Lecture 16 Conservation Laws

We know about vectors and tensors, how they transform, some of their properties (principal axes, eigenvalues + eigenfunctions), and how to use them to describe flow (material + eulerian coordinates, deformation + rotation, stream functions). We are now ready to put in the physics.

⇒ The big secret is the physics is really simple and the same for everything from your bank account to cows in a corral to conservation of mass, momentum, and energy (the heart of fluid dynamics).

Consider, for example, fluid density. A material element of fluid with density ρ must change its volume if its density changes.

②

Definition of divergence:

$$\frac{1}{\rho}\frac{D\rho}{Dt} \equiv -\frac{1}{\gamma}\frac{D\gamma}{Dt} = -\underline{\nabla}\cdot\underline{u} .$$

$$\underline{\nabla}\cdot\underline{u} = \frac{\text{Volume change}}{\text{unit volume}} = \text{divergence of fluid} \quad \left[\frac{1}{\text{sec}}\right] \cong$$

If A_{fm} is a fluid source in $\frac{\text{kg fluid}}{\text{kg fluid-sec}}$, we $\quad \frac{\text{kg fluid}}{\text{kg fluid-sec}}$

A_{fm}: fm ← fluid, mass unit

can express conservation of mass:

$$\frac{A_f}{m} \equiv \frac{\text{kg}/\text{m}^3\text{-sec}}{\text{kg}/\text{m}^3} =$$

see flow — Note, Dash starting generally Leibniz rule to dot the results

(5-1a) **Conservation of fluid mass**

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot u = \frac{A_f}{\rho}} \quad \frac{A_f}{\rho} \left\{ \begin{array}{l} \hat{=}\ \text{kg}/\text{m}^3\text{sec} \\ \hat{=}\ \text{sec}^{-1} \end{array} \right.$$

This is the conservation of mass equation for a **material element**. If there are no fluid sources, so A = 0, material element expansion ($\underline{\nabla}\cdot\underline{u}$) must decrease the density of the material (assuming of course same mass of as material coordinates material is considered)

It's worth doing this in Eulerian coordinates to gain added insight. For a volume fixed in

Space:

Corral fixed in space

kg of fluid in volume (cows)

flux out of volume

addition of mass in volume

just logic of cows in corral

$$\frac{d}{dt}\int_V \rho \, dv + \int_A \rho u \cdot dA = \int_V A_f \, dV$$

$\equiv$ kg/m³-sec

Since:

$$\boxed{\frac{d}{dt}\int_V \rho \, dv = \int_V \frac{\partial \rho}{\partial t} \, dv},$$

depending on t

d/dt

Fixed vol so [illegible]

and:

$$\int_A \rho u \cdot dA = \int_V \nabla\cdot(\rho u) \, dv,$$

by Green's law

Then

$$\int_V \frac{\partial \rho}{\partial t} \, dv + \int_V \underline{\nabla}\cdot(\rho \underline{u}) dV = \int_V A_f \, dV$$

$$= \int_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot\rho u - A_f \right) dV = 0$$

Since this must apply everywhere,

Continuity equation (no sources or sinks of fluid)

(5-1b)
$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot\rho \underline{u} = A_f.}$$

This is same as (5-1), since $\underline{\nabla}\cdot\rho\underline{u} = u\cdot\nabla\rho + \rho\nabla\cdot u$

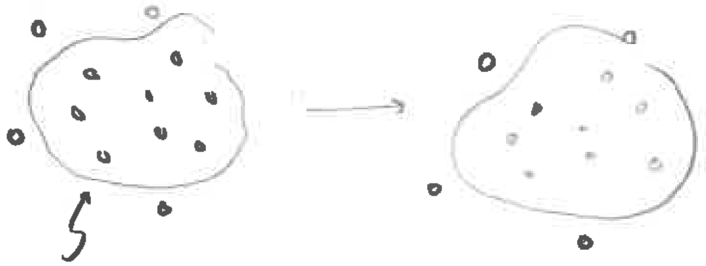
and $\frac{D\rho}{Dt} = \cancel{\frac{\partial\rho}{\partial t}} + u\cdot\nabla\rho$ so: $\frac{D\rho}{Dt} + \rho\nabla\cdot u = A$, as before

