waves allow energy to be transmitted faster than allowed by the fluid motion.

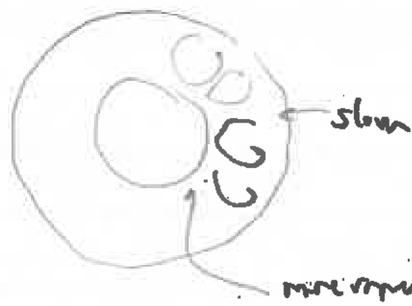
Another consequence comes from energy consideration. Magnetic lines of force depict the energy in the magnetic field. When densely packed, the energy density is high, and magnetic field is strong. It is reasonable these dense tubes of energy would push outward and reduce the density of the fluid in the tubes. Toroidal field of sun is therefore buoyant and tends to rise and break out at the surface.



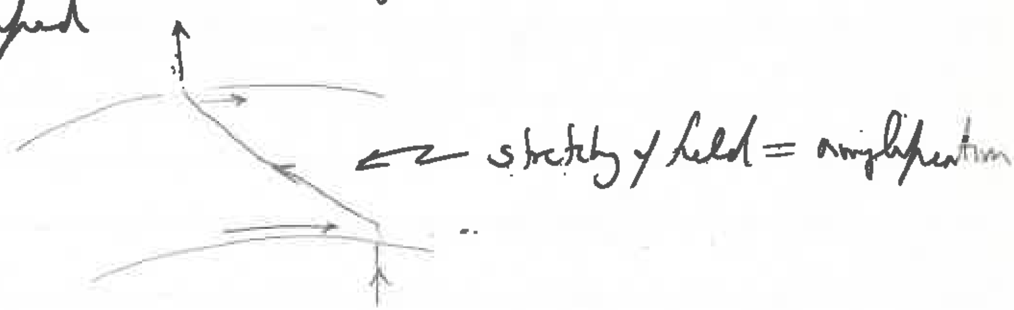
B. The Elsasser Dynamo – The logic

A. lines of magnetic force frozen in fluid

B. Rotating fluid that is convecting will rotate less rapidly (than rigid rotation) at the outer surface and more rapidly than the average rotation at the inner surface



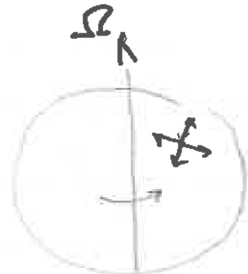
C. Then magnetic field will be amplified



D. An initial poloidal (dipole) field will be wrapped torroidally like rubber bands around a golf ball.

E. Torroidal bands will be buoyant and will be twisted by the corriolis force. where the fluid diverge or converge.

$$F_{coriolis} = -2\Omega \times u$$



- divergence twists anticyclonically (fluid left behind as move north)
- convergence twists cyclonically ...

divergence
anticyclonic twist
reverse field

cyclonic twist
convergence
reinforce

Think of a rotating disk at north
locally of [illegible]

Thus net twist of a rising magnetic toroid will be cyclonic if the deep convergence counts more, and anticyclonic if the surface or near surface divergence counts more. A cyclonic twist will rejuvenate the original poloidal field; an anticyclonic twist will destroy the original poloidal field and replace it with a reversed field. This can be most easily seen in the case of the sun. The

Bipolar magnetic regions (BMR's) are twisted anti-cyclonically, and, as described by Babcock (Astrophysical Journal, 133, p572 – 1961), this twist first neutralized the polloidal field and then replaces it with one of reversed polarity.

The N pole of the sun is replaced with a S pole, etc.

