Lecture 15. Fourier Transform techniques

(1)
$$\boxed{\phi_n(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta}$$

For any function $f(\theta)$, require $\phi(\theta) \overset{\text{e.g. matches}}{=} f(\theta)$ for which:

(2)
$$\int_0^{2\pi} \phi_n(\theta) \begin{matrix}\cos\\ \sin\end{matrix} n\theta \, d\theta = \int_0^{2\pi} f(\theta) \begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix} \, d\theta$$

for $n = 0, 1, \ldots n$. The left side equals $\pi \begin{matrix}a_n\\ b_n\end{matrix}$, $n = 1, \ldots n$ and $2\pi a_0$ for $k = 0$. The Fourier coefficients a_k, b_k are thus:

(3)
$$\boxed{\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \begin{matrix}\cos\\ \sin\end{matrix} n\theta \, d\theta} \quad , n = 1, 2 \ldots n$$

Orthogonality relationship

(4)
$\phi(\theta) = f(\theta)$ because $\displaystyle\int_0^{2\pi} \cos p\theta \cos q\theta \, d\theta = \int_0^{2\pi} \cos p\theta \sin m\theta \, dx$

$$= \int_0^{2\pi} \sin p\theta \sin q\theta = 0$$ where p, q, m are integers and $p \neq q$. If $p = q$ the integral $= \pi$.

Thus the left side equals $\pi \begin{pmatrix} a_n \\ b_n \end{pmatrix}$, and (2) follows.

Since $\sin n\theta = \dfrac{e^{in\theta} - e^{-in\theta}}{2i}$ and the $\sin n\theta$ term will select odd functions where $f(\theta) = -f(-\theta)$, we

Can express (1):

$$\phi_n(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n i \sin n\theta$$

$$= \sum_{n=0}^{\infty} a_n \frac{e^{i\theta n} + e^{-i\theta n}}{2} + \sum_{n=-\infty}^{-1} \underset{-a_n}{\underset{\shortparallel}{b_n}}\, \cancel{i}\, \frac{e^{i\theta n} - e^{-i\theta n}}{2\cancel{i}}$$

(1b) $$\phi_n(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{i\theta n} \qquad \theta = \frac{2\pi j}{p}$$

We can let $\theta = \frac{2\pi x}{L}$, and note the equivalent of (4) is

(4b) $$\int_0^L e^{-2\pi i m x/L} e^{2\pi i n x/L} = L\delta_{mn}.$$

Thus the equivalent of (2) can be written

$$\int_0^L \phi_n(x) e^{-2\pi i n x/L} dx = \int_0^L f(x) e^{-2\pi i n x/L} dx$$

$$\sum_{n=-\infty}^{\infty} a_n \int_0^L e^{-2\pi i n x/L} e^{2\pi i m x/L} dx = \int_0^L f(x) e^{-2\pi i m x/L} dx$$

(3b) $$L a_n = \int_0^L f(x) e^{-2\pi i n x/L} dx$$

Now if we define $k = n/L$, we can express

The transforms more symmetrically:

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x/L} = \int_{-\infty}^{\infty} a_n e^{2\pi i n x/L}\, dn$$

$$\parallel$$

$$f(x) \underset{\text{since } dk = dn/L}{=} L \int_{-\infty}^{\infty} a_n e^{2\pi i k x}\, dk$$

If we now define $L a_n = F(k)$, define the integral in (3b) between $-L/2$ and $L/2$, and let $L \to \infty$, we find

$$\boxed{\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(k)\, e^{2\pi i k x}\, dk \\ F(k) &\underset{(3b)}{=} \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx \end{aligned}} \qquad (5)$$

This is a beautifully symmetric transform pair. The first equation is the inverse $(+i)$ Fourier transform. The second equation $(-i)$ is the forward Fourier transform. k is the oscillation frequency which equals $1/\lambda$ where λ is the wavelength.

If we define $k' = \frac{2\pi n}{L} = 2\pi k = 2\pi/\lambda$ we find

(6)
$$\boxed{\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(k') \, e^{ik'x} \, dk' \\ F(k') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) \, e^{-ik'x} \, dx \end{aligned}}$$

inverse FT

forward FT

where $k' = 2\pi/\lambda$. This transform pair is more physically intuitive but involves a normality constant $1/2\pi$ that is mathematically displeasing. We'll use this transform pair, since we need physically intuitive results. Notice that we place the normality constant in the forward transform, rather than the inverse, where it would naturally appear. In either case the meaning of the transforms is clear. The forward transform gives the amplitude of cos (real) and sin (complex) wave numbers $(2\pi/\lambda)$ that must be added together to produce $f(x)$. These amplitudes are expressed as a function $F(k)$. For some particular wavelength λ, the $\cos \frac{2\pi x}{\lambda}$ coefficient is the real part of $F(2\pi/\lambda)$, and the $\sin \frac{2\pi x}{\lambda}$ coefficient is the complex part of $F\left(\frac{2\pi}{\lambda}\right)$. Programs exist to do the fourier transforms, and these numerical transforms have been cleverly optimized so that they are very fast.

In two dimensions we define wave numbers in both the x and y directions. Call them k_x and k_y. The transform pair is then

(7)
$$\boxed{\begin{aligned} \bar{u}(k_x, k_y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y)\, e^{-ik_x x}\, e^{-ik_y y}\, dx\, dy \\ u(x,y) &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{u}(k_x, k_y)\, e^{ik_x x}\, e^{ik_y y}\, dx\, dy \end{aligned}}$$

where the exponential products might also be written

$$e^{ik_x x}\, e^{ik_y y} = e^{i\underline{k}\cdot\underline{r}}.$$