See next page

The Lagrangian and Eulerian description are the same and can go back + forth:

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \underline{\nabla}\cdot\underline{u} = 0}$$

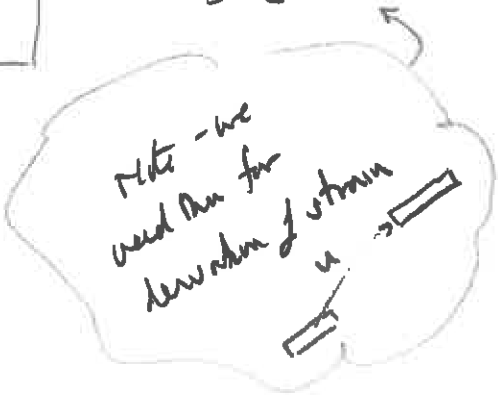
Lagrangian



enclose same material

Expand

Same material in larger volume



note - we used this for derivation of strain

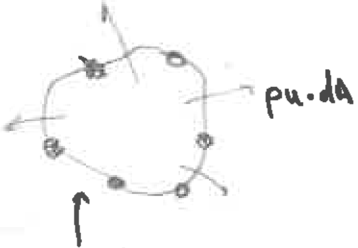
— Therefore density decreased

Chain Rule

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + u\cdot\underline{\nabla}}$$

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} + \frac{u\cdot\nabla\rho}{\rho} + \nabla\cdot u = 0$$

$$\frac{\partial\rho}{\partial t} + \underline{u}\cdot\nabla\rho + \rho\nabla\cdot\underline{u} = 0$$



Circle fixed in space

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\underline{u} = 0}$$

Eulerian

Flux out of fixed circle determines change in density

Conservation of a chemical species

Now consider a fluid with a dissolved chemical in it. Let $C \equiv \frac{\text{kg chemical}}{\text{kg fluid}}$ = ~~mass~~ fraction C, then by the same Eulerian logic:

$$\int_V \frac{\partial \rho c}{\partial t} = -\int_A c\rho \underline{u} \cdot d\underline{A} + \int_V A_c \left[\frac{\text{kg}\,C}{\text{m}^3\text{-sec}}\right] dV$$

$$\frac{\partial \rho c}{\partial t} + \underline{\nabla} \cdot c\rho\underline{u} - A_c = 0$$

$$\downarrow$$

$$\partial_i c\rho u_i$$

$$\boxed{\rho\frac{\partial c}{\partial t}} + \underbrace{c\frac{\partial \rho}{\partial t} + c\,\partial_i \rho u_i}_{} + \boxed{\rho u_i \partial_i c} - A_c = 0$$

b/c (5-11)

$$\rho\frac{Dc}{Dt} + \left(c\,A_f - A_c\right) = 0.$$

$\left[\frac{\text{kg}\,c}{\text{kg fld}}\right]\left[\frac{\text{kg fluid}}{\text{m}^3\text{-sec}}\right]$ $\left(\frac{\text{kg}\,c}{\text{m}^3\text{-sec}}\right)$

(same as A_c)

$$\boxed{\rho\frac{Dc}{Dt} = \left(A_c - c\,A_f\right)}$$

(Equation 4.6 above)

Rule: $\frac{D}{Dt}$ moves inside volume integral but operates only on C (not ρ)

$$\boxed{\frac{D}{Dt}\int_V c\rho\, dV = \int_V \rho\frac{Dc}{Dt}\, dV}$$

If no fluid source, $A_f \to 0$ and increase in C produced by introducing C at $\frac{\text{kg}\,C}{\text{m}^3\text{-sec}} = A_c$

