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Lecture 3 Vectors, Tensors, Gauss + Stokes Theorems

Differential Calculus of Vector Fields

Feynman Physics Volume:

2–1 Understanding physics

Review: Chapter 11, Vol. I, *Vectors*

The physicist needs a facility in looking at problems from several points of view. The exact analysis of real physical problems is usually quite complicated, and any particular physical situation may be too complicated to analyze directly by solving the differential equation. But one can still get a very good idea of the behavior of a system if one has some feel for the character of the solution in different circumstances. Ideas such as the field lines, capacitance, resistance, and inductance are, for such purposes, very useful. So we will spend much of our time analyzing them. In this way we will get a feel as to what should happen in different electromagnetic situations. On the other hand, none of the heuristic models, such as field lines, is really adequate and accurate for all situations. There is only one precise way of presenting the laws, and that is by means of differential equations. They have the advantage of being fundamental and, so far as we know, precise. If you have learned the differential equations you can always go back to them. There is nothing to unlearn.

It will take you some time to understand what should happen in different circumstances. You will have to solve the equations. Each time you solve the equations, you will learn something about the character of the solutions. To keep these solutions in mind, it will be useful also to study their meaning in terms of field lines and of other concepts. This is the way you will really "understand" the equations. That is the difference between mathematics and physics. Mathematicians, or people who have very mathematical minds, are often led astray when "studying" physics because they lose sight of the physics. They say: "Look, these differential equations—the Maxwell equations—are all there is to electrodynamics; it is admitted by the physicists that there is nothing which is not contained in the equations. The equations are complicated, but after all they are only mathematical equations and if I understand them mathematically inside out, I will understand the physics inside out." Only it doesn't work that way. Mathematicians who study physics with that point of view—and there have been many of them—usually make little contribution to physics and, in fact, little to mathematics. They fail because the actual physical situations in the real world are so complicated that it is necessary to have a much broader understanding of the equations.

What it means really to understand an equation—that is, in more than a strictly mathematical sense—was described by Dirac. He said: "I understand what an equation means if I have a way of figuring out the characteristics of its solution without actually solving it." So if we have a way of knowing what should happen in given circumstances without actually solving the equations, then we "understand" the equations, as applied to these circumstances. A physical understanding is a completely unmathematical, imprecise, and inexact thing, but absolutely necessary for a physicist.

Ordinarily, a course like this is given by developing gradually the physical ideas—by starting with simple situations and going on to more and more complicated situations. This requires that you continuously forget things you previously learned—things that are true in certain situations, but which are not true in general. For example, the "law" that the electrical force depends on the square of the distance is not *always* true. We prefer the opposite approach. We prefer to take first the *complete* laws, and then to step back and apply them to simple situations, developing the physical ideas as we go along. And that is what we are going to do.

So now we set the mathematician under our belt.

Outline

- Vectors + Tensors — defn
- Manipulation
- Eigenfunctions
- Rotation + vorticity
  symm, antisymm
- Invariants

Lecture 3 – Vectors, Tensors, Green + Stokes Theorems

Structure of mathematics reflects fundamentals

perfect differential

structure of thermodynamics

Here we consider vectors + tensors

I. Vectors + Tensors

Vector – any quantity which transforms like a vector under rotation of a coordinate system – eg pressure length etc

Tensor – any quantity that rotates like a vector (set of vectors)

Physical basis – physical properties should be the same no matter what reference frame we use

Rotate axes

$$\underset{\substack{\uparrow\ \ \uparrow \\ \text{row}\ \ \text{column}}}{C_{ij}} = \begin{pmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{pmatrix}$$