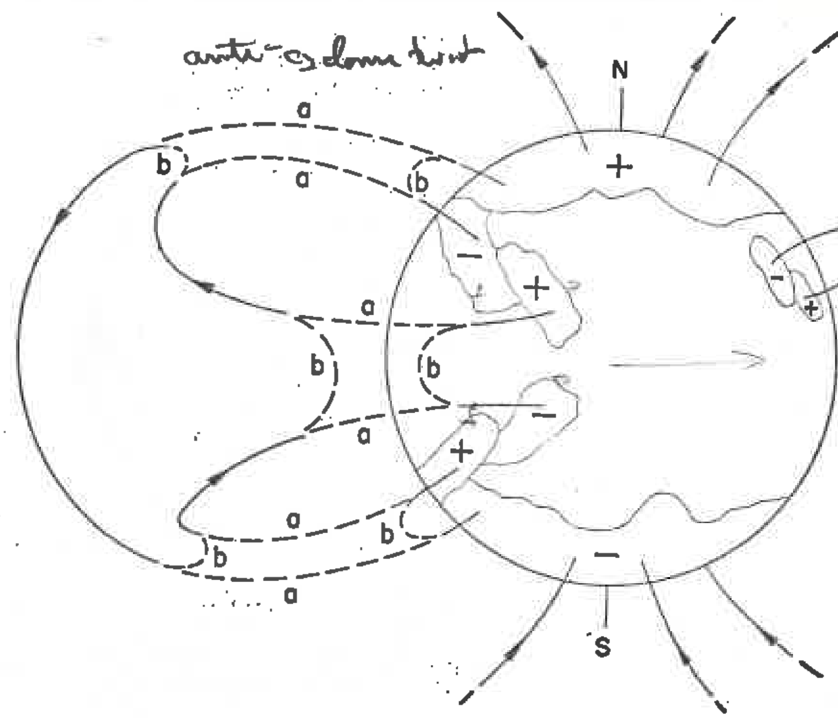


FIG. 8.—The expanding lines of force above older BMR's move out to approach the lines of force of the main dipole field. Severing and reconnection gradually occur, so that parts *a* are replaced by parts *b* and a portion of the main field is neutralized. Also a large flux loop of low intensity is liberated in the corona. Continuation of the process results in the formation of a new main dipolar field of reversed polarity.

The smooth / contrary surface of the sun is its top surface so anti-cyclonic circulation rather sense.

For the earth, the lower surface (inner core / outer core boundary) or the lower zone below the core mantle boundary seems to be controlling. If convergence at depth controls the twisting of a rising toroidal tube in the outer core, the twist of that loop will be cyclonic and the feedback to the poloidal field reinforcing. Then the earth's field will only reverse when the twists exceed 90°. The earth's poloidal field will reverse irregularly.

Support comes from meteorology where flow in or out of a pressure centre should be concentrated at the earth's surface (and centrally) whilst the outflow in the upper parts of the atmosphere is diffused and not centrally. Thus inflow to a low pressure system along the earth's surface will produce a cyclonic circulation despite the weak anti-cyclonic circulation higher in the atmosphere.

Elsasser points out these assymetries are required to produce the feedbacks that make the magnetic fields of the earth + sun possible.

assymetry of centrally simpler

Finally, of course the westward drift of the magnetic isopores and the faster rotation of the solid inner core are expected / predicted.

Why does the mantle not rotate as

slowly ~~as~~ the outer part of the outer core? This is due to electromagnetic coupling. The torroidal field is produced by the non-uniform rotation, and by Lenz's law this field must try to eliminate its cause (e.g. speed up the outer layers and overlying mantle). Electrical currents in the mantle provide the "friction" to keep it rotating faster than the outer-most outer core.

dishonour & Rubbish of E

We thus have a complete + good explanation of the generation of the earth's magnetic field. It is a non-linear, turbulent convection semi-quantitative conceptual model – like a description of a convecting pot of water. What could a numerical simulation of the process add?

magnetic lines of force, frozen into large conducting body, can be sheared (magnetic Intensity amplified) by Convection, Producing a very strong torroidal field.