If inject fluid at A_f kg fld/kg fld-sec

$$A_c = C_{inj}\,A_f$$

makes sense! If $C_{inj} = C$, $A = 0$ as should be → same form as $A_f\left(C_{inj} - C\right)$

1. Generalization - all conservation equations have the same form; (5

So, we can generalize. Consider any conserved quantity^{G} Ψ

Whether G is advected component kg i / kg fluid, momentum measured in momentum / kg fluid, or energy measured in energy units / kg fluid etc, we can write a general conservation equation for Ψ :=

Ψ ≡ conserved quantity (e.g. chemical mass, momentum, energy).

$G = \Psi / \text{kg fluid}$

J'_Ψ = non-advective flux of Ψ across a unit area

A_Ψ = source of Ψ per unit volume of space-s

Then conservation of Ψ can be written:

add flux out (eg, diffusion)

$$\left(\frac{D}{Dt}\int_{V} G\rho\, dV = \int_{V}\left(\rho\frac{DG}{Dt} + \left(GA_f - A_\Psi\right) dV\right)\right) + \int_{A} J_\Psi \cdot dA = 0$$

Rule → V

note DG/Dt as in book but going just G

generalized source

$$\shortparallel$$

Gauss Thy $+ \int_V \nabla \cdot J_\Psi \, dV$

⑥

So:

$$\int_{\forall}\left(\rho\frac{DG}{Dt}+\nabla\cdot J_{\boxed{\mathfrak{G}}}+\left(G A_f - A_{\boxed{\mathfrak{G}}}\right)\right)d\forall = 0$$

Since this relation must hold everywhere

$$\boxed{\rho\frac{DG}{Dt}+\nabla\cdot J_{\boxed{\mathfrak{G}}}+\left(\underbrace{G}_{\boxed{\mathfrak{G}}/\text{kg fluid}}\underbrace{A_f}_{\frac{\text{kg fluid}}{m^3\text{-sec}}} - A_{\boxed{\mathfrak{G}}}\right)=0}$$

$\equiv \left[\boxed{\mathfrak{G}}/m^3\text{-sec}\right]$ $\equiv \left[\boxed{\mathfrak{G}}/m^3\text{-sec}\right]$

(5-2) General Conservation equation

Now consider some particular examples; with

A. Conservation of chemical mass (eqn.)

$\boxed{\mathfrak{G}}$ = kg chemical c = M_c

$G = \dfrac{\text{kg chemical c}}{\text{kg fluid}} = C = \dfrac{M_c}{M_f}$

chemical mass

$A_{\boxed{\mathfrak{G}}} = \dfrac{\text{kg chemical c}}{\text{unit volume}} \Big/ \text{unit time}$

$J_{\boxed{\mathfrak{G}}}$ = kg of c diffusing across an area per unit time $= \underline{\underline{D}}\cdot\nabla\overbrace{\rho c}$

diffuse in space

$M_c/m^3 = M_f/m^3 \cdot M_c/M_f$

$m^2/\text{sec}\ \frac{1}{m}\ \frac{\text{kg } i}{m^3} = \frac{\text{kg } i}{m^2\text{-sec}}$

where c = mass fraction of c

$\underline{\underline{D}}$ = diffusion tensor in m^2/sec.

(5-3) conservation of chemical mass

$$\boxed{\rho\frac{Dc}{Dt}+\nabla\cdot\underline{\underline{D}}\cdot\nabla\rho c+\left(\underbrace{cA_f - A_c}\right)=0}$$