α_{ij} ← rotated axes

original axes

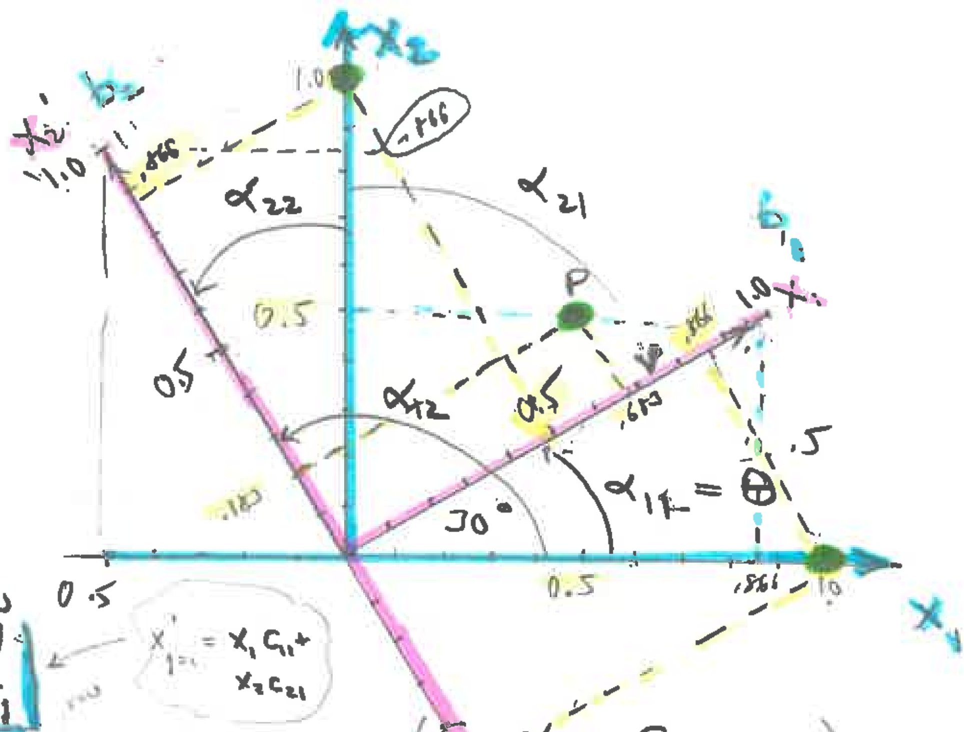
Simple Example of rotational invariance

row ↓ ↓ column

α_{ij}

↑ Rotated axis

orig axis



$\alpha_{11} = \theta$

$\alpha_{12} = \theta + 90°$

$\alpha_{21} = 90° - \theta$

$\alpha_{22} = \theta$

RULE

Rotation rule

row

$$\boxed{x'_j = x_i C_{ij}}$$

$x'_{j=1} = x_1 C_{11} + x_2 C_{21}$

row

$$x'_j = (C^T)_{ji}\, x_i$$

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$C_{ij} = \begin{pmatrix} \cos\alpha_{11} & \cos\alpha_{12} \\ \cos\alpha_{21} & \cos\alpha_{22} \end{pmatrix}$$

$$\underset{\alpha_{11}=30°}{=} \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$$

$$\begin{array}{cc} \| & \| \\ b_1 & b_2 \end{array}$$

rotated axes in old coord system

Rule

$$\boxed{\underline{x}' = \underline{\underline{C}}^T \underline{x}}$$

$$\underline{\underline{C}}^T \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.683 \\ 0.183 \end{pmatrix}$$

P old coord $\qquad$ P' new coord

$$\underline{\underline{C}}^T \underset{x \text{ axis}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \underset{x' \text{ axis}}{\begin{pmatrix} 0.866 \\ -0.5 \end{pmatrix}}$$

$$\underline{\underline{C}}^T \underset{y \text{ axis}}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \underset{y' \text{ axis}}{\begin{pmatrix} 0.5 \\ 0.866 \end{pmatrix}}$$

points stay in same position. Just coordinates differ in rotated + un-rotated axes

all just looking at same point in old or new, rotated axis

③

$$\underline{\underline{C}}\,\underline{x}' = \underline{\underline{C}}\,\underline{\underline{C}}^T \underline{x} = x \;;\quad \underline{\underline{C}}\,\underline{\underline{C}}^T = \underline{\underline{I}}$$

Now the reverse

$$\boxed{\underline{x} = \underline{\underline{C}}\,\underline{x}'}$$

just sum over repeated index

$$\boxed{x'_j = C_{ij}\, x_i}$$

$$\underline{\underline{C}} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{new axis}} = \underbrace{\begin{pmatrix} 0.866 \\ 0.5 \end{pmatrix}}_{\text{old axis}}$$

Which by inspection is clearly correct

sum rows of $\underline{\underline{C}}$ along a column

So vectors transform:

sum over first index

(1)

$$\boxed{\begin{aligned} \underline{x}' &= \underline{\underline{C}}^T \underline{x} \quad \text{or} \quad x'_j = x_i \underbrace{C_{ij}}_{\text{row}\;\text{col}} = (C^T)_{ji} x_i \\ \underline{x} &= \underline{\underline{C}}\,\underline{x}' \quad \text{or} \quad x_i = C_{im} x'_m \end{aligned}}$$

for matrix multiplication need to avoid another of transforms

sum cols of $\underline{\underline{C}}$ along a row

sum over repeated index

Order of multiplication does not matter for index method

$$\begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\underbrace{(C^T)_{ji}}_{\text{row}\;j\;i\;\text{col}}\, x_i$$

$$\underline{\underline{\tau}}' = \underline{\underline{C}}^T \underline{\underline{\tau}}\, \underline{\underline{C}}$$

Generalization — Tensors + higher order tensors transform like vectors

a tensor is just a higher order vector

(2)

tensors transform like vectors

$$\boxed{\begin{aligned} \tau'_{mn} &= C_{im}\, C_{jn}\, \tau_{ij} \\ A'_{mnpq} &= C_{im}\, C_{jn}\, C_{kp}\, C_{lq}\, A_{ijkl} \end{aligned}}$$