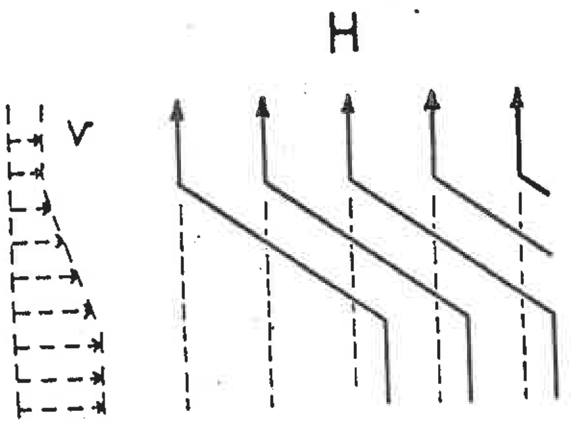
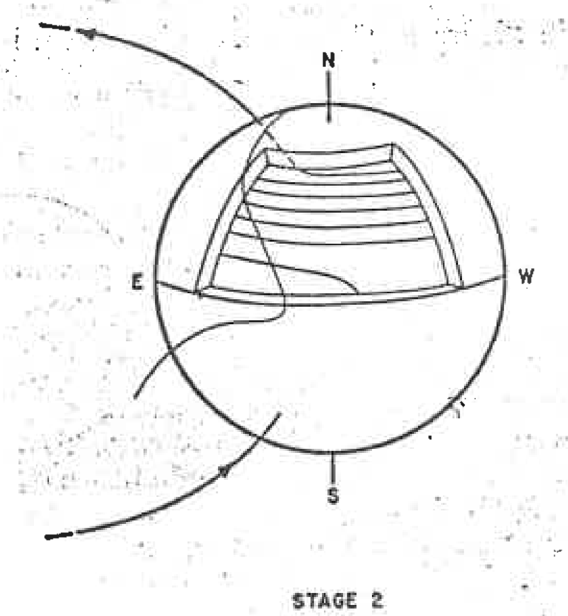


FIG. 2. Amplification of magnetic field by a linear velocity shear normal to field.



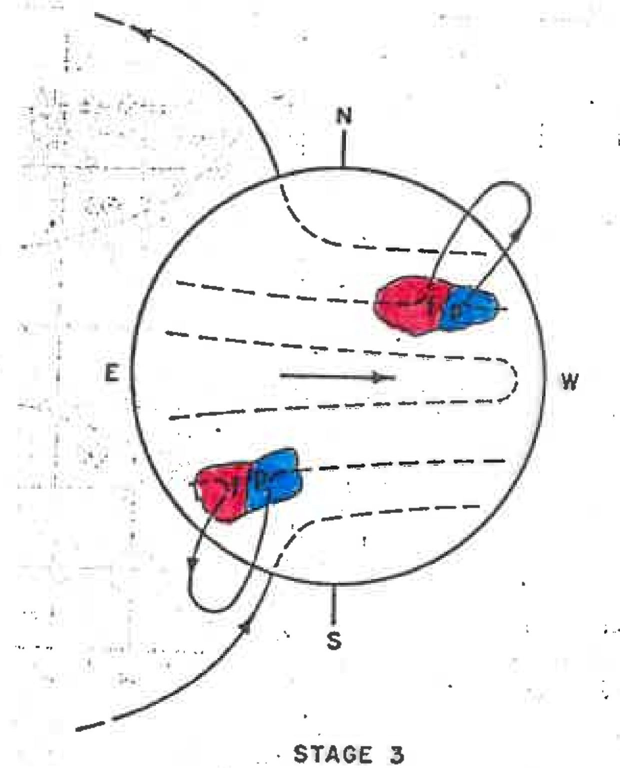


FIG. 2.—The submerged lines of force have been drawn out in longitude and wrapped around the sun by the differential rotation, with a consequent amplification of field strength that depends on latitude.

FIG. 3.—Bipolar magnetic regions (BMR's) are formed where buoyant flux loops of the submerge toroidal field are brought to the surface. The BMR's continue to expand, and the flux loops rise high into the corona.

Torroidal field breaks through as sunspots, which rotate anticyclonically and produce poloidal field of opposite direction.

$\vec{F} = -2\vec{\Omega} \times \vec{v}$

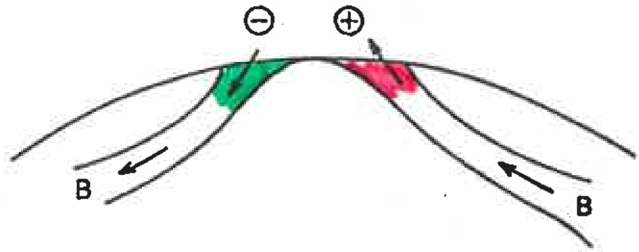


FIG. 10. Formation of a sunspot pair.