But first let's complete the story by sketching E conservation

B. The Energy Equation

$$\mathcal{G} = \text{energy} = \left(\frac{1}{2}u_i^2 + e\right)\rho$$

$$= \left(\frac{1}{2}\left(u_1^2+u_2^2+u_3^2\right) + e\right)\rho$$

where e = internal energy of molecular vibration + rotation

Energy

$$G = \frac{\text{Energy}}{\text{kg}} = \frac{1}{2}u_i^2 + e$$

$$A_{\mathcal{G}} = \text{source of energy} = \underset{\text{work}}{u_i\frac{\partial \tau_{ij}}{\partial x_j}} + \underset{\text{potential}}{\rho u_i g_i}$$

$$J_{\mathcal{G}} = j \qquad (\text{diffusive heat flux})$$

So $$\rho\frac{D\left(\frac{1}{2}u_i^2 + e\right)}{Dt} + \nabla\cdot j - u_i\frac{\partial \tau_{ij}}{\partial x_j} - \rho u_i g_i = 0$$

reworking terms for physical insight yields (Flow #2):

$$\underbrace{\rho\frac{D\left(\frac{1}{2}u_i^2 + gz + e\right)}{Dt}}_{\text{rate of change of kinetic, potential, and internal energy}} + \underbrace{\nabla\cdot j}_{\text{diffusion of heat}} = \underbrace{\frac{\partial}{\partial x_j}\left(u_i\tau_{ij}\right)}_{\text{working by surface forces}} + \underbrace{p\nabla\cdot u}_{\text{work of volume expansion}} - \underbrace{\phi}_{\text{work of viscous dissipation}}$$

If the flow speeds are small compared to the speed of sound, so kinetic energy small, and we neglect changes in gravitational potential, and heating due to viscous stress and dilatational work are negligible, then:

$$\rho \frac{De}{Dt} = -\underline{\nabla} \cdot \underline{j}$$

Internal Energy available at constant pressure

but $\frac{De}{Dt} = C_p \frac{DT}{Dt}$

and $\underline{j} = -k \nabla T$ (Fourier's law).

Thus

$$\rho C_p \frac{DT}{Dt} = k \nabla^2 T$$ (eq 4.88 in book)

(5-17) $$\boxed{\frac{DT}{Dt} = \kappa \nabla^2 T}$$

$\kappa = \frac{k}{\rho C_p}$ = Thermal diffusivity $\overset{m}{=}$ m^2/s .

Note heat flows relative to material matrix .

B. Conservation of fluid momentum

momentum

$\underline{G}$ = fluid momentum = $\rho \underline{u}$

G = $\frac{\text{fluid momentum}}{\text{kg of fluid}}$ = $\underline{u}$

Gauss's Thm: $A = \frac{\partial \tau_{ij}}{\partial x_j}$

$A_{\underline{G}}$ = $\frac{\text{unbalanced force}}{\text{unit volume}}$ = $\int_{\partial V} \underline{\tau} \cdot d\underline{a} + \rho g$

area, ∂V not V

$J_{\underline{G}}$ = diffusion of momentum = 0
(could think of as viscous dissipation – but we will include this in $A_{\underline{G}}$)

(5-4) conservation of momentum (Cauchy's equation of motion) Eq 4.15 in book

$$\boxed{\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}} - \underline{u} A_f \quad \text{if } A_f \neq 0$$

fluid source in kg fluid / kg fluid / m³-sec

we've used Gauss Thm.

We will spend quite a lot of time on this equation. We'll just illustrate conservation of momentum in simplest direct terms. Then derive Navier-Stokes equation for viscous flow (parallel flow, porous medium flow). Then specialize to Bernoulli equation. Then put in rotating frame, then consider vorticity.

II. Momentum Examples

A. Momentum Theorem for fixed Eulerian volume:

(1) $$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

Cauchy Equation
(Consv of Momentum)

(2) $$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

add u_i times conservation of mass, and integrate over volume

(3) $$+ \quad u_i \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) \right) = 0$$

Consv mass
(Continuity)