Can get a tensor by multiplying vectors, taking gradient, or defining quantities (such as stress) which involve two directions (e.g. Force acting on an oriented surface). Consider Stress:

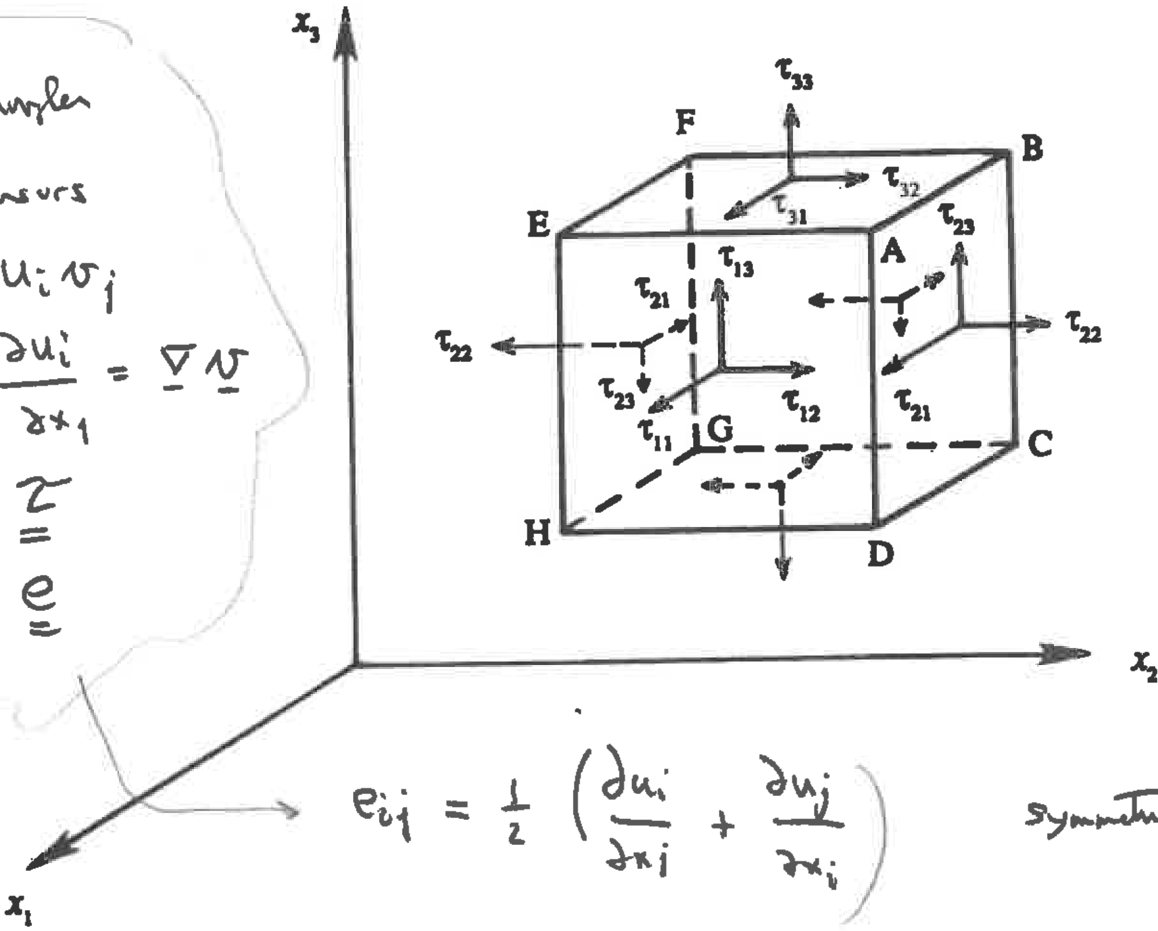
Examples of Tensors

$u_i v_j$

$\frac{\partial u_i}{\partial x_1} = \underline{\nabla}\,\underline{v}$

$\underline{\underline{\tau}}$

$\underline{\underline{e}}$



$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ symmetric

τ_{ij}

i ← outward normal

j ← direction of force

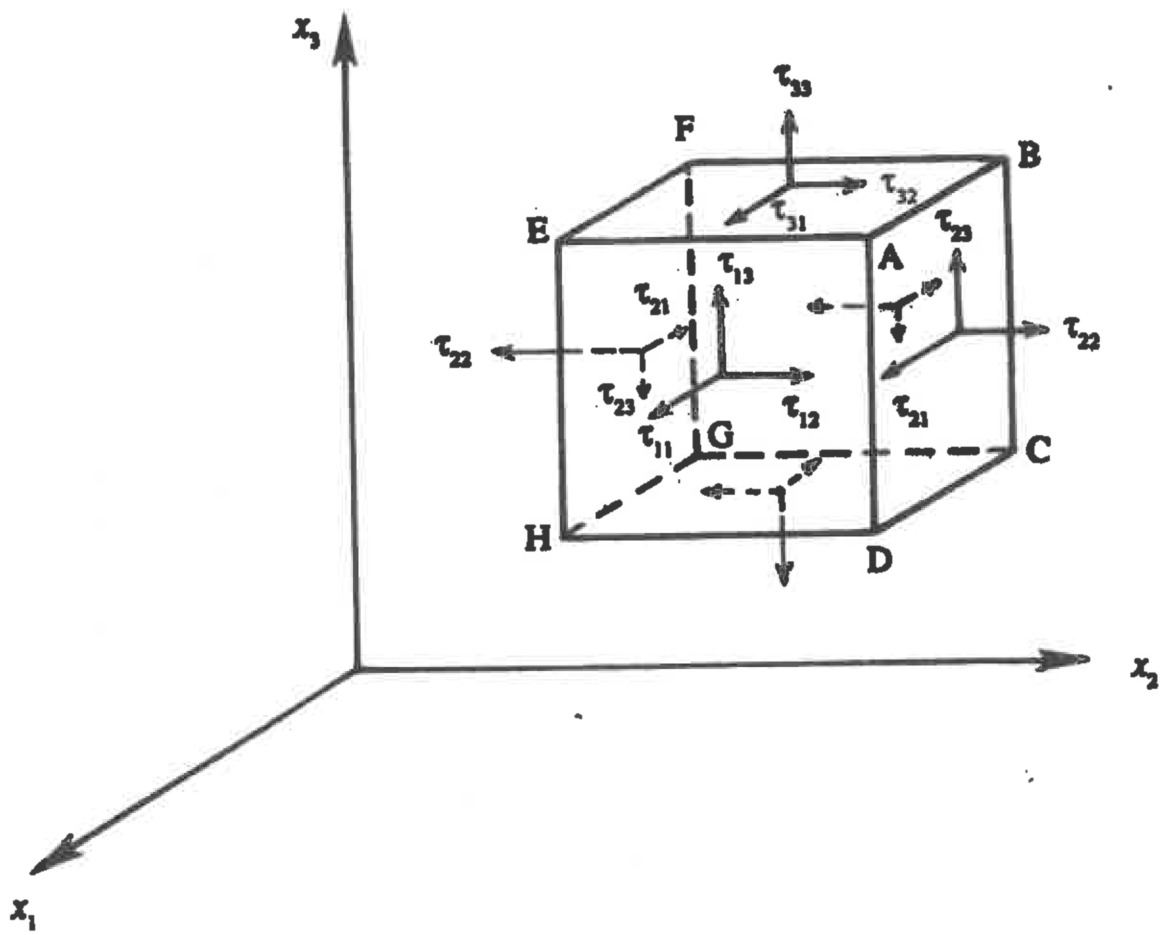
if $\hat{n}$ has opposite sign all reversed —

Compare ABCD with EFGH

$\underline{f} = \hat{n} \cdot \underline{\underline{\tau}}$

Note stress is balanced by pairs of forces acting in an infinitesimally small space (a pt)

$\tau_{22} \longleftarrow \longrightarrow \tau_{22}$ positive stress is tensile



x3
τ33
F
B
τ32
E
τ31
τ23
A
τ13
τ21
τ22
τ22
τ23
τ12
τ21
τ11
G
C
H
D
x2
x1

II. Multiplication and contraction

Higher order tensors can be formed by multiplying lower order tensors

$$u_i v_j = B_{ij}$$
$$P_{ijkl} = A_{ij} B_{kl}$$

If parts transform as tensors, product will also.

Contraction – set two indices equal + sum – reduces order of a tensor. e.g. dot product of 2 vectors = scalar, dot product of a vector and tensor is a vector.

Two 2nd order tensors

outer $$A_{ij} B_{ki} = B_{ki} A_{ij} = (\underline{\underline{B}} \cdot \underline{\underline{A}})_{kj}$$

inner $$A_{ij} B_{jk} = (\underline{\underline{A}} \cdot \underline{\underline{B}})_{ik}$$

left $$A_{ij} B_{ik} = (A^T)_{ji} B_{ik} = (\underline{\underline{A}}^T \cdot \underline{\underline{B}})_{jk}$$

right $$A_{ij} B_{kj} = A_{ij} (B^T)_{jk} = (\underline{\underline{A}} \cdot \underline{\underline{B}}^T)_{ik}$$

