Lecture 16 Glacial Rebound Equations and Their Solution

I. The Equations

For slow deformations where momentum is unimportant, $\rho \frac{Du_i}{Dt} = 0$, and the Cauchy conservation of momentum equation (5-4 in notes) becomes:

(1) $$\rho \underline{g} + \underline{\nabla} \cdot \underline{\underline{\tau}} = 0$$

The material we consider (the earth) lies in a self-generated gravitational field and the earth is <u>hydrostatically pre-stressed</u> (eg pressure increases with depth). This pre-stress causes no deformation and is thus conveniently subtracted out. The stresses are applied to the earth's surface by redistributions/load on the surface and this load-redistribution causes the deformation. If we let

$$\tau_{ij} = -p_0(z)\,\delta_{ij} + \sigma_{ij}$$

Deviatoric stress tensor — stress tensor with hydrostatic pressure subtracted out

Then (1) becomes:

(2) $$\rho \underline{g} - \underline{\nabla} p_0 + \underline{\nabla} \cdot \underline{\underline{\sigma}}_{ij} = 0$$

From our previous discussion we know this equation must apply differently to an elastic solid and a fluid. A load floats in a fluid because the deformed fluid moves through the equilibrium (rest state) pressure field. A deformed solid carries its zero-order pressure field with it when it deforms. There is no elastic buoyancy. Love realized this in 1911. Pressure is advected in elastic deformation. If the pressure in the elastic body is p and a load is applied at t_0, a very short time, δt, later, after the elastic deformation is completed, the pressure will be

$$p_0\Big|_{t+\delta t} = p_0\Big|_{t_0} - \underline{u}^e \cdot \underline{\nabla} p_0 \tag{3}$$

where the superscript e indicates elastic displacement.

Now linearize the problem by expanding ρ and $\underline{g}$ into rest state (no deformation) and perturbations

(4)
$$\rho(\underline{x},t) = \rho_0(x_1) + \rho_1(\underline{x},t)$$
$$\underline{g}(\underline{x},t) = g_0(x_1) + g_1(x,t).$$

Note from (1) that when $\underline{\underline{\sigma}}=0$, the rest state is

(5)
$$\underline{\nabla} p_0 - \rho_0 \underline{g_0} = 0 \quad .$$

Substitution into (2) for the elastic equation where

$p_0\big|_{t+\delta t} = p_0\big|_{t_0} - \underline{u}^e \cdot \underline{\nabla} p_0$ yields

$$\underline{\nabla}\cdot\underline{\underline{\sigma}} \underbrace{\underbrace{-\nabla p_0} + \underline{\nabla}(\underline{u}^e\cdot\nabla p_0) + \underbrace{\rho_0 \underline{g_0}}}_{\underset{(4,5)}{=0}} + \rho_1 \underline{g_0} + \rho_0 \underline{g_1} =$$

The viscous equation is:

$$\underline{\nabla}\cdot\underline{\underline{\sigma}} \underbrace{- \nabla p_0 + \rho_0 \underline{g_0}}_{\overset{\shortparallel}{0}} + \rho_1 \underline{g_0} + \rho_0 \underline{g_1} = 0$$

Thus

Elastic Eqn (6)
$$\boxed{\underline{\nabla}\cdot\underline{\underline{\sigma}} + \underline{\nabla}(\underline{u}^e\cdot\underline{\nabla} p_0) + \rho_1 \underline{g_0} + \rho_0 \underline{g_1} = 0}$$

viscous Eqn (7)
$$\boxed{\underline{\nabla}\cdot\underline{\underline{\sigma}} + \rho_1 \underline{g_0} + \rho_0 \underline{g_1} = 0}$$

These equations can be simplified by expressing ρ_1 in another form. To first order the continuity equation is

$\frac{\partial \rho_1}{\partial t} + \underline{\nabla} \cdot \rho_0 \underline{v} = 0$. Integrating over δt we find

$$\rho_1\Big|_{t+\delta t} = \underline{\nabla} \cdot \rho_0 \underline{u}^e = \underline{u}^e \cdot \nabla \rho_0 + \rho_0 \underline{\nabla} \cdot \underline{u}_e = u_1^e \frac{\partial \rho_0}{\partial x_1} + \rho_0 \underline{\nabla} \cdot \underline{u}_e .$$

The second term in the elastic equation can be expressed:

$$\nabla \left(u^e \cdot \nabla p_0 \right) = \nabla \left(u^e \rho_0 \underline{g}_0 \right) = - \nabla u_1 \rho_0 g_0$$

$$= -\rho_0 \underline{\nabla} u_1 g_0 - g_0 u_1 \frac{\partial \rho_0}{\partial x_1} \hat{x}_1 .$$

Then

$$\underline{g_0} \rho_1 + \nabla (u^e \cdot \nabla p_0) = \hat{x}_1 g_0 \left(\cancel{u_1^e \frac{\partial \rho_0}{\partial x_1}} + \rho_0 \nabla \cdot u_e \right) - \rho_0 g_0 \nabla u_1^e - \cancel{g_0 u_1 \frac{\partial \rho_0}{\partial x_1}}$$

$$= \rho_0 g_0 \nabla \cdot u^e \, \hat{x}_1 - \rho_0 \nabla u_1^e g_0$$

Thus (6) becomes

Elastic (6a)

$$\boxed{\underline{\nabla} \cdot \underline{\underline{\sigma}} - \rho_0 \underline{\nabla} u_1^e g_0 + \rho_0 g_0 (\nabla \cdot u^e) \hat{x}_1 + \rho_0 g_1 = 0}$$

And by similar methods the viscous equation (7) is

viscous (7a)

$$\boxed{\underline{\nabla} \cdot \underline{\underline{\sigma}} + \left(\rho_0 g_0 \nabla \cdot u^e + g_0 u_1^v \frac{\partial \rho_0}{\partial x_1} \right) \hat{x}_1 + \rho_0 g_1 = 0}$$

Notice that we assume change in material density is due to elastic deformation, while buoyancy arises entirely from viscous movements through the zero-order pressure field. Remember $x_1 = \hat{r}$ or $\hat{z}$. In fact $\frac{\partial \rho_0}{\partial x_1}$ is the non-adiabatic density gradient. If the fluid changes density and temperature so it is exactly similar to its surroundings, no buoyant forces arise. Finally, notice that pressure does not occur in these equations. We apply stresses to the surface of the earth by re-distributing loads and these stresses cause elastic and viscous deformation. Avoiding the complication of pressure, a scalar separate from the stress tensor, is a conceptual advantage.