(4) $$\begin{array}{cccc} \int_V \Big(\frac{\partial \rho u_i}{\partial t} & + \frac{\partial \rho u_i u_j}{\partial x_j} & = \rho g_i & + \frac{\partial \tau_{ij}}{\partial x_j} \Big) dV \\ \| & \| & \| & \| \\ \frac{\partial}{\partial t}\int \rho u_i \, dV & \int_A \rho u_i u_j \, dA_j & & \int_A \tau_{ij} \, dA_j \\ \| & \| & & \| \\ \frac{\partial M_i}{\partial t} & + \dot{M}_{out} & = F_{bi} & + F_S \end{array}$$

body force

(5) Newton's Law

$$\boxed{\begin{array}{c} \underline{F} = \frac{d\underline{m}}{dt} + \underline{\dot{m}}_{out} \\ \text{Force on volume} = \text{chg internal momentum} \quad \text{export of momentum} \\ \underline{T} = \frac{dH}{dt} + \dot{H}_{out} \end{array}}$$

(6)

Momentum
Theorem for fixed
(Eulerian) volume
(3 indep. components)

Torque – first kind of
momentum, $\underline{H} = \underline{r} \times \underline{u} \, dm$

B. Example - Wind force on a tank

$$\underline{F} = \frac{d\underline{M}}{dt} + \dot{M}_{out}$$

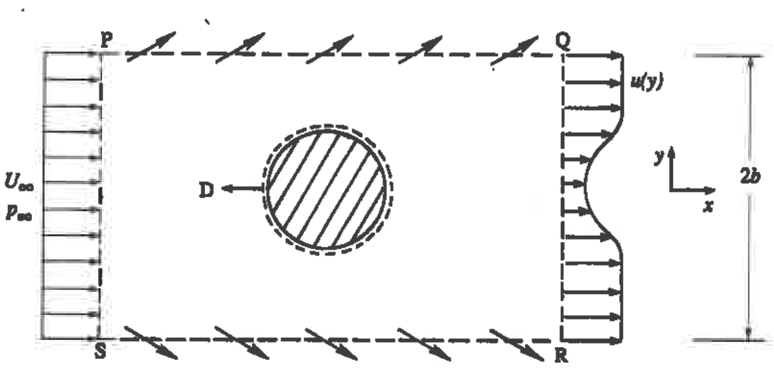


Figure 4.8 Momentum balance of flow over a body (Example 4.1).

boundaries distant enough that $p \sim p_\infty$

QR distant enough that streamlines ∥ and $V \sim 0$

steady state

consv momentum (steady state) $\quad \therefore \quad F_x = \dot{M}_x^{out}$

$$\dot{M}_x^{PS} = \int_{-b}^{b} U_\infty (\rho U_\infty \, dy) = -2b\rho U_\infty^2$$

$$\dot{M}_x^{QR} = \int_{-b}^{b} u(\rho u \, dy) = \rho \int_{-b}^{b} u^2 dy$$

Conservation of mass (mass flux out sides must equal mass flux deficit):

Consv mass $\quad$ mass $\quad \dot{m}^{PQ} + \dot{m}^{SR} = \rho \int_{-b}^{b} (U_\infty - u)\, dy$

momentum $\quad \dot{M}_x^{PQ} + \dot{M}_x^{SR} = \rho U_\infty \int_{-b}^{b} (U_\infty - u)\, dy$

$$\overset{In}{M_x^{PS}} + \overset{out}{M_x^{QR}} + \overset{sides}{\left(\dot{M}_x^{PQ} + \dot{M}_x^{SR}\right)} = -2b\rho U_\infty^2 + \rho \int_{-b}^{b} u^2 dy + \rho U_\infty \int_{-b}^{b} (U_\infty - u)\, dy$$

$$\boxed{\underline{F_x} = \rho \int_{-b}^{b} u(U_\infty - u)\, dy}$$

Force on tank

momentum deficit downwind of Tank

① note how use both consv momentum and consv mass

② Logic - flow out sides!