Second order tensor and vector

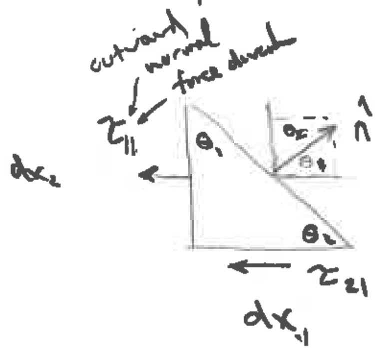
$$A_{ij} u_j = (\underline{\underline{A}} \cdot \underline{u})_i$$
$$A_{ij} u_i = (A^T)_{ji} u_i = (\underline{\underline{A}}^T \cdot \underline{u})_j \overset{?}{=} (\underline{u} \cdot \underline{\underline{A}})_j$$

double contraction two 2nd order tensors

$$A_{ij} B_{ji} = \underline{\underline{A}} : \underline{\underline{B}}$$
$$A_{ij} B_{ij} = A_{ij} (B^T)_{ji} = \underline{\underline{A}} : \underline{\underline{B}}^T$$

contraction example – force per unit area on a surface

$$\left.\frac{\text{force}}{\text{area}}\right|_{\text{surface}} = \hat{n}\cdot\underline{\underline{\tau}}$$



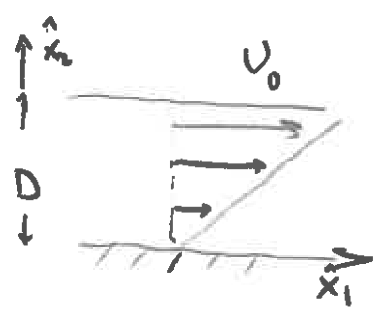
$$\frac{F_1}{ds} = \tau_{11}\underbrace{\frac{dx_2}{ds}}_{=\cos\theta_1 = \hat{n}_1} + \tau_{21}\underbrace{\frac{dx_1}{ds}}_{=\cos\theta_2 = \hat{n}_2}$$

$$f = \tau_{ij}\, n_i$$

symmetric

$$\underline{f} = \underset{\text{row}}{\tau_{j}}{}_{\underset{\text{col}}{i}}\, n_i = \underline{\underline{\tau}}\cdot\hat{n}$$

III: Views in rotated coordinate systems



Consider a case of 2D flow in the $\hat{x}$ direction that is only a function of $\hat{y}$

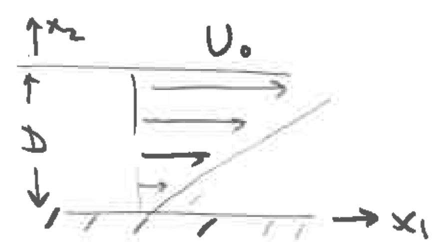
so $\underline{u} = \begin{bmatrix} u_1(x_2) \\ 0 \end{bmatrix}$

$$e_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\underline{e} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \partial u_1/\partial x_1 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

If $u_1 = \dfrac{U_0 x_2}{D}$



$$\underline{\underline{e}} = \begin{bmatrix} 0 & \frac{1}{2}\frac{U_0}{D} & 0 \\ \frac{1}{2}\frac{U_0}{D} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

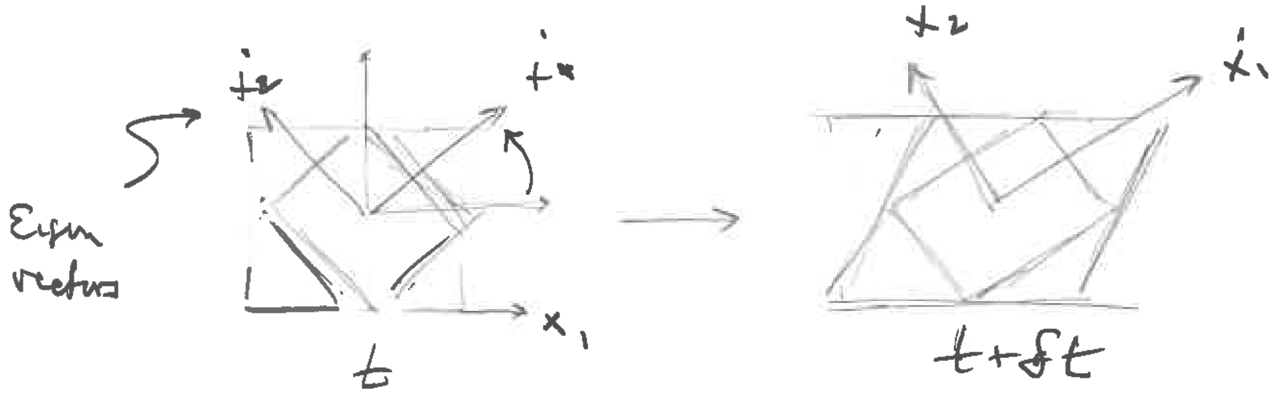
$$\underline{\underline{e}}_{2D} = \begin{bmatrix} 0 & \Gamma \\ \Gamma & 0 \end{bmatrix}, \quad \Gamma = \frac{1}{2}\frac{\partial u_1}{\partial x_2} = \frac{1}{2}\frac{U_0}{D}$$