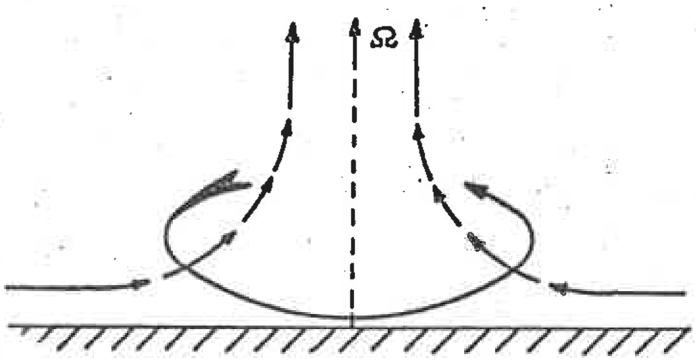


FIG. 5. Creation of cyclonic circulation by convergence of the fluid in a plane normal to the axis of rotation.

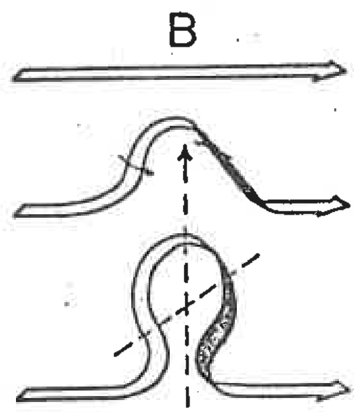
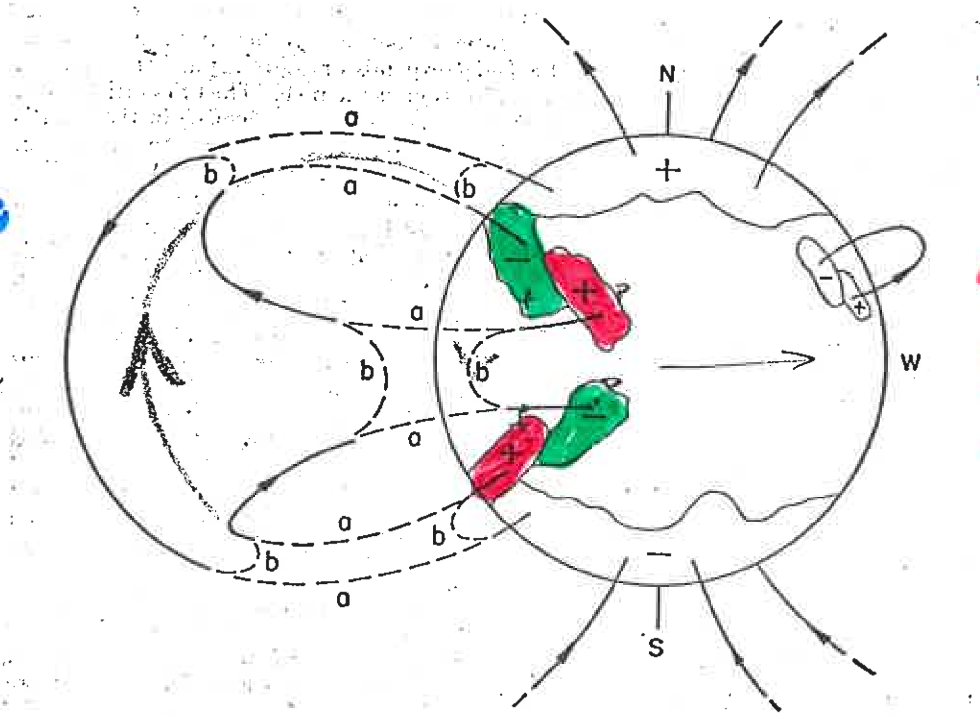


FIG. 6. A strand of toroidal field (top) is lifted (middle) and twisted (bottom) giving rise to a loop in the meridional plane normal to the original field.