B. Tank example by contour Integration

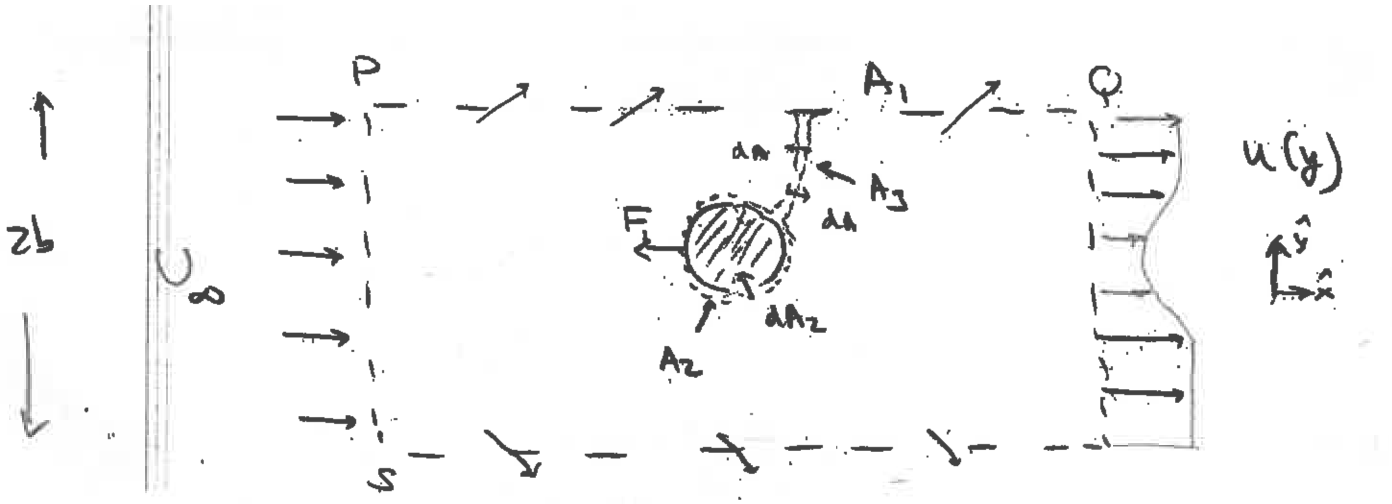
go back to (4) and consider steady state and neglect gravity:

(7) $$\int_V \frac{\partial}{\partial x_j}\left(\rho u_i u_j - \tau_{ij}\right) = 0$$

by stokes theorem

(8) $$\boxed{\int_A \left(\rho \underline{u}\,\underline{u} - \underline{\underline{\tau}}\right)\cdot d\underline{A} = 0}$$ conserv momentum

(9) $$\boxed{\int_A \rho \underline{u}\cdot d\underline{A} = 0}$$ conservation of mass



$$0 = \int_{A_2}\left(\overset{\text{can't flow through solid}\ 0}{\cancel{\rho u u}} - \underline{\underline{\tau}}\right)\cdot dA_2 + \int_{A_3}\left(\rho u u - \underset{0}{\cancel{\tau}}\right)\cdot dA_3 + \int_{A_1}\left(\rho \underline{u}\,\underline{u} - \underline{\underline{\tau}}\right)\cdot dA_1$$

A_3 collapses when ended → 0

$$\parallel$$

$$\int_{A_2} \underline{\underline{\tau}}\cdot dA_2$$

$$\parallel$$

force on F_x

Examine last term:

$$F_{A_1} = -\int_{A_1} (\rho \underline{u}\underline{u} - \tau)\cdot dA_1$$

τ = const pressure - cancels out around circuit

$$= -\int_{A_1} \rho \underline{u}\underline{u} \cdot A_1$$

A_1 is outward pointing normal so

$$\int_{A_1^{in}} \rho uu\cdot dA_1 = -\int_{A_1^{in}\,y=-b}^{b} \rho \cancel{U}_\infty^2 \, dy$$

$$\int_{A_1^{sides}} \rho uu\cdot dA_1 = \rho U_\infty \int_{y=-b}^{b} \underbrace{(\cancel{U}_\infty - u(y))}_{\text{deficit must go out sides}} dy$$