If we rotate:

$$\underline{\underline{e}} = \begin{bmatrix} 0 & \Gamma \\ \Gamma & 0 \end{bmatrix}$$

by 45° it becomes:

$$\underline{\underline{e}}' = \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix}$$



Fluid element stretched in x_1' direction and squashed in x_2' direction. x_1' and x_2' are principal axes of deformation tensor $\underline{\underline{e}}$.

```
      8 2⍕rotation_matrix_2D 0
no rot
1.00   0.00
0.00   1.00
      8 2⍕rotation_matrix_2D 45
45°
0.71   ¯0.71
0.71    0.71
                  Cᵀ                                        C
      8 2⍕(⍉rotation_matrix_2D 45)+.×((2 2⍴0 1 1 0)+.×rotation_matrix_2D 45)
Diagonal
1.00    0.00
0.00   ¯1.00
       (2 2⍴0 1 1 0)                    [0 1]
 0 1                                    [1 0]
 1 0

      8 2⍕(⍉rotation_matrix_2D 0)+.×((2 2⍴0 1 1 0)+.×rotation_matrix_2D 0)
0°
0.00    1.00
1.00    0.00
      8 2⍕(⍉rotation_matrix_2D 10)+.×((2 2⍴0 1 1 0)+.×rotation_matrix_2D 10)
10°
0.34    0.94
0.94   ¯0.34
      8 2⍕(⍉rotation_matrix_2D 20)+.×((2 2⍴0 1 1 0)+.×rotation_matrix_2D 20)
20°
0.64    0.77
0.77   ¯0.64
      8 2⍕(⍉rotation_matrix_2D 30)+.×((2 2⍴0 1 1 0)+.×rotation_matrix_2D 30)
30°
0.87    0.50
0.50   ¯0.87
      8 2⍕(⍉rotation_matrix_2D 40)+.×((2 2⍴0 1 1 0)+.×rotation_matrix_2D 40)
40°
0.98    0.17
0.17   ¯0.98
      8 2⍕(⍉rotation_matrix_2D 45)+.×((2 2⍴0 1 1 0)+.×rotation_matrix_2D 45)
45°
1.00    0.00
0.00   ¯1.00
      8 2⍕(⍉rotation_matrix_2D 50)+.×((2 2⍴0 1 1 0)+.×rotation_matrix_2D 50)
50°
 0.98   ¯0.17
¯0.17   ¯0.98
      8 2⍕(⍉rotation_matrix_2D 60)+.×((2 2⍴0 1 1 0)+.×rotation_matrix_2D 60)
60°
 0.87   ¯0.50
¯0.50   ¯0.87
```

(9)

We can show that $\underline{C}$ relates $\underline{\underline{e}}$ to $\underline{\underline{\bar{e}}}$:

$$C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\boxed{\bar{e}_{mn} = C_{im} C_{jn} e_{ij}}$$

because $E_{11} = E_{22} = 0$

$$E'_{12} = C_{i1} C_{j2} E_{ij} = C_{11} C_{22} E_{12} + C_{21} C_{12} E_{21}$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Gamma + \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}}\Gamma = 0$$

$$e = \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}$$

$$E'_{21} = C_{i2} C_{j1} E_{ij} = 0 \text{ also}$$

$$E_{11} = C_{i1} C_{j1} E_{ij} = C_{11} C_{21} E_{12} + C_{21} C_{11} E_{21}$$

$$= \left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right)\Gamma = \Gamma$$

$$C_{45^\circ} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$E_{22} = C_{i2} C_{j2} E_{ij} = C_{12} C_{22} E_{12} + C_{22} C_{12} E_{21}$$

$$= \left(\frac{-1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}}\right)\Gamma = -\Gamma$$

alternately:

$$\underline{\underline{\bar{e}}} = \underline{C}^T \underline{\underline{e}} \underline{\underline{C}}$$

$$= C^T \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \Gamma C^T \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$= \Gamma \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So it all makes perfect sense!

