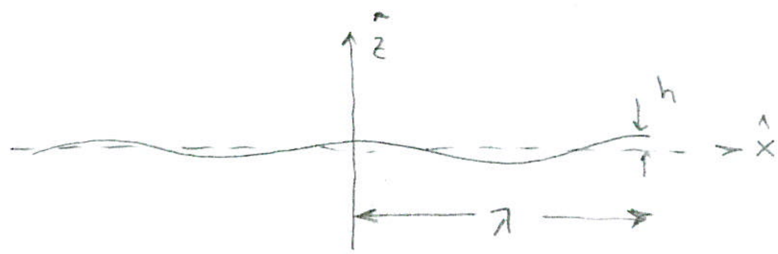
Lecture 17 Exploration + rough Applications

D. What the Solutions Mean

We now have all the theory we need to interpret glacial rebound and infer the viscosity of the earth's mantle, but the meaning of these solutions are not yet clear. To make them clear we will first examine the solutions we have obtained in some detail. We will then develop a simple method for examining glacial rebounds of various scales to infer the viscosity profile of the earth. Finally we will return to some critical details and present some computer solutions. The sequence of extensive theory → insights + simplifications → back of the envelope analysis of physical processes → modeling + testing of subtler insights is a common cycle in my experience.

1. Flow in a uniform viscosity half-space

Equation (22) describes the flow produced by a $1\ \mathrm{N\,m^{-2}}$ harmonic load applied to the surface of a fluid half space. To visualize more physically, consider the harmonic load $\bar{\tau}_{zz}$ to be a small deformation of the surface of amplitude h:



The vertical stress is then $\tau_{zz} = -\rho g h \cos kx$, where $k = 2\pi/\lambda$. The fourier transform $\bar{\tau}_{zz} = -\rho g h$, and equation (22) becomes:

$$
\begin{bmatrix} 2\eta^* i k \bar{v}_x \\ 2\eta^* k \bar{v}_z \\ i\,\bar{\tau}_{xz} \\ \bar{\tau}_{zz} \end{bmatrix} = \rho g h\, e^{kz} \left[\begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} + kz \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right] \qquad (22a)
$$

The flow response is the real part of the Fourier transform (6):

$$f(x) = \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk.$$

Since $e^{ikx} = \cos kx + i \sin kx$, for a single value of k:

$$v_x = -\frac{\rho g h}{2\eta^* k}\, kz\, e^{kz} \sin kx$$

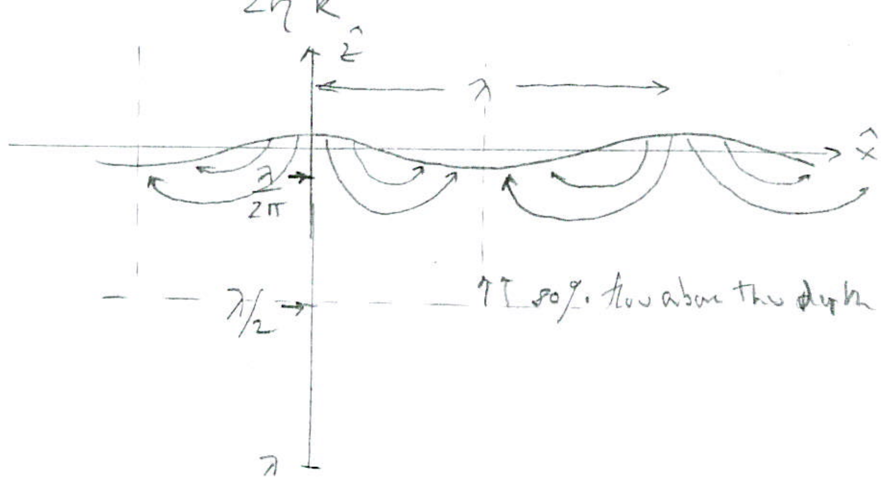
(22b)

$$v_z = \frac{\rho g h}{2\eta^* k} (kz - 1) e^{kz} \cos kx .$$

The horizontal and vertical velocities decrease nearly exponentially with depth. v_x is zero at the surface and at $x = 0, \lambda/2 \ldots$ but increases to a maximum $\underbrace{\text{at } x = \lambda/4, 3\lambda/4 \ldots}_{\text{at } z = \lambda/2\pi}$ At the surface v_z is a maximum at $x = 0, \pi, 2\pi, \ldots$

The response under a load (eg. $x=0$) is

$v_z(x=0, t=0) = -\frac{\rho g h}{2\eta^* k}$. Flow lines can be sketched:



The flow described by (22a,b) is an instantaneous flow rate. At $x=0, z=0, t=0$ $v_z(x=0, z=0, t=0) = \frac{-\rho g h}{2\eta^+ k}$

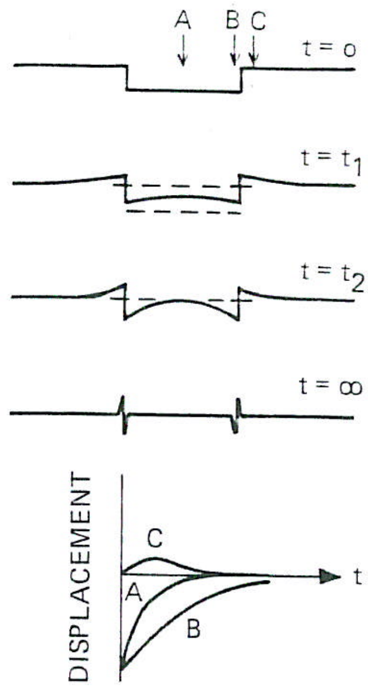
Since $v_z = \frac{\partial h}{\partial t}$ it is clear the amplitude of the initial harmonic deformation of the fluid surface will decay exponentially:

(26) Deep flow

$$\boxed{h = h_0 e^{-t/\tau}, \quad \tau = \frac{2\eta^+ k}{\rho g}}$$

Since $k = 2\pi/\lambda$, it is clear from (26) that longer wavelength surface deformations will have proportionally shorter decay times (τ). Since high frequencies of a square edged initial surface deformation are composed of short wavelengths (high frequencies), uplift of an area loaded by a glacier that has melted will look sorely:

DEEP FLOW



The depth to which significant flow persists can be easily estimated. From 22b

$$\frac{v_z}{v_z(z=0)} = -(zk-1)\, e^{kz}$$

which can be evaluated:

zk	$\frac{v_z}{v_z(z=0)}$	
0	1	
−1	.736	
−2	.406	← 50% @ $zk = -1.66$
−3	.199	← 80% @ $zk = -3$

Fifty percent of the flow beneath a load of wavenumber $\frac{2\pi}{k}$ occurs above a depth of $zk = -1.66$; 80% above a depth of $zk = -3$. For $zk = -3$,

$z = \frac{-3\lambda}{2\pi} \cong \frac{1}{2}\lambda$. Eighty percent of the flow occurs above a depth of $\frac{1}{2}\lambda$.

2. Flow in a Viscous Channel

For a viscous channel that is thin compared to λ, we saw from (23) that the vertical flow becomes

$$\bar{v}_z = \frac{\rho g h}{2\eta^* k}\left(\frac{2}{3}(kD)^3\right) = \frac{\rho g k^2 D^3}{2\eta} h$$

Thus

(27) Channel flow
$$\boxed{h = h_0 e^{-t/\tau_c}, \quad \tau_c = \frac{2\eta}{\rho g k^2 D^3}}$$

The short wavelengths decay much faster because here $\tau \propto \lambda^2$, and in fact heat conduction a very good analogue for the