Anti-cyclonic twist = negative feedback



Smooth Surface = top

⇒ Anticyclones

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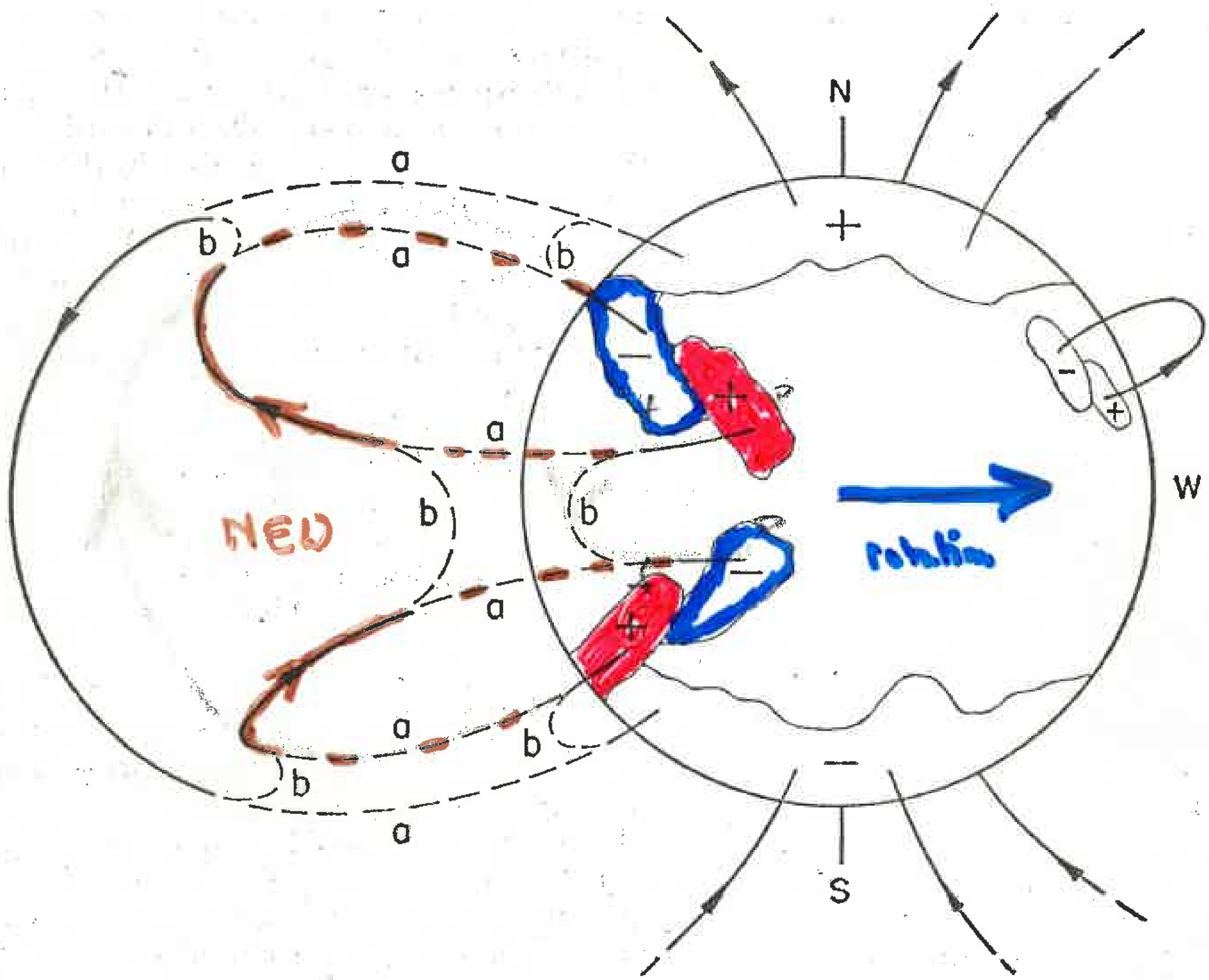

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