Can we get there directly? Yes — the diagonal coordinate system is the eigenvectors of $\underline{\underline{e}}$ and the dilation/contraction the eigenvalues (11)

The eigenvalues for this matrix are determined from

$$\det\left[e_{ij} - \lambda\delta_{ij}\right] = 0$$

$$\begin{vmatrix} -\lambda & \Gamma \\ \Gamma & -\lambda \end{vmatrix} = \lambda^2 - \Gamma^2 = 0$$

$$(\lambda - \Gamma)(\lambda + \Gamma) = 0$$

$$\lambda = \pm\Gamma$$

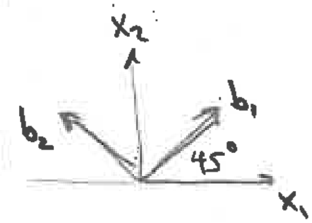
$$e' = \begin{bmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{bmatrix}$$

In which

eigenvectors of strain rate tensor are the coordinate axes in which strain is pure extension or contraction (circle → ellipsoid)

The eigenvectors are determined from

$$\left(e_{ij} - \lambda\delta_{ij}\right) b_j = 0$$



for $\lambda = +\Gamma$ $\quad \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \Gamma \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

which is satisfied by $b_1 = b_2 = 1$

Eigenvectors are new coord axes

for $\lambda = -\Gamma$ $\quad \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = -\Gamma \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

which gives $b_1 = -1$, $b_2 = 1$

$$\begin{array}{cc} \lambda = \Gamma & \lambda = -\Gamma \end{array}$$
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The normalized eigenvectors are also the rotation matrix

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \leftarrow$$

rotation matrix is matrix whose eigenvectors are columns

rotated λ coordinate are columns in rotation matrix

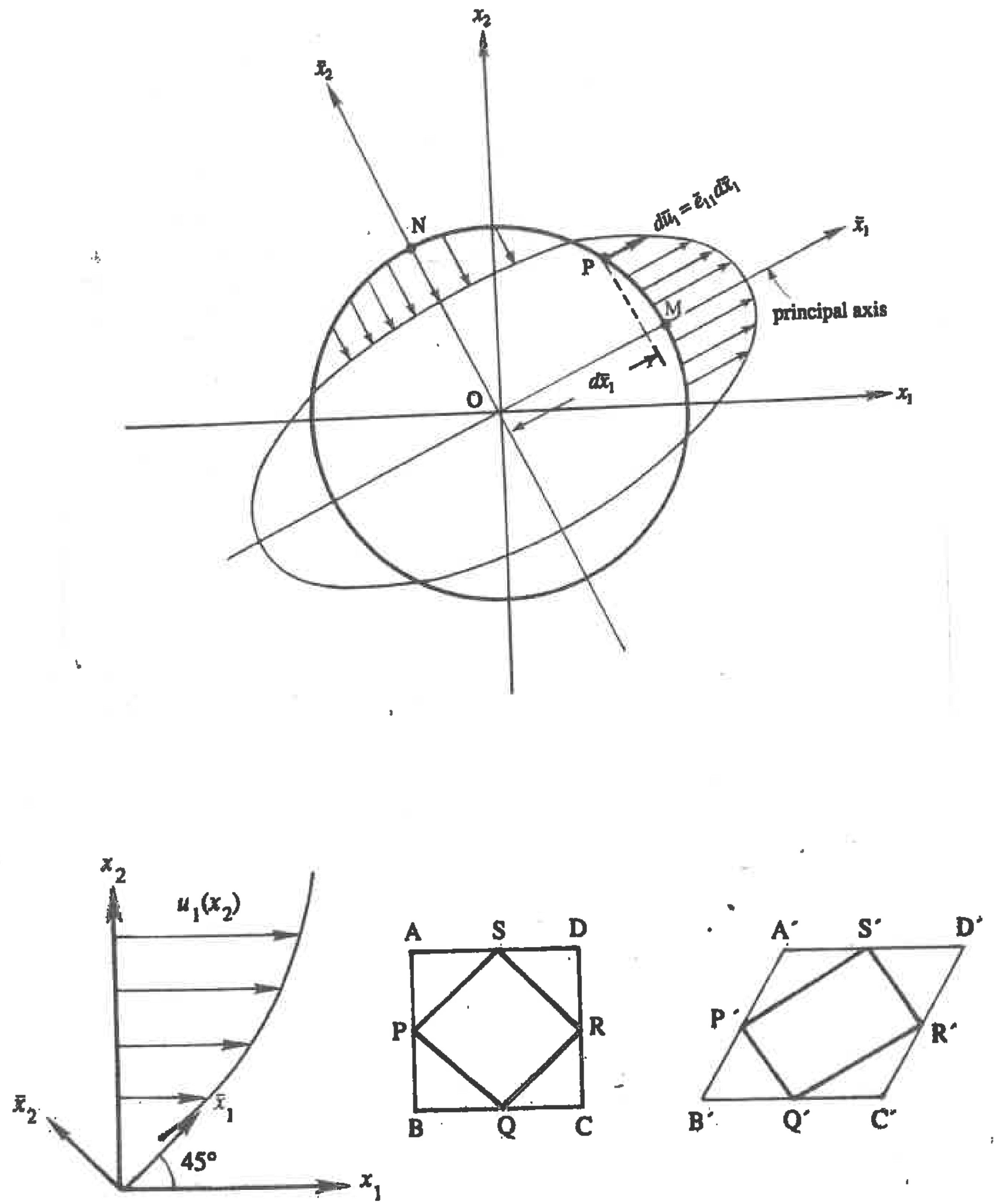


Figure 3.14 Deformation of elements in a parallel shear flow. The element is stretched along the principal axis $\bar{x}_1$ and compressed along the principal axis $\bar{x}_2$.