momentum flux out side

by conservation of mass this is fluid exiting the sides

$$\int_{A_1^{out}} \rho uu\cdot dA = \rho \int_{y=-b}^{b} u^2(y)\, dy$$

$$\|$$

$$\rho \int_{y=-b}^{b} (U_\infty u - u^2)\, dy$$

Since it has a velocity close to U_∞ the momentum is as indicated

$$\boxed{\therefore\ F_8 = -\rho \int_{y=-b}^{b} u(U_\infty - u)\, dy}$$

Momentum flux leaving system

Force on object in flow field can be measured by measuring y-velocity across a transect.

b. Sprinkler Example

Sprinkler example is beautiful example of method

flux of momentum

T = torque =

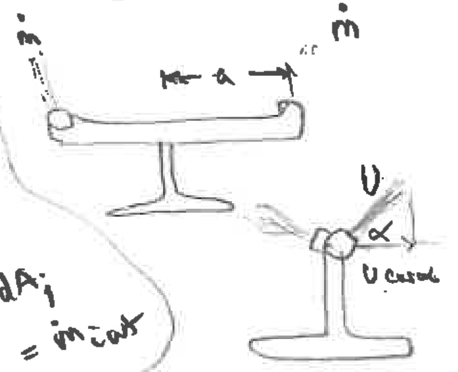
$= Fr$

$$H = \text{angular momentum} = \int \underline{r} \times \underline{u} \, dm$$

$$T = \frac{dH}{dt} + \dot{H}^{out}$$

outflow of momentum

equiv of $\int_R \rho u u_i \, dA_j = \dot{m}$ out



Sprinkler steady state

transient change in angular momentum

$$\therefore \quad T = \dot{H}^{out}$$

area of nozzle

$\dot{m} = \rho A U \doteq$ kg/sec

$$\dot{H}_z^{out} = \left[(\dot{m} U \cos\alpha) a \right] 2$$

momentum flux leaving system (= force at two ends)

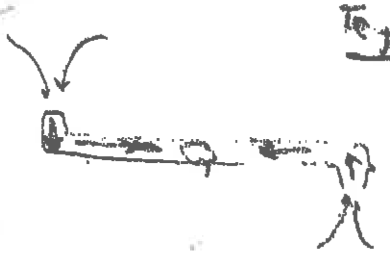
$$\boxed{T = 2a\rho A U^2 \cos\alpha}$$

At steady state F must balance momentum flux. This is the meaning of (J-4 example).

sign of flow does not matter since squared!

student point - vg A+

Brain Teaser



Feynman problem

② $\dot{m}u$ $\dot{m}a$

↑ ↑

neg neg

① # will go same direction

Force has [illegible]

Flows 9 - Revy book derivation

Unless they reproduce so

ALL PHYSICS HERE

Indicates material volume following particles

$$\boxed{\frac{D}{Dt}\int_{\mathcal{V}} \rho \, d\mathcal{V} = \int A\left[\frac{\text{cells}}{m^3}\right] d\mathcal{V}}$$

or $\frac{kg}{m^3}$

CELLS IN ELASTIC CORRALE

(material volume ≡ $\mathcal{V}$)

Now in spatial coordinates the density / cells decreases if the volume expands. By the generalized Leibnitz rule VG

[illegible] VOLUME

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho \, d\mathcal{V} = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} + \int_{A} u\rho \cdot dA = 0$$

material volume · fixed region in space

By the Gauss Theorem (applied to $\partial\mathcal{V}$ or $\mathcal{V}$)

$$\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot u\rho\right) d\mathcal{V} = 0$$

must apply to any volume

Then

$$\frac{\partial \rho}{\partial t} + \underbrace{\nabla \cdot u\rho}_{u\cdot\nabla\rho + \rho\nabla\cdot u} = 0$$

(Equation 2.5)

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot u = 0}$$

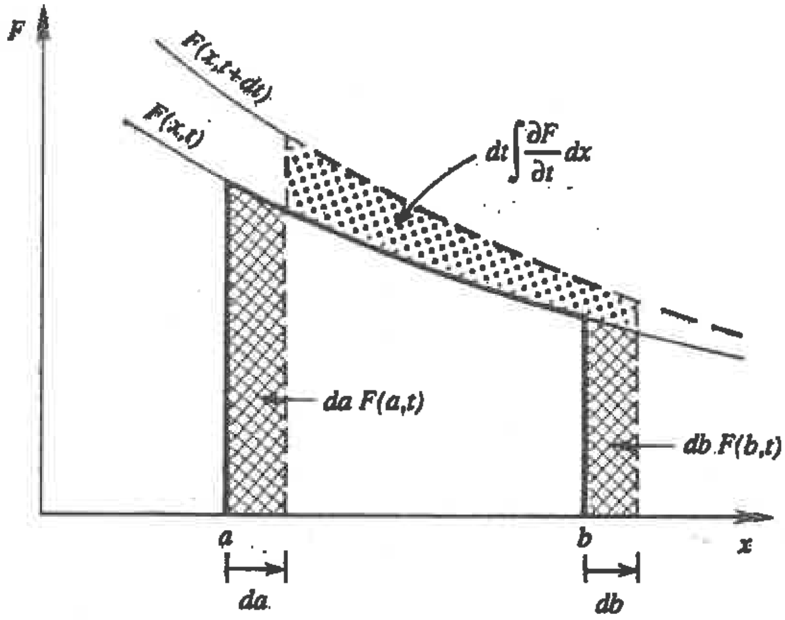


Figure 4.1 Graphical illustration of Leibniz's theorem.

$$\frac{d}{dt}\int_{V(t)} F(\mathbf{x},t)\,dV = \int_{V(t)} \frac{\partial F}{\partial t} dV + \int_{A(t)} \mathbf{dA}\cdot\mathbf{u}_A F, \qquad (4.3)$$

Class #2 Lecture 6

$$\rho \frac{D(\frac{1}{2}u_i^2 + e)}{Dt} + \nabla\cdot j - u_i \frac{\partial \tau_{ij}}{\partial x_j} - \rho u_i g_i = 0$$

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(u_i \tau_{ij}\right) - \tau_{ij} e_{ij}$$

work by surface forces

$$\rho u_i g_i = \frac{D}{Dt}\rho g z = \rho \cancelto{0}{\frac{\partial z}{\partial t}} + \rho g u_i \partial_i z$$

scalar (not g_i)

$$\downarrow$$

Navier-Stokes $\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu(\nabla\cdot u)\delta_{ij}$

$$\tau_{ij}e_{ij} = -p\nabla\cdot u + \underbrace{2\mu e_{ij}e_{ij} - \frac{2}{3}\mu(\nabla\cdot u)^2}_{\text{viscous deformation work}}$$

$$= 2\mu\left[e_{ij} - \frac{1}{3}(\nabla\cdot u)\delta_{ij}\right]^2$$

$$\equiv \phi$$

rate of viscous dissipation

Can show this is true by evaluating $[\ ]^2 = \frac{1}{9}(\nabla\cdot u)^2 \overbrace{\delta_{ij}\delta_{ij}}^{3} + e_{ij}e_{ij} - \frac{2}{3}(\nabla\cdot u)^2$

$= e_{ij}e_{ij} - 1/3(\nabla\cdot u)^2$

So collecting terms:

$$\rho \frac{D(\frac{1}{2}u_i^2 + gz + e)}{Dt} + \nabla\cdot j = \frac{\partial}{\partial x_j}\left(u_i \tau_{ij}\right) + p\underline{\nabla}\cdot\underline{u} - \phi$$

rate of change of kinetic, potential and internal energy

efflux of heat

work by surface forces

vol expansion work

viscous dissip. rate