V. Invariants

A sum of tensor elements that does not change when the coordinate system is rotated is called an invariant of the tensor. A 2nd order tensor has 3 invariants:

Example - bulk strain rate | Invariant System

$I_1 = A_{ii}$ is a scalar which is clearly invariant

$$I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}$$

$I_3 = \det A_{ij}$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Bulk strain, A_{ii}, is a good example. It is easy to see that bulk volume change should look the same in all coordinate systems.

Why are invariants important? Clearly invariants, being independent of coordinate system, are particularly good ways to describe a flow field. We do not need to look at the flow from a particular perspective. The tensor sums will be the same in any coordinate system.

$$\boldsymbol{A} \cdot \boldsymbol{B} = \text{scalar} = A_x B_x + A_y B_y + A_z B_z \tag{2.1}$$

$$\boldsymbol{A} \times \boldsymbol{B} = \text{vector} \tag{2.2}$$

$$(A \times B)_z = A_x B_y - A_y B_x$$

$$(A \times B)_x = A_y B_z - A_z B_y$$

$$(A \times B)_y = A_z B_x - A_x B_z$$

$$\boldsymbol{A} \times \boldsymbol{A} = 0 \tag{2.3}$$

$$\boldsymbol{A} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = 0 \tag{2.4}$$

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C} \tag{2.5}$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}) \tag{2.6}$$

review $\underline{\underline{e}}' = \underline{\underline{C}}^T \underline{\underline{e}}\, \underline{\underline{C}}$

III Gauss Theorem

$$\int_V \partial_i \varphi \; dV = \int_A \varphi \, dA_i$$

φ is a scalar, vector, or tensor field

$$\int_V \partial_i \varphi_i = \int_V \underbrace{\underline{\nabla}\cdot\underline{\varphi}}_{\text{divergence}} = \int_A \underline{\varphi}\cdot d\underline{a}$$

IV. Stokes Theorem

$$\int_A (\nabla\times\underline{u})\cdot dA = \int_C \underline{u}\cdot ds$$

view from outside in direction of $\underline{ds}$, interior is to left

curl (circled)
flux of $\nabla\times u$
thru C

circulation of u around C
(integral of $\underline{u}$ along boundary curve)

We will discuss this more later

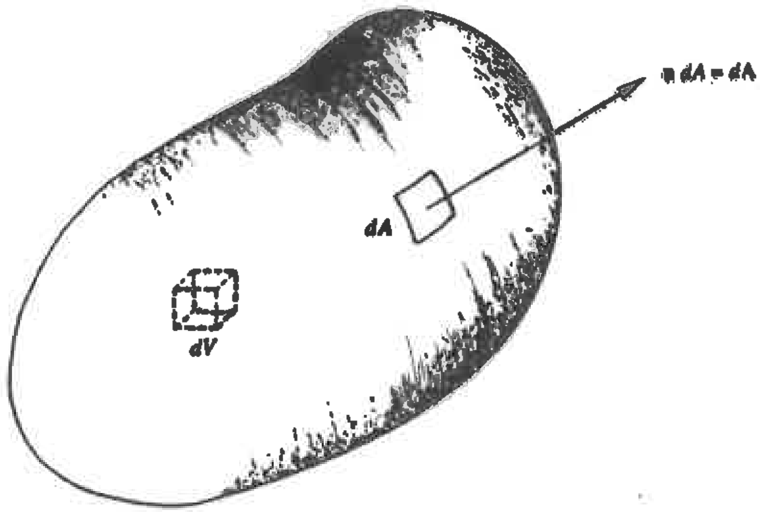


Figure 2.10 Illustration of Gauss' theorem.

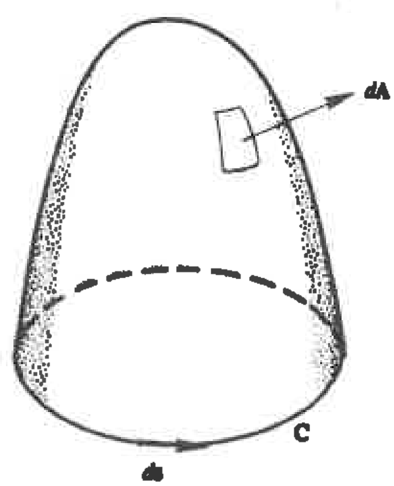


Figure 2.11 Illustration of Stokes